MATH 141, FALL 2015, A few examples of how to use de L'Hopital's Rule

In class and in your textbook there are detailed examples of limits illustrating most of indeterminate forms, with th eexception of two of them: ∞^0 and $\infty - \infty$.

1) ∞^0: $\lim_{x\to\infty} x^{\frac{1}{x}}$

First, by continuity of the natural exponential function, we rewrite

$$\lim_{x\to+\infty} x^{\frac{1}{x}} = \lim_{x\to+\infty} e^{\ln(x)\frac{1}{x}} = e^{\lim_{x\to+\infty}\ln(x)\frac{1}{x}},$$

and note that the exponent $\lim_{x\to+\infty}\ln(x)\frac{1}{x}$ can be written in the form

$$\lim_{x\to+\infty}\ln(x)\frac{1}{x} = \lim_{x\to+\infty}\frac{\ln(x)}{x},$$

where

$$\lim_{x\to+\infty}\ln(x) = +\infty$$

and

$$\lim_{x\to+\infty} x = +\infty.$$

Hence this example fits into the settings of the classical de L'Hopital's Rule. What remains to check is first:

$$\frac{d}{dx}\ln(x) = \frac{1}{x} \neq 0,$$

and

$$\frac{d}{dx}x = 1 \neq 0,$$

as $x \to +\infty$.

Next, we compute

$$\lim_{x\to+\infty}\frac{\frac{1}{x}}{1} = \lim_{x\to+\infty}\frac{1}{x} = 0.$$

Hence, the assumptions are all satisfied and we can conclude that

$$\lim_{x\to+\infty} x^{\frac{1}{x}} = e^{\lim_{x\to+\infty}\ln(x)\frac{1}{x}} = e^0 = 1.$$

2) $\infty - \infty$: $\lim_{x\to 0^+}(\csc(x) - \cot(x))$

We write:

$$\lim_{x\to 0^+}(\csc(x) - \cot(x)) = \lim_{x\to 0^+}\left(\frac{1}{\sin(x)} - \frac{\cos(x)}{\sin(x)}\right) = \lim_{x\to 0^+}\frac{1-\cos(x)}{\sin(x)}.$$

Now we observe that

$$\lim_{x\to 0^+}(1-\cos(x)) = 0$$

and

$$\lim_{x\to 0^+}\sin(x) = 0,$$

Hence this example fits into the settings of the classical de L'Hopital's Rule. What remains to check is first:

$$\frac{d}{dx}(1-\cos(x)) = \sin(x) \neq 0,$$

and

$$\frac{d}{dx}\sin(x) = \cos(x) \neq 0,$$

as $x \to 0^+$ (remember this implies in our notation that $x \neq 0$).

Next, we compute

$$\lim_{x\to 0^+}\frac{\sin(x)}{\cos(x)} = 0.$$

Hence, the assumptions are all satisfied and we can conclude that

$$\lim_{x\to 0^+}(\csc(x) - \cot(x)) = \lim_{x\to 0^+}\frac{1-\cos(x)}{\sin(x)} = 0.$$