

## Exam 4 Problem 1 Solution

1. 
$$\sum_{n=3}^{\infty} (-1)^{k+2} \frac{3^{k-1}}{4^{2k}}$$

$$\sum_{k=3}^{\infty} (-1)^{k+2} \frac{3^{k-1}}{4^{2k}} = \sum_{k=3}^{\infty} \frac{1}{3} \frac{(-1)^k 3^k}{16^k} = \sum_{k=3}^{\infty} \frac{1}{3} \left( \frac{-3}{16} \right)^k \quad (3 \text{ points})$$

This is a geometric series with  $r = -\frac{3}{16}$ . Since  $|r| < 1$ , the series converges. (3 points)

Recall the geometric series formula:

$$\sum_{k=m}^{\infty} cr^k = \frac{cr^m}{1-r} \quad (2 \text{ points})$$

Then the sum of the series is

$$\sum_{k=3}^{\infty} \frac{1}{3} \left( \frac{-3}{16} \right)^k = \frac{\frac{1}{3} \left( \frac{-3}{16} \right)^3}{1 - \left( \frac{-3}{16} \right)} = -\frac{9}{16^2 \cdot 19} = -\frac{9}{4864} \quad (3 \text{ points})$$

Now we need to determine whether the series converges absolutely. Consider the series  $\sum |a_n|$ .

$$\sum_{k=3}^{\infty} \left| (-1)^{k+2} \frac{3^{k-1}}{4^{2k}} \right| = \sum_{k=3}^{\infty} \left| \frac{1}{3} \left( \frac{-3}{16} \right)^k \right| = \sum_{k=3}^{\infty} \frac{1}{3} \left( \frac{3}{16} \right)^k$$

This is a geometric series with  $r = \frac{3}{16}$ . Since  $|r| < 1$ , the series converges, and hence the original series converges absolutely. (4 points)

Generalized ratio or root test can be used to prove that the series converges absolutely (hence converges). But to find the sum, you need to identify the series as geometric series. Alternating series test can't tell if the series converges absolutely.

2. 
$$\sum_{n=2}^{\infty} \frac{2}{n(n+1)}$$

This is a telescoping series.

Notice that we can use partial fractions on each term to get

$$\frac{1}{n(n+2)} = \frac{2}{n} - \frac{2}{n+1} \quad (6 \text{ points})$$

Now we need to write down the partial sums for this series.

$$\begin{aligned}
 s_j &= \sum_{n=2}^j \frac{2}{n(n+1)} \\
 &= \sum_{n=2}^j \left( \frac{2}{n} - \frac{2}{n+1} \right) \\
 &= \frac{2}{2} - \frac{2}{3} + \frac{2}{3} - \frac{2}{4} + \frac{2}{4} - \frac{2}{5} + \dots + \frac{2}{j-1} - \frac{2}{j} + \frac{2}{j} - \frac{2}{j+1} \\
 &= 1 - \frac{2}{j+1}
 \end{aligned}
 \tag{4 points}$$

We can determine the convergence of this series by taking the limit of the partial sums.

$$\sum_{n=2}^{\infty} \frac{2}{n(n+1)} = \lim_{j \rightarrow \infty} s_j = \lim_{j \rightarrow \infty} 1 - \frac{2}{j+1} = 1
 \tag{2 points}$$

So the series converges. For positive series, converges means converges absolutely. (Because when you add the absolute value sign, nothing changes.) (3 points)

Ratio test can't be applied because the ratio is 1. You can use the (limit) comparison test to show it converges absolutely, but then you lost 5 points for not finding the sum.

If you want to use the integral test, you need to show it's decreasing by finding the derivative. Also there is a trick at last.

$$\lim_{c \rightarrow \infty} \ln(c) - \ln(c+1) = \lim_{c \rightarrow \infty} \ln \frac{c}{c+1} = \ln 1 = 0$$

So the integral actually converges.

Note that integral test doesn't say the value of the integral is equal to the sum of the series.