SOLUTION TO PROBLEM 5 ON MIDTERM 1

You can start like this:

$$\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)^{x^2} = \ (2\ \textit{points})\ \ \lim_{x\to\infty} e^{x^2\ln(1+1/x^2)} = \ (2\ \textit{points})\ \ e^{\lim_{x\to\infty} x^2\ln(1+1/x^2)}.$$

We now consider $\lim_{x\to\infty}\big(x^2\ln(1+1/x^2)\big)$:

$$\lim_{x\to\infty} x^2\ln\left(1+\frac{1}{x^2}\right) = \ (2\ \textit{points})\ \ \lim_{x\to\infty}\frac{\ln(1+1/x^2)}{\frac{1}{x^2}}$$

Next observe that de l'Hopital's rule applies to the limit above on the right. Thus, by applying it we have:

$$\lim_{x\to\infty}\frac{\ln(1+1/x^2)}{\frac{1}{x^2}} = \lim_{x\to\infty}\frac{\frac{1}{1+1/x^2}\cdot\frac{(-2)}{x^3}}{\frac{(-2)}{x^3}}$$

Each correctly computed derivative is worth 4 *points* (for the total of 8 points). I take away 2 points for each incorrect constant in the computation of either of the derivatives. Further, we notice that

$$\lim_{x\to\infty}\frac{\frac{1}{1+1/x^2}\cdot\frac{(-2)}{x^3}}{\frac{(-2)}{x^3}} = \lim_{x\to\infty}\frac{1}{1+1/x^2} = 1,$$

which is also worth 4 *points*.

Last, we plug this observation into the original problem to obtain:

$$\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)^{x^2} = e^{\lim_{x\to\infty} x^2\ln(1+1/x^2)} = e^1 = e.\ \ (2\ \textit{points})$$