Problem 1

Verify if the following series are convergent or divergent. If convergent, specify whether the convergence is absolute or conditional. Find the sum of the series whenever it is possible.

(a)

$$\sum_{k=3}^{\infty} \frac{(-1)^{k+1} 4^{k-1}}{5^{(2k)}},$$

(b)

$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}}.$$

Solution:

(a) Notice that

$$\sum_{k=3}^{\infty} \frac{(-1)^{k+1} 4^{k-1}}{5^{(2k)}} = \sum_{k=3}^{\infty} \left(\frac{-1}{4}\right)\left(\frac{-4}{25}\right)^{k}.$$

This is clearly a geometric series with $r = -4/25$. Since $|r| < 1$, the series converges (5 pts). Recall the geometric series formula:

$$\sum_{k=m}^{\infty} cr^{k} = \frac{cr^{m}}{1-r}.$$

Then the sum of the series is

$$\sum_{k=3}^{\infty} \left(\frac{-1}{4}\right)\left(\frac{-4}{25}\right)^{k} = \frac{\left(\frac{-1}{4}\right)\left(\frac{-4}{25}\right)^{3}}{1 - \left(\frac{-4}{25}\right)}$$

$$= \frac{16}{(25^{2})(29)} = \frac{16}{18125}. \qquad (5 \text{ pts})$$

Now we need to determine whether the series converges absolutely. There are a number of ways to do this, the simplest being to consider the series $\sum |a_n|$:

$$\sum_{k=3}^{\infty} \left|\frac{(-1)^{k+1} 4^{k-1}}{5^{(2k)}}\right| = \sum_{k=3}^{\infty} \left(\frac{1}{4}\right)\left(\frac{4}{25}\right)^{k}.$$

This is also a geometric series, this time with $r = 4/25$. Since $|r| < 1$ this series converges, and hence the original series converges absolutely (5 pts).

The last part can be proven in a number of ways, including the ratio test and the root test. In fact,

proving absolute convergence alone is sufficient to prove convergence. However, in order to compute the sum you needed to identify the series as geometric and state that it is summable since $|r| < 1$. Regardless of your methods your logic needed to be clear and you needed to present a coherent argument.

(b) Consider

$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}}.$$

Let $a_n = 1/(n\sqrt{\ln(n)})$. Notice that $a_n > 0$ for all $n \geq 2$, and that since $n\sqrt{\ln(n)}$ is monotone increasing a_n will be monotone decreasing. Let

$$f(x) = \frac{1}{x\sqrt{\ln(x)}}.$$

Then for all natural numbers n we have $f(n) = a_n$, so the function $f(x)$ agrees with the sequence a_n. Therefore we may apply the integral test (3 pts). This test says that

$$\int_2^{\infty} \frac{1}{n\sqrt{\ln(x)}} \, dx$$

converges to a finite number if and only if the series converges.

Let's compute the integral:

$$\int_2^{\infty} \frac{1}{n\sqrt{\ln(x)}} \, dx = \lim_{b \to \infty} \int_{\ln 2}^{\ln b} u^{-1/2} \, du$$

$$= \lim_{b \to \infty} 2\sqrt{\ln b} - 2\sqrt{\ln 2} = \infty. \quad\quad (5 \text{ pts })$$

Then the integral diverges. By the integral test, therefore, the series

$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$$

diverges as well (2 pts).

Note that the original question asked about the convergence of the series, not of the integral. Therefore the answer is incomplete without a conclusion stating that the result of this convergence test indicates that the original series diverges.