Problem 3 (25 points)

For center of gravity problems we need four major pieces, the limits of the region, the area of the region, and the two moments. Lastly, we take these components to form the center of gravity.

Intersection (3+2 points):

$$\begin{aligned}
\text{Find: } & f(x) = g(x) \\
& x^2 = -x^2 + 4x \\
& 2x^2 - 4x = 0 \\
& 2x(x-2) = 0 \\
& x = 0\,,\ x = 2.
\end{aligned}$$

Area (4+2 points): The integral of the difference of $f(x)$ and $g(x)$ gives the area between the two curves. It's important to note that the area will be non-negative, so we must figure out which function is "higher." In this case, $g(x) = -x^2 + 4x$ is the upper bound of the region.

$$\begin{aligned}
A &= \int_0^2 (-x^2 + 4x) - (x^2)\,dx \\
&= \int_0^2 -2x^2 + 4x\,dx \\
&= \left[-\frac{2}{3}x^3 + 2x^2\right|_0^2 \\
&= -\frac{16}{3} + 8 \\
&= \frac{8}{3}.
\end{aligned}$$

Moment about x $[M_x]$ **(4+2 points):** Because we are working in the x

domain, the moment about the x axis, our formula is as follows.

$$\begin{aligned}
M_x =& \int_0^2 \frac{1}{2}[(-x^2+4x)^2-(x^2)^2]\,dx \\
=& \frac{1}{2}\int_0^2 x^4-8x^3+16x^2-x^4\,dx \\
=& \frac{1}{2}\int_0^2 -8x^3+16x^2\,dx \\
=& \frac{1}{2}\left[-2x^4+\frac{16}{3}x^3\right|_0^2 \\
=& \frac{1}{2}\left(-32+\frac{128}{3}\right) \\
=& \frac{32}{6}.
\end{aligned}$$

Moment about y $[M_y]$ (4+2 points):

$$\begin{aligned}
M_y =& \int_0^2 x[(-x^2+4x)-(x^2)]\,dx \\
=& \int_0^2 -x^3+4x^2-x^3\,dx \\
=& \int_0^2 -2x^3+4x^2\,dx \\
=& \left[-\frac{1}{2}x^4+\frac{4}{3}x^3\right|_0^2 \\
=& -8+\frac{32}{3} \\
=& \frac{8}{3}.
\end{aligned}$$

Final Solution (2 points):

$$(\bar{x},\bar{y}) = \left(\frac{M_y}{A},\frac{M_x}{A}\right) = \left(\frac{\frac{8}{3}}{\frac{8}{3}},\frac{\frac{32}{6}}{\frac{8}{3}}\right) = (1,2).$$