

#2. For any positive integer n establish the following formula:

$$\int (\ln(x))^n dx = x (\ln(x))^n - n \int (\ln(x))^{n-1} dx.$$

Answer. Integrate by parts, using $u = (\ln x)^n$ and $dv = dx$. That gives the following table:

$$\begin{array}{ll} u = (\ln x)^n & v = x \\ du = n(\ln x)^{n-1} \cdot \frac{1}{x} dx & dv = dx \end{array} .$$

The integration by parts formula gives us

$$\begin{aligned} \int u dv &= uv - \int v du \\ \int (\ln x)^n dx &= x(\ln x)^n - \int \left(n(\ln x)^{n-1} \cdot \frac{1}{x} \right) x dx \\ &= x(\ln x)^n - \int (n(\ln x)^{n-1}) dx \\ &= x(\ln x)^n - n \int ((\ln x)^{n-1}) dx, \end{aligned}$$

as we were asked to show. □

The short version of the rubric was three points for integrating by parts anywhere for any reason, fifteen points for the correct table, and seven points for the execution.

The most common mistake by far was omission of the factor of $\frac{1}{x}$ coming from the chain rule when $(\ln x)^n$ is differentiated. That was a five-point mistake, and typically led to bigger errors to attempt to cover up the fact that the resulting formula differed from the intended one. Typically I gave twenty if you just left well enough alone and got the wrong formula and went down as far as fifteen if your method of correcting the mistake you introduced was to purposefully use the wrong integration by parts formula, or even worse (some people attempted to fix the problem by saying $1 = (\ln x)^{n-1}$, for instance).

There were a number of flagrant abuses of logarithms and exponents. It is not true that $(\ln x)^n = n \ln x$, and anyone who said so was punished with fury. Some people managed not only to do that but work in the equally-false $e^{AB} = e^A e^B$. (The correct equations are $\ln(x^n) = n \ln x$ and $e^{A+B} = e^A e^B$. Neither $(\ln x)^n$ nor e^{AB} simplify!) Some people attempted to antidifferentiate $(\ln x)^n$; if you could do that there would be no need for the formula we're trying to show. As such everyone who tried that got it wrong. Everyone who tried to rewrite $(\ln x)^n = e^{n \ln(\ln x)}$ got it very wrong, as it just obfuscated the intended direction of the problem. Most people who attempted to solve the problem after "simplifying" the left side wound up with somewhere between five and seven. Those who started in with a u -substitution and made no forward progress received no credit.

Lastly there were some people who did it basically correctly but only did it for a special case, like $n = 3$ or $n = 5$. The point of the problem was to get the formula for *arbitrary* exponents n . Arbitrary does not mean you get to pick which one! It means it must work in general. If it was done correctly and the only thing that was necessary to do to change the numbers to n and $n - 1$, I was lenient.

One last note: a few people were able to derive the formula by differentiating both sides and deriving that they're equal. Kudos to them.