Exam 2 Problem 3 Solution

$$L = \lim_{x\to\infty} \frac{\log_2(\log_3(x^2))}{\log_4(x^2)}$$

Grading scheme 1

$$\log_2(\log_3(x^2)) = \frac{\ln(\log_3(x^2))}{\ln 2} \qquad \text{(2 points)}$$

$$\log_4(x^2) = \frac{\ln(x^2)}{\ln 4} \qquad \text{(2 points)}$$

$$L = \frac{\ln 4}{\ln 2} \lim_{x\to\infty} \frac{\ln(\log_3(x^2))}{\ln(x^2)} \qquad \text{(2 points)}$$

As $x \to \infty$, $\ln(\log_3(x^2)) \to \infty$, $\ln(x^2) \to \infty$ (2 points)

It has the intermediate form ∞/∞, and $(\ln(x^2))' = \frac{2x}{x^2} \neq 0$ near ∞. Hence we can use the L'Hospital's Rule. (2 points)

$$\begin{aligned}
\lim_{x\to\infty} \frac{(\ln(\log_3(x^2)))'}{(\ln(x^2))'} &= \lim_{x\to\infty} \frac{\frac{1}{\log_3(x^2)}(\log_3(x^2))'}{\frac{2x}{x^2}} && \text{(4 points)} \\
&= \lim_{x\to\infty} \frac{\frac{\ln 3}{\ln(x^2)}\left(\frac{1}{\ln 3} \cdot \frac{2x}{x^2}\right)}{\frac{2x}{x^2}} && \text{(4 points)} \\
&= \lim_{x\to\infty} \frac{1}{\ln(x^2)} && \text{(3 points)} \\
&= 0 && \text{(2 points)}
\end{aligned}$$

Hence by L'Hospital's Rule,

$$L = \frac{\ln 4}{\ln 2} \lim_{x\to\infty} \frac{(\ln(\log_3(x^2)))'}{(\ln(x^2))'} = \frac{\ln 4}{\ln 2} \cdot 0 = 0 \qquad \text{(2 points)}$$

Grading scheme 2

As $x \to \infty$, $\log_2(\log_3(x^2)) \to \infty$, $\log_4(x^2) \to \infty$ (2 points)

It has the intermediate form ∞/∞, and $(\log_4(x^2))' = \frac{2x}{\ln 4 \cdot x^2} \neq 0$ near ∞. Hence we can use the L'Hospital's Rule. (2 points)

$$\lim_{x\to\infty} \frac{(\log_2(\log_3(x^2)))'}{(\log_4(x^2))'} = \lim_{x\to\infty} \frac{\dfrac{1}{\ln 2}\cdot\dfrac{1}{\log_3(x^2)}(\log_3(x^2))'}{\dfrac{1}{\ln 4}\cdot\dfrac{2x}{x^2}} \qquad \text{(10 points)}$$

$$= \lim_{x\to\infty} \frac{\dfrac{1}{\ln 2}\cdot\dfrac{\ln 3}{\ln(x^2)}\left(\dfrac{1}{\ln 3}\cdot\dfrac{2x}{x^2}\right)}{\dfrac{1}{\ln 4}\cdot\dfrac{2x}{x^2}} \qquad \text{(4 points)}$$

$$= \frac{\ln 4}{\ln 2}\lim_{x\to\infty}\frac{1}{\ln(x^2)} \qquad \text{(3 points)}$$

$$= 0 \qquad \text{(2 points)}$$

Hence by L'Hospital's Rule,

$$L = \lim_{x\to\infty} \frac{(\log_2(\log_3(x^2)))'}{(\log_4(x^2))'} = 0 \qquad \text{(2 points)}$$