MATH 141 - Quiz 6, Solution

1 pm

1. (5 points) Determine whether the following series converges or diverges. Explain what test(s) you use and how each test applies.

$$\sum_{n=1}^{\infty} \frac{(2n)!}{2^n n!}.$$

Solution: Apply ratio test on this series.

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(2n+2)!}{(2n)!}}{\frac{2^{n+1}}{2^n}\frac{(n+1)!}{n!}} = \frac{(2n+2)(2n+1)}{2(n+1)}$$

Take the limit,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{2(n+1)} = \infty$$

So, the ratio tends to be infinity, which is greater than 1. By ratio test, the series diverges.

2. (5 points) Find the radius of convergence of the series

$$\sum_{n=1}^{\infty} \frac{3^n}{(2n)!} x^n.$$

Solution: Use generalized ratio test on this series.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{3^{n+1}}{3^n}}{\frac{(2n+2)!}{(2n)!}} \frac{|x|^{n+1}}{|x|^n} = \frac{3}{(2n+2)(2n+1)}|x|$$

When $n \to \infty$, the ratio tends to be 0. So, the radius of convergence is $R = \infty$.

3. (5 points) Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{2^{n+1}}{3^n}.$$

Solution: Rewrite the series like below.

$$\sum_{n=1}^{\infty} \frac{2^{n+1}}{3^n} = \sum_{n=1}^{\infty} 2 \left(\frac{2}{3}\right)^n$$

Then, we can apply formula for geometric series (as $|r| = 2/3 < 1$)

$$\sum_{n=1}^{\infty} \frac{2^{n+1}}{3^n} = \frac{2 \cdot \left(\frac{2}{3}\right)^1}{1 - \frac{2}{3}} = 4.$$

4. (5 points) Find the interval of convergence of the series

$$\sum_{n=1}^{\infty} 2^n x^n.$$

Solution: Use generalized ratio test on this series.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{3^{n+1}}{3^n} \frac{|x|^{n+1}}{|x|^n} = 3|x|$$

The series will converge if $3|x| < 1$, and will diverge if $3|x| > 1$. So, the radius of convergence is $R = 1/3$.

Specially, when $x = 1/3$,

$$\sum_{n=1}^{\infty} 3^n x^n = \sum_{n=1}^{\infty} 3^n \left(\frac{1}{3}\right)^n = \sum_{n=1}^{\infty} 1 = \infty$$

So, $1/3$ is excluded in the interval of convergence.

Also, when $x = -1/3$,

$$\sum_{n=1}^{\infty} 3^n x^n = \sum_{n=1}^{\infty} 3^n \left(-\frac{1}{3}\right)^n = \sum_{n=1}^{\infty} (-1)^n$$

This series diverge by alternating series test. ($\lim_{n\to\infty} a_n = 1$) So, $-1/3$ is also excluded in the interval of convergence.

To sum up, the interval of convergence should be $(-1/3, 1/3)$.