Exam 3 Problem 1 Solution

$$\int_0^\infty x2^{-x}\ dx$$

Grading scheme 1

Since $x2^{-x}$ is bounded near 0 and continuous on $[0, \infty)$

$$\int_0^\infty x2^{-x}\ dx = \lim_{c\to\infty}\int_0^c x2^{-x}\ dx \qquad \text{(2 points)}$$

Now we use integration by parts.

$$\begin{aligned} u &= x \qquad dv = 2^{-x}dx = e^{-(\ln 2)x}\ dx \\ du &= dx \quad\ v = -\frac{1}{\ln 2}e^{-(\ln 2)x} \end{aligned} \qquad \text{(4 points)}$$

Hence

$$\begin{aligned} \lim_{C\to\infty}\int_0^C x2^{-x}\ dx &= \lim_{C\to\infty} -\frac{xe^{-(\ln 2)x}}{\ln 2}\Big|_0^C - \int_0^C -\frac{1}{\ln 2}e^{-(\ln 2)x}\ dx && \text{(3 points)} \\ &= \lim_{C\to\infty} -\frac{xe^{-(\ln 2)x}}{\ln 2}\Big|_0^C - \frac{1}{(\ln 2)^2}e^{-(\ln 2)x}\Big|_0^C && \text{(3 points)} \\ &= \lim_{C\to\infty} -\Big(\frac{Ce^{-(\ln 2)C}}{\ln 2} - 0\Big) - \Big(\frac{1}{(\ln 2)^2}e^{-(\ln 2)C} - \frac{1}{(\ln 2)^2}\Big) && \text{(2 points)} \end{aligned}$$

We know that

$$\lim_{C\to\infty}\frac{1}{(\ln 2)^2}e^{-(\ln 2)C} = 0 \qquad \text{(1 point)}$$

But $Ce^{-(\ln 2)C}$ has indeterminate form $\infty \cdot 0$. So by L'Hopital's Rule,

$$\lim_{C\to\infty} Ce^{-(\ln 2)C} = \lim_{C\to\infty}\frac{C}{e^{(\ln 2)C}} = \lim_{C\to\infty}\frac{1}{\ln 2 \cdot e^{(\ln 2)C}} = 0 \qquad \text{(2 points)}$$

Therefore

$$\lim_{C\to\infty}\int_0^C x2^{-x}\ dx = \frac{1}{(\ln 2)^2} \qquad \text{(1 point)}$$

So the integral converges. (2 points)