MATH 630, Spring 2007, SOLUTION SKETCHES for SAMPLE FINAL

1) Let $f : [0,1] \to \mathbb{R}$ be an increasing right continuous function with the property that

$$\forall\, g \in C([0,1]), \qquad \int_0^1 g\, df = 0.$$

Prove that f is a constant function.

Answer: Assume by contradiction that f is not constant. This implies $f(0) < f(1)$. Because g is continuous everywhere, Theorem 3.5.4 implies that it is Riemann-Stieltjes integrable with respect to f. Compute $S_P^f(g)$ (see (3.26)) for $g = 1$ on $[0,1]$ and for an arbitrary partition P. Observe that it is independent of P and compute the upper Riemann-Stieltjes integral in terms of f. Note the contradiction with the assumption we made at the beginning.

2) Let $f : [a,b] \to \mathbb{R}$ be a differentiable function and suppose $f' \in BV([a,b])$. Prove that $f' \in C([a,b])$.

Answer: Recall the Darboux property of the derivative (i.e., that is assumes all the values between any two of its values). Assume by contradiction that f' is not continuous and draw a contradiction between this and the fact that f' has the Darboux property and is of bounded variation.

3) Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $1 \le p \le \infty$, $1/p + 1/q = 1$. Assume that $\{f_n : n = 1, \ldots\} \subseteq L^p_\mu(X)$, $\{g_n : n = 1, \ldots\} \subseteq L^q_\mu(X)$ are such that $f_n \to f$ in $L^p_\mu(X)$ and $g_n \to g$ in $L^q_\mu(X)$. Prove that $f_n g_n \to fg$ in $L^1_\mu(X)$.

Answer: Since we say that $f_n \to f$ in $L^p_\mu(X)$, it means that $\|f_n - f\|_p \to 0$. In particlar, $f_n - f \in L^p_\mu(X)$, and so $f \in L^p_\mu(X)$. Same for g. To show $f_n g_n \to fg$ in $L^1_\mu(X)$ write

$$\int_X |f_n g_n - fg|\, d\mu = \int_X |(f_n g_n - f_n g) + (f_n g - fg)|\, d\mu \le \int_X |f_n||g_n - g|\, d\mu + \int_X |g||f_n - f|\, d\mu.$$

Use the assumptions on convergence of f_n's and g_n's to complete the proof. (Note that convergence in norm implies norm convergence, and boundedness of truncated sequences of norms.)

4) Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Assume that $\{f_n : n = 1, \ldots\} \subseteq L^{2007}_\mu(X)$ is bounded in the norm $L^{2007}_\mu(X)$ and that $f_n \to f$ μ-*a.e.* Show that $f_n \to f$ in $L^1_\mu(X)$.

Answer: Let $p = 2007$. Let q be the adjoint. Use Fatou lemma to observe that $f \in L^1_\mu(X)$. Thus, wlog wma that $f_n \to 0$ μ-*a.e.*. Fix $\epsilon > 0$. Use Egorov's theorem to find a set $A = A(\epsilon)$ such that $\mu(A) < \epsilon^q$ and f_n converges uniformly to 0 on $A^\sim$. Write:

$$\int_X |f_n|\, d\mu \leq \int_A + \int_{A^\sim} |f_n|\, d\mu \leq \mu(X)\|f_n\|_\infty + \int_X \chi_A |f_n|\, d\mu$$

Estimate the first summand in the above, by choosing n sufficiently large so that $\|f_n\|_\infty \leq \epsilon/(2\mu(X))$. Estimate the second summand by first noting that $\int \chi_A |f_n|\, d\mu \leq \|\chi_A\|_q \|f_n\|_p \leq \mu(A)^{1/q}\|f_n\|_p$. Use the boundedness of f_n's to conclude your estimates.

5) Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Show that $L^p_\mu(X) \subseteq L^r_\mu(X)$, for any $1 \leq r \leq p \leq \infty$. Show that the assumption $\mu(X) < \infty$ is necessary.

Answer: Write $X = A \cup A^\sim$, where $A = \{x \in X : |f(x)| \leq 1\}$. To show that $L^p_\mu(X) \subseteq L^r_\mu(X)$, it is suppficient to show that $\|f\|_p < \infty$ implies $\|f\|_r < \infty$. You do this as follows:

$$\int_X |f|^r\, d\mu \leq \int_A + \int_{A^\sim} |f|^r\, d\mu \leq \int_A\, d\mu + \int_{A^\sim} |f|^p\, d\mu \leq \mu(X) + \|f\|_p^p.$$

If $\|f\|_p < \infty$, so is $\|f\|_r$.