Math 410. HW 5 Solutions

1. First, observe that since the functions $f(x)$ and $\sin(x)$ have contact of order 99 at 0, their Taylor polynomials of degree 3 coincide. Observing that $\sin(0) = 0$, $\frac{d}{dx}\sin(x)|_{x=0} = \cos(0) = 1$, $\frac{d^2}{dx^2}\sin(x)|_{x=0} = -\sin(0) = 0$ and $\frac{d^3}{dx^3}\sin(x)|_{x=0} = -\cos(0) = -1$, $p_3(x) = x - \frac{x^3}{3!}$.

2. We compute it from the definition.

$$\begin{aligned}
f(x) &= x^6 - 3x^4 + 2x - 1, & f(1) &= -1\\
f'(x) &= 6x^5 - 12x^3 + 2, & f'(1) &= -4\\
f''(x) &= 30x^4 - 36x^2, & f''(1) &= -6\\
f'''(x) &= 120x^3 - 72x, & f'''(1) &= 48\\
f^{(4)}(x) &= 360x^2 - 72, & f^{(4)}(1) &= 288\\
f^{(5)}(x) &= 720x, & f^{(5)}(1) &= 720.
\end{aligned}$$

   Then, $p_5(x) = -1 - 4(x-1) - 3(x-1)^2 + 8(x-1)^3 + 12(x-1)^4 + 6(x-1)^5$.

3. Using the information above, we have that $f(-1) = -5, f'(-1) = 8, f''(1) = -6, f'''(1) = -48, f^{(4)}(-1) = 288$ and $f^{(5)}(-1) = -720$. Moreover, $f^{(6)}(x) = 720$ and $f^{(n)}(x) = 0$ for all $n \geq 7$. Therefore, the Taylor series of $x^6 - 3x^4 + 2x - 1$ at $x_0 = -1$ is

$$-5 + 8(x+1) - 3(x+1)^2 - 8(x+1)^3 + 12(x+1)^4 - 6(x+1)^5 + (x+1)^6.$$

4. To have a shorter notation, we will write f instead of $f(x)$. We start with $f' = 1 + f^2$, so $f'(0) = 1$. Taking further derivatives and evaluating at 0, we get

$$\begin{aligned}
f'' &= 2ff' = 2f(1+f^2) = 2(f+f^3), & f''(0) &= 0,\\
f''' &= 2f'(1+3f^2) = 2(1+f^2)(1+3f^2) = 2(1+4f^2+3f^4), & f'''(0) &= 2,\\
f^{(4)} &= 2f'(8f+12f^3) = 2(1+f^2)(8f+12f^3) = 2(8f+20f^3+12f^5), & f^{(4)}(0) &= 0,\\
f^{(5)} &= 2f'(8+60f^2+60f^4), & f^{(5)}(0) &= 16,\\
f^{(6)} &= 2f''(8+60f^2+60f^4) + 2f'(120ff' + 240f^3f'), & f^{(6)}(0) &= 0.
\end{aligned}$$

   Thus, $p_6(x) = x + \frac{2}{3!}x^3 + \frac{16}{5!}x^5$.

5. We differentiate and observe that the first three derivatives of f satisfy $f'(x) = 2xf(x) + 1, f''(x) = 2xf'(x) + 2f(x), f'''(x) = 2xf''(x) + 4f'(x)$. Using induction, we can show that $f^{(n+1)} = 2xf^{(n)} + 2nf^{(n-1)}$ for all $n \in \mathbb{N}$. Indeed, the base case, $n = 1$, follows from the computations above, and if the proposition holds for some $k \geq 1$, we take derivatives and get $f^{(k+1)} = (f^{(k)})' = \left(2xf^{(k-1)} + 2(k-1)f^{(k-2)}\right)' = 2xf^{(k)} + 2kf^{(k-1)}$, as desired. Hence, $f^{(n+1)}(0) = 2nf^{(n-1)}(0)$ for all $n \in \mathbb{N}$. Since $f(0) = 0$, we get that $f^{(2n)}(0) = 0$ for all $n \in \mathbb{N}$. Again by induction we show that $f^{(2n+1)}(0) = 4^n n!$ for all $n \geq 0$. The base case, $n = 0$, holds since $f'(0) = 1 = 4^0 0!$. Assuming $f^{(2k+1)}(0) = 4^k k!$ holds for some $k \geq 0$, we have that $f^{(2(k+1)+1)}(0) = 2(2k+2)f^{(2k+1)}(0) = 4(k+1)4^k k! = 4^{k+1}(k+1)!$, as desired. Hence the Taylor series of f is $\sum_{n=0}^{\infty} \frac{4^n n!}{(2n+1)!}x^{2n+1}$.