Problem 3 Solution

Question :

Find the Taylor series of $f(x) = x\,ln(1+x^2)$.

Solution :

First, if not memorized, one must derive the Taylor series for $ln(1+x)$.

$$\begin{aligned}
&f(x) = ln(1+x) && f(0) = 0 \\
&f'(x) = (1+x)^{-1} && f'(0) = 1 \\
&f''(x) = -(1+x)^{-2} && f''(0) = -1 \\
&f^{(3)}(x) = 2(1+x)^{-3} && f^{(3)}(0) = 2 \\
&f^{(4)}(x) = -6(1+x)^{-4} && f^{(4)}(0) = -6 && \text{(6 pts)}
\end{aligned}$$

Then one can deduce that for $n \geq 1$

$$f^{(n)}(x) = (-1)^{n+1}\,(n-1)!\,(1+x)^{-n} \qquad f^{(n)}(0) = (-1)^{n+1}\,(n-1)!. \qquad \text{(4 pts)}$$

The formula for the Taylor Series of $ln(1+x)$ is then

$$ln(1+x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}. \qquad \text{(4 pts)}$$

Then the Taylor series of $ln(1+x^2)$ is given by

$$ln(1+x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{2n+2}, \qquad \text{(6 pts)}$$

and the Taylor series for $x\,ln(1+x^2)$ is

$$x\,ln(1+x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{2n+3}. \qquad \text{(5 pts)}$$