Math 410. HW 4 Solutions

1. First, we note that there are several ways of expressing the solution to this problem. They are of course equivalent, and the difference between them is absorbed into the constant of integration. We present one: Observe that

$$\frac{1}{\sqrt{(x-a)(x-b)}} = \frac{1}{\sqrt{x-a}+\sqrt{x-b}} \frac{\sqrt{x-a}+\sqrt{x-b}}{\sqrt{(x-a)(x-b)}} = \frac{1}{\sqrt{x-a}+\sqrt{x-b}} (\frac{1}{\sqrt{x-a}} + \frac{1}{\sqrt{x-b}}).$$

Let $u = \sqrt{x-a} + \sqrt{x-b}$. Then $du = \frac{dx}{2\sqrt{x-a}} + \frac{dx}{2\sqrt{x-b}}$. Hence,

$$\int \frac{dx}{\sqrt{(x-a)(x-b)}} = \int \frac{1}{\sqrt{(x-a)}+\sqrt{(x-b)}} (\frac{1}{\sqrt{x-a}} + \frac{1}{\sqrt{x-b}}) dx =$$
$$\int \frac{2du}{u} = 2\ln(u) + C = 2\ln(\sqrt{(x-a)} + \sqrt{(x-b)}) + C. \quad \square$$

2. Note that $\sqrt{\frac{a+x}{a-x}} = \frac{a+x}{\sqrt{a^2-x^2}}$. Using the substitution $x = a\sin(\theta)$ we get

$$\int \sqrt{\frac{a+x}{a-x}} dx = \int \frac{a+x}{\sqrt{a^2-x^2}} dx = \int \frac{a+a\sin(\theta)}{a\cos(\theta)} a\cos(\theta) d\theta$$
$$= a\theta - a\cos(\theta) + C = a\arcsin(\frac{x}{a}) - \sqrt{a^2-x^2} + C. \quad \square$$

3. By the Second Fundamental Theorem (Differentiating integrals) and its corollary (6.32), the derivatives of the two functions coincide:

$$\frac{d}{dx} a \int_1^x \frac{1}{t} dt = \frac{a}{x},$$
$$\frac{d}{dx} \int_1^{x^a} \frac{1}{t} dt = \frac{1}{x^a} a x^{a-1} = \frac{a}{x}.$$

Moreover, evaluating at $x = 1$ we obtain that $a \int_1^1 \frac{1}{t} dt = 0 = \int_1^{1^a} \frac{1}{t} dt$. Hence, by the Identity Criterion (4.20), the two functions coincide, yielding

$$\frac{d}{dx} a \int_1^x \frac{1}{t} dt = \frac{d}{dx} \int_1^{x^a} \frac{1}{t} dt. \quad \square$$

4. Recall that Riemann integrable functions are bounded. Hence, there exists a constant $C > 0$ such that $-C \leq f(t) \leq C$ for all $t \in [a, b]$. Then, by additivity and monotonicity properties of the integral,

$$|\int_a^x f(t)dt - \int_a^y f(t)dt| = |\int_y^x f(t)dt| \leq |\int_y^x C dt| = C|x-y|. \quad \square$$

5. By the triangle inequality, $|f(x) + g(x)| \leq |f(x)| + |g(x)| \; \forall x \in [a, b]$. Therefore, by monotonicity and linearity properties of the integral,

$$\int_a^b |f(x) + g(x)| dx \leq \int_a^b |f(x)| + |g(x)| dx = \int_a^b |f(x)| dx + \int_a^b |g(x)| dx. \quad \square$$