MATH 141 - Quiz 6, Solution

12 noon

1. (5 points) Determine whether the following series converges or diverges. Explain what test(s) you use and how each test applies.

$$\sum_{n=2}^{\infty} \frac{2^n n!}{(2n)!}.$$

Solution: Apply ratio test on this series.

$$\frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1}}{2^n}\frac{(n+1)!}{n!}}{\frac{(2n+2)!}{(2n)!}} = \frac{2(n+1)}{(2n+2)(2n+1)}$$

Take the limit,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2(n+1)}{(2n+2)(2n+1)} = 0$$

So, the ratio is 0, which is less than 1. By ratio test, the series converges.

2. (5 points) Find the radius of convergence of the series

$$\sum_{n=1}^{\infty} \frac{2^n}{n!} x^n.$$

Solution: Use generalized ratio test on this series.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{2^{n+1}}{2^n}}{\frac{(n+1)!}{n!}} \frac{|x|^{n+1}}{|x|^n} = \frac{2}{n+1}|x|$$

When $n \to \infty$, the ratio tends to be 0. So, the radius of convergence is $R = \infty$.

3. (5 points) Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{2^n}{3^{n+1}}.$$

Solution: Rewrite the series like below.

$$\sum_{n=1}^{\infty} \frac{2^n}{3^{n+1}} = \sum_{n=1}^{\infty} \frac{1}{3} \left(\frac{2}{3}\right)^n$$

Then, we can apply formula for geometric series (as $|r| = 2/3 < 1$)

$$\sum_{n=1}^{\infty} \frac{2^n}{3^{n+1}} = \frac{\frac{1}{3} \cdot \left(\frac{2}{3}\right)^1}{1 - \frac{2}{3}} = \frac{2}{3}.$$

4. (5 points) Find the interval of convergence of the series

$$\sum_{n=1}^{\infty} 2^n x^n.$$

Solution: Use generalized ratio test on this series. (Generalized root test is also suitable.)

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}}{2^n} \frac{|x|^{n+1}}{|x|^n} = 2|x|$$

The series will converge if $2|x| < 1$, and will diverge if $2|x| > 1$. So, the radius of convergence is $R = 1/2$.

Specially, when $x = 1/2$,

$$\sum_{n=1}^{\infty} 2^n x^n = \sum_{n=1}^{\infty} 2^n \left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} 1 = \infty$$

So, $1/2$ is excluded in the interval of convergence.

Also, when $x = -1/2$,

$$\sum_{n=1}^{\infty} 2^n x^n = \sum_{n=1}^{\infty} 2^n \left(-\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} (-1)^n$$

This series diverge by alternating series test. ($\lim_{n\to\infty} a_n = 1$) So, $-1/2$ is also excluded in the interval of convergence.

To sum up, the interval of convergence should be $(-1/2, 1/2)$.