PRELUDE TO A KISS.

DMITRY DOLGOPYAT

ABSTRACT. We consider expanding maps of the circle with an almost neutral fixed point c. In this setting the ergodic averages of a smooth observable A satisfy the Central Limit Theorem. If $A(c) = 0$ we show that the expectation and the variance behave as the square root of the multiplier of c times rapidly oscillating functions of parameters.

1. INTRODUCTION.

Several authors investigated the regularity of parameter dependence of various dynamical characteristics (see [11, 12, 3, 4, 14, 15, 16, 17, 6, 5, 8, 7, 9], etc.). In general one can obtain nice dependence if the system is uniformly (partially) hyperbolic for all values of the parameters. However, for general systems one should expect much less regularity of the dynamical invariants. Therefore an important problem is to develop methods for obtaining optimal regularity in the case when the invariants are not smooth.

Here we investigate a much simpler question. Recall that a map F of a compact Riemannian manifold is called **expanding** if there are constants $C > 0$ and $\lambda > 1$ such that for any tangent vector v

$$||dF^n(v)|| \geq C\lambda^n ||v||$$

(equivalently, F locally increases distances with respect to some Riemannian metric). We consider a one parameter family f_ε of expanding maps of S^1, $\varepsilon > 0$, which loses its hyperbolicity at $\varepsilon = 0$ via a saddle-node bifurcation and study how the mean and the variance in the Central Limit Theorem change when hyperbolicity detiorates. Let us give a more precise description of families we consider.

Let $f : S^1 \to S^1$ be a C^∞ map with the following properties.

(A) There exists a partition of the circle into intervals $S^1 = \bigcup_{j=1}^p Q_j$ such that $f : Q_j \to S^1$ is one-to-one, $j = 1, \dots, p$.

(B) There is a constant $\lambda > 1$ such that $f'(x) \geq \lambda$ for all $x \in \bigcup_{j=2}^p Q_j$.

(C) $Q_1 = [d, b_0]$ where d is a hyperbolic fixed point.

(D) Inside Q_1 we have $f(x) > x$ except for a neutral fixed point c which is non-degenerate in the sense that

$$\alpha := \frac{1}{2}\frac{d^2 f}{dx^2}(c) > 0. \tag{1}$$

Let a_0 be the leftmost point in Q_1 where $f'(a) = 1$ (observe that $f' \geq \lambda$ on ∂Q_1 due to (B) and that $f'(x) < 1$ for x immediately to the left of c due to (C). Let $a_1 = f(a_0), I_k = f^k([a_0, a_1])$, ($k$ can be either positive or negative), $b_{-1} = f^{-1}b_0$, $J_k = f^{-k}[b_{-1}, b_0]$. (Here and below we use f^{-1} to denote the branch of the inverse of f with range $[d, b_0]$.) Given a segment I, let H_I denote the increasing affine map $I \to [0, 1]$. For $x \in [a_0, a_1]$ set

$$g_1(x) = \lim_{k\to\infty} H_{I_k} f^k(x)$$

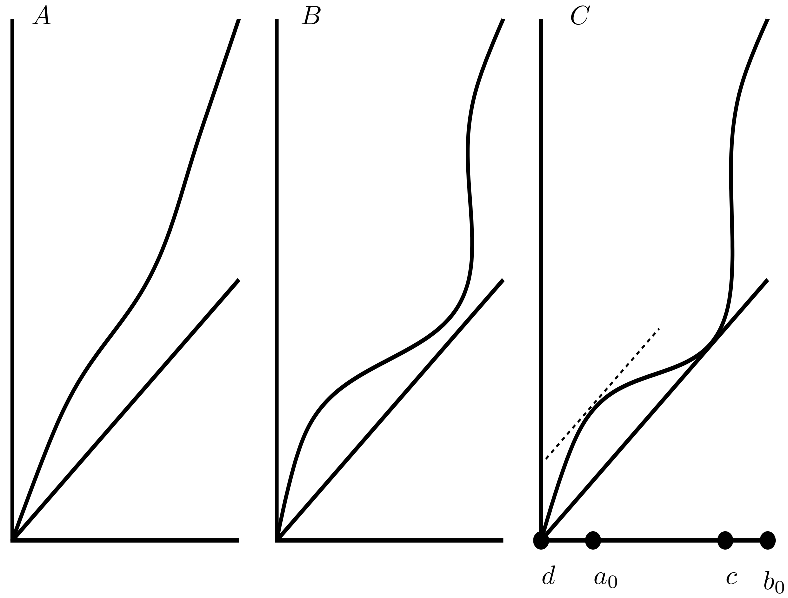

FIGURE 1. f_ε restricted to Q_1 for (A) large positive ε, (B) positive ε near 0, (C) $\varepsilon = 0$. As the title indicates we are interested in what happens slightly before the tangency appears (Figure 1(B)).

(this limit exists because for all r the C^r distance between $H_{I_k} f^k$ and $H_{I_{k+1}} f^{k+1}$ is less than $\frac{C}{k^2}$ by Lemma 2 below). Extend g_1 to (d, c) by setting $g_1(x) = g_1 f^{-k}(x)$ for $x \in I_k$. Similarly for $x \in [b_{-1}, b_0]$ set

$$g_2 = \lim_{k\to\infty} H_{J_k} f^{-k}.$$

Our last condition on f reads

$$(E) \quad \inf_{x\in(d,c)} (g_1'(x)) \max_{x\in J_0} (g_2'(x)) > 1.$$

Let $\{f_\varepsilon\}$ be a one parameter family such that $f_0 = f$ which is non-degenerate in the sense that $\frac{d}{d\varepsilon}|_{\varepsilon=0} f_\varepsilon(c) \neq 0$. We normalize f_ε by requiring that $\frac{d}{d\varepsilon}(f_\varepsilon(c)) = 1$, for ε near 0. Thus

$$(F) \quad f_\varepsilon(c+h) = c+h+\varepsilon+\alpha h^2+o(\varepsilon+h^2).$$

Let us explain the meaning of the above conditions. (A) and (C) imply that f has a Markov partition consisting of d and its preimages. Condition (B) implies that f is expanding outside of Q_1. By (D) for small ε most points spend most of the time near c and so if they pick up a strong contraction where. Hence under conditions (A)–(D) and (F) f_ε may fail to be expanding in spite of (B). Condition (E) ensures overall expansion.

Proposition 1. *If $\{f_\varepsilon\}$ satisfies (A)–(F) then f_ε is expanding for small positive ε.*

Proposition 1 implies that f_ε has an absolutely continuous invariant measure μ_ε. $(f_\varepsilon, \mu_\varepsilon)$ is exponentially mixing and satisfies the Central Limit Theorem. Namely, if $A : S^1 \to \mathbb{R}$ is

Holder continuous then

$$\text{mes}\left(x \in S^1 : \quad \frac{\sum_{j=0}^{n-1} A(f_\varepsilon^j x) - n\mu_\varepsilon(A)}{\sqrt{D_\varepsilon(A)n}} \le z\right) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-s^2/2} ds$$

where mes is Lebesgue measure on S^1 and

$$D(A, f_\varepsilon) = \mu_\varepsilon(A^2) - (\mu_\varepsilon(A))^2 + 2\sum_{j=1}^{\infty} \left[\mu_\varepsilon(A(\cdot)(A \circ f_\varepsilon^j)) - (\mu_\varepsilon(A))^2\right]$$

(this sum is finite because of the exponential mixing). Moreover by [12] if A is C^∞ then the maps $\varepsilon \to \mu_\varepsilon(A)$ and $\varepsilon \to D_\varepsilon(A)$ are C^∞ for $\varepsilon > 0$.

We say that a function $\phi : \mathbb{R}^+ \to \mathbb{R}$ is **weakly periodic** if there exists a periodic function ϕ_0 and a function $\alpha(t)$ such that for all T

$$\lim_{t\to\infty} \sup_{s\in[t,t+T]} |\phi(s) - \phi_0(s - \alpha(t))| = 0.$$

Theorem 1. *If $\{f_\varepsilon\}$ satisfies (A)–(F), then there are functionals $\omega_i : C^\infty(S^1, \mathbb{R}) \times \mathbb{R}^+ \to \mathbb{R}$, $i = 1, 2$ weakly periodic in the second variable such that*

$$\mu_\varepsilon(A) - A(c) \sim \sqrt{\varepsilon}\omega_1(\sqrt{1/\varepsilon}, A),$$
$$D_\varepsilon(A) \sim \sqrt{\varepsilon}\omega_2(\sqrt{1/\varepsilon}, A).$$

(Here and below "$\sim$" means that the ratio of the left hand side to the right hand side approaches 1 as $\varepsilon \searrow 0$.)

The factors $\sqrt{\varepsilon}$ in the above asymptotics reflect the fact that a typical orbit spends time of order $\sqrt{\varepsilon}$ outside a small neighborhood of c. Indeed outside such a neighborhood a point has a positive probability of hitting $[d, c]$ and points from $[d, c]$ quickly enter a small neighborhood of c where they get stuck for time $\pi/\sqrt{\alpha\varepsilon}$ (cf. Lemma 9). Observe that the dependence of ω_1 and ω_2 on ε is quite irregular. Such a phenomenon is of course well known for logistic maps (see e.g. [13]) but here we exhibit an example in hyperbolic setting.

Section 2 contains preliminary facts about f itself. In Section 3 we treat positive ε and prove Theorem 1.

Acknowledgments. The first studies of parameter dependence of dynamical characteristics were performed by Tolya Katok and his collaborators motivated by Katok's entropy rigidity conjecture. It is a pleasure to dedicate this work to Tolya on occasion of his 60th birthday. I am grateful to Brian Hunt for explaining to me Mather's invariant, to Omri Sarig for explaining to me the dynamics of systems with indifferent fixed points, to Kolya Chernov for discussions of similar problems in a different context and to Misha Brin for reading carefully the preprint version of this paper and suggesting numerous improvements. This work is partially supported by IPST, NSF and Sloan Foundation.

2. Tangency.

Here we collect some properties of f used in the proof of Theorem 1. In order to simplify the computations below we assume from now on that $A(c) = 0$.

Let $L = \bigcup_{j=2}^{p} Q_j$ be the union of the elements of the Markov partition not containing c. We want to approximate the first return map $F_\varepsilon : L \to L$ by a map $F : L \to L$. The problem is that for $\varepsilon = 0$, almost all points are eventually attracted to c so not all of them return to L. Fix $s \in [0, 1]$. Let $R_s(y) = y + s \mod 1$ be a rotation of $[0, 1]$. We define $\tilde{F}_s : L \to L$

as follows. If the f-orbit of $x \in L$ returns to L we let $\tilde{F}_s x$ be the point of the first return. If not, then $f(x) \in I_m$ for some m (positive or negative). Let $k_1(\delta)$ be the first moment k when length$(I_k) < \delta$. Let $k_2(\delta)$ be the first moment when length$(J_k) < \delta$. We define

$$\begin{aligned} \tilde{F}_s(x) &= \lim_{\delta \to 0} f^{k_2(\delta)+1} H^{-1}_{J_{k_2(\delta)}} R_s H_{I_{k_1(\delta)}} f^{k_1(\delta)-m+1}(x) \\ &= f g_2^{-1} R_s g_1 f(x). \end{aligned} \tag{2}$$

We say that $\{\tilde{F}_s\}$ is the **Mather family** associated to $\{f_\varepsilon\}$. Constructions similar to this are used in [1, 2, 10, 18]. We refer to these papers for a more detailed exposition of some computations similar to the ones presented below.

Recall (1). Given x let $x_n = f^n x$, set $t_n(x) = \alpha n + \frac{1}{|x-c|}$ where $\alpha = \frac{1}{2}\frac{d^2 f}{dx^2}(c)$. Let $x_{-n} = f^{-n}x$, where f^{-1} denotes the branch with range $[d, b_0]$. We also let $x_{n,\varepsilon} = f_\varepsilon^n x$ where again n can be either positive or negative.

Lemma 2. *If $x \in [a_0, c]$ then*

$$x_n = c - \frac{1}{t_n(x)} + \xi_n(x), \ \ \textit{where} \ \ |\xi_n(x)| \leq \mathrm{Const}\frac{\ln t_n(x)}{t_n^2(x)}.$$

Proof. Consider the auxiliary equation

$$\dot{z} = -\alpha(z^2 + \beta z^3).$$

Its solution $z(t)$ satisfies

$$\frac{1}{z} + \beta \ln\left(\frac{z + \frac{1}{\beta}}{z}\right) = \alpha t + \frac{1}{z(0)} + \beta \ln\left(\frac{z(0) + \frac{1}{\beta}}{z(0)}\right).$$

Thus, we have

$$\frac{1}{z} = \left[\alpha t + \frac{1}{z(0)}\right] + O\left(\ln\left(\alpha t + \frac{1}{z(0)}\right)\right). \tag{3}$$

uniformly for β in a bounded interval. Let $z_n = c - x_n$. There are β_1 and β_2 such that z_n is between the time n maps of the vector fields

$$\dot{z} = -\alpha(z^2 + \beta_1 z^3) \text{ and } \dot{z} = -\alpha(z^2 + \beta_2 z^3).$$

Now the result follows from (3). □

Corollary 3. *If $x \in [a_0, c)$, then*

$$\sum_{j=0}^{n-1} A(x_j) = -\frac{A'(c)\ln t_n}{\alpha} + A^*(x) + O\left(\frac{\ln t_n}{t_n^2}\right),$$

where $A^*(x) = \frac{\ln|c-x|}{\alpha} + O(1)$. Similarly if $x \in (c, b_0)$, then

$$\sum_{j=0}^{n-1} A(f^{-j}x) = \frac{A'(c)\ln t_n}{\alpha} + A^{**}(x) + O\left(\frac{\ln t_n}{t_n^2}\right),$$

where $A^{**}(x) = -\frac{\ln|x-c|}{\alpha} + O(1)$.

For $x \in [d, a_0]$ let $m(x)$ denote the first m when $x_m \in [a_0, a_1]$. Then

$$A^+(x) := \sum_{j=0}^{m(x)-1} A(x_j) = \frac{|\ln x| A(d)}{\ln f'(d)} + O(1)$$

Proof. The first part follows from Lemma 2 and the asymptotics $A(x_n) = A'(0)(x_n - c) + O((x_n - c)^2)$. To get the second part, apply the first part to f^{-1}. To prove the last estimate we observe that for $x \in [a_0, a_1]$

$$f^{-j}(x) \sim C(x) \left(\frac{1}{f'(d)} \right)^j$$

since near d the map f is conjugates to $x \to d + f'(d)(x - d)$. □

We now apply Corollary 3 to $(f^n)'$. Suppose $x \in [a_0, c]$. We have

$$\ln (f^n)'(x) - \ln (f^{n-1})'(x) \sim -2\alpha(c - x_n)$$

Thus by Corollary 3

$$\ln (f^n)'(x) \sim -2 \left[\ln \left(n + \frac{1}{c - x} \right) - \ln \frac{1}{c - x} \right].$$

In other words we have

Corollary 4. *There exists a constant K such that if $x \in [a_0, c]$ then*

$$\frac{1}{K} \le \frac{(f^n)'(x)(c - x_n)^2}{(c - x)^2} \le K$$

and if $x \in [c, b_0]$ then

$$\frac{1}{K} \le \frac{(f^{-n})'(x)(x_{-n} - c)^2}{(x - c)^2} \le K$$

Corollary 4 implies the following weak distortion bound: if $x', x'' \in I_k$ then

$$\frac{1}{\hat{C}} \le \frac{(f^n)'(x')}{(f^n)'(x'')} \le \hat{C}$$

with $\hat{C}$ independent of k, n. We will strengthen this bound. We say that a family of piecewise C^2 maps $\{\phi_\alpha\}$ has **bounded distortion** if there is a constant C such that for all x, α

$$|\phi''_\alpha(x)| \le C |\phi'_\alpha(x)|^2. \tag{4}$$

Observe that (4) implies in particular that

$$|\ln \phi'_\alpha(x') - \ln \phi'_\alpha(x'')| \le C \, |\phi_\alpha(x') - \phi_\alpha(x'')|$$

which is a standard distortion bound. Also if $\{\phi_\alpha\}$ has bounded distortion and is uniformly expanding in the sense that there exists $\lambda > 1$ such that $|\phi'(\alpha)| \ge \lambda$ then the family consisting of all compositions of maps from $\{\phi_\alpha\}$ also has bounded distortion (with a different constant C). When we say below that a family $\{F_\sigma\}$ has bounded distortion we mean that (4) holds uniformly for all x and σ.

For $\gamma \in (0, 1)$, let $C^\gamma(L)$ denote the space of Holder continuous functions with Holder exponent γ on L and $|| \cdot ||_\gamma$ denote the norm on $C^\gamma(L)$.

Lemma 5. *The family $\{\tilde{F}_s\}$, $s \in [0,1]$, is uniformly expanding with bounded distortion. Hence each $\tilde{F}_s$ has an absolutely continuous invariant measure ν_s, and there are constants $C > 0$ and $\theta \in (0,1)$ such that for $A, B \in C^\gamma(L)$*

$$\left| \int A(\tilde{F}_s^n x) B(x) dx - \nu_s(A) \int B(x) dx \right| \leq C ||A||_{C^\gamma} ||B||_{C^\gamma} \theta^n. \tag{5}$$

Proof. Expansion follows from the the second line of (2) and (E).

To estimate distortion let $x \in [a_0, c]$, say $x \in I_m$. Recall the first line of (2). We shall check that (4) holds uniformly for $\delta > 0$. Denote $\Delta_{s,\delta} = H^{-1}_{J_{k_2(\delta)}} \circ R_s \circ H_{I_{k_1}}$. Observe that $\Delta_{s,\delta}$ is affine, so $\Delta''_{s,\delta} = 0$. We have

$$\frac{d^2}{dx^2} \left(f^{k_2(\delta)+1} \Delta_{s,\delta} f^{k_1(\delta)-m+1} \right) = \tag{6}$$

$$(f^{k_2(\delta)+1})' \Delta' (f^{k_1(\delta)-m+1})'' + (f^{k_2(\delta)+1})'' \left[\Delta' (f^{k_1(\delta)-m+1})' \right]^2.$$

Now,

$$f^{k_2(\delta)+1} = f \circ f^{k_2(\delta)} = f \circ \left(f^{k_2(\delta)} \circ H^{-1}_{J_{k_2(\delta)}} \right) \circ H_{J_{k_2(\delta)}}.$$

The expression in parenthesis is C^2–close to g_2^{-1}. Thus

$$|(f^{k_2(\delta)+1})''| \leq \text{Const } \delta^{-2}. \tag{7}$$

By Corollary 4

$$(f^{k_2(\delta)+1})' \geq \text{Const } \delta^{-1} \tag{8}$$

Next, let us estimate $(f^{k_1-m+1})''(x)$. Denote $\xi_j(x) = (f^j)''(x)$. Then

$$\xi_{j+1} \leq \xi_j (1 - 2\alpha(c - x_j)) + \text{Const}[(f^j)'(x)]^2.$$

Observe that $(f^j)'(x)$ satisfies the corresponding homogenuous equation

$$(f^{j+1})'(x) = (f^j)'(x)(1 - 2\alpha(c - x_j)).$$

Thus the Gronwall inequality, Corollary 4 and the fact that

$$(f^{k_1(\delta)-m+1})'(x) \leq \text{Const} \frac{\delta}{(c-x)^2}$$

by the definition of $k_1(\delta)$ imply that

$$\xi_{k_1(\delta)-m+1}(x) \leq \text{Const } \frac{\delta}{(c-x)^4}. \tag{9}$$

Next, by Corollary 4

$$(f^{k_1(\delta)-m+1})'(x) \geq \text{Const} \frac{\delta}{(c-x)^2} \tag{10}$$

Combining (6)–(10) we get

$$\left| \frac{d^2}{dx^2} \left(f^{k_2(\delta)+1} \Delta_{s,\delta} f^{k_1(\delta)-m+1} \right) \right| \leq \frac{\text{Const}}{(c-x)^4}$$

On the other hand by Corollary 4

$$\left| \frac{d}{dx} \left(f^{k_2(\delta)+1} \Delta_{s,\delta} f^{k_1(\delta)-m+1} \right) \right| \geq \frac{\text{Const}}{(c-x)^2}.$$

This proves the needed distortion estimate for $x \in [a_0, c]$. For $x \in [c, b_0]$ say $x \in J_m$ we represent

$$f_\varepsilon^{k_2(\delta)+1-m}(x) = f_\varepsilon^{k_2(\delta)+1}(f_\varepsilon^{-m} x)$$

and proceed as before. Finally for $x \in [d, a_0]$ the distortion bound follows from the distortion estimates on $[a_0, a_1]$ and the discussion preceding the statement of Lemma 5.

Mixing follows from expansion, distortion and the fact that the image of each continuity interval is all of L (see e.g. [19]). $\square$

3. Unfolding.

Here we prove Theorem 1. In order to simplify the computations below we assume that $f_\varepsilon|_{\partial Q_j} = f|_{\partial Q_j} = d$, $j = 1, \dots, p$ (this can be achieved by an ε-dependent coordinate change).

Recall that F_ε denotes the first return to L.

Lemma 6. *F_ε are uniformly expanding with bounded distortion. Moreover (5) holds.*

Observe that Lemma 6 implies Proposition 1 because the only way an orbit may fail to return quickly to L is by spending a lot of time near d but for such orbits expansion is obvious.

Proof. We begin with expansion. For a large R, divide the trajectory of a point x into three intervals (some of which might be empty)

(1) before the first visit to $[c - R\sqrt{\varepsilon}, c + R\sqrt{\varepsilon}]$.
(2) passage of $[c - R\sqrt{\varepsilon}, c + R\sqrt{\varepsilon}]$.
(3) return to L.

To handle the derivative on interval (1) we let $\delta_{n,\varepsilon} = \frac{df_\varepsilon^n}{dx}$. Then

$$\delta_{n+1,\varepsilon} = \delta_{n,\varepsilon}\left(1 + 2\alpha\left(x_{n,\varepsilon} - c\right)\right) + O\left(\varepsilon + (x_{n,\varepsilon} - c)^2\right).$$

Since $(f^n)'$ are uniformly bounded on interval (1), we get by induction that $x_{n,\varepsilon} - x_n = O(\varepsilon n)$. This gives

$$\ln \delta_{n+1,\varepsilon} - \ln \delta_{n,\varepsilon} \sim 2\alpha(x_n - c) + O(\varepsilon n^2)$$

(the term $O\left(\varepsilon + (x_{n,\varepsilon} - c)^2\right)$ gives a smaller contribution.) Since the main term does not depend on ε we get

$$(f_\varepsilon^n)' = (f^n)'(1 + \tilde{\gamma}_n) \tag{11}$$

where $|\tilde{\gamma}_n|$ can be made arbitrary small by choosing R sufficiently large. By considering f^{-1} we see that a similar estimate holds for interval (3). It remains to show that the derivative on interval (2) is close to 1. Introduce $y_n = \frac{x_{n,\varepsilon} - c}{\sqrt{\varepsilon}}$. Then for $|x_n| \leq R\sqrt{\varepsilon}$ we have

$$y_{n+1} - y_n = \sqrt{\varepsilon}(\alpha y_n^2 + 1) + O(\varepsilon) \tag{12}$$

We now use the following standard fact.

Proposition 7. *Suppose y_n satisfy (12). Suppose that y_0 be close to $-R$. Fix $t < \frac{\pi}{2\sqrt{\alpha}} + \frac{\arctan(\sqrt{\alpha})R}{\sqrt{\alpha}}$ and let $n = [t/\sqrt{\varepsilon}]$. Then as $\varepsilon \to 0$ the map $y_0 \mapsto y_n$ converges in C^r to time t map of*

$$\dot{y} = \alpha y^2 + 1 \tag{13}$$

This proposition shows that the derivative on interval (2) is converges to 1 (=the derivative of time $2\arctan(\sqrt{\alpha}R)/\sqrt{\alpha}$ map of (13)). In view of (11) expansion now follows from Lemma 5.

The proof of the distortion estimate is now similar to the proof of Lemma 5 except that now we have an extra term $\frac{\sqrt{\varepsilon^3}}{(c-x)^4}$ in the estimate of f'' (equation (6)) coming from the term

$$(f^{k_2+1})' \, \frac{\partial^2 y_n}{\partial y_0^2} \, [(f^{k_1-m})']^2$$

where n is the passage time from $-R\sqrt{\varepsilon}$ to $R\sqrt{\varepsilon}$. However this does not spoil the main term.

Mixing follows as before. □

Let $\tau_\varepsilon(x)$ denote the first return time to L and set $\tilde{A}(x,\varepsilon) = \sum_{j=0}^{\tau_\varepsilon - 1} A(f_\varepsilon^j x)$. Let δ be a small positive number.

Lemma 8.

$$\tilde{A}(x,\varepsilon) = A^\dagger(x,\varepsilon) + o(1) + \sigma_\varepsilon(x)$$

where

$$A^\dagger(x,\varepsilon) = A(x) + A^+(fx) + A^*(f^{m(x)}x) + A^{**}(F_\varepsilon)(x),$$

and $\mathrm{mes}(\sigma_\varepsilon \neq 0) \leq \mathrm{Const}\varepsilon^{1/2-\delta}$. *(Here* $A^+ = 0$ *if* $f(x) \notin [d, a_0]$, $A^* = 0$ *if* $fx \notin [d, c]$ *and* $A^{**} = 0$ *if* $fx \in L$.*)*

Proof. The proof is similar to the proof of Lemma 6. □

Define

$$\hat{\tau} = \frac{\pi}{\sqrt{\alpha}} 1_{[d,c]}(fx)$$

Lemma 9. *The difference between* $\sqrt{\varepsilon}\tau_\varepsilon(x)$ *and* $\hat{\tau}(x)$ *is uniformly small outside a set of measure* $\mathrm{Const}\varepsilon^{1/2-\delta}$.

Proof. The proof is similar to the proof of Lemma 6. The main contribution comes from passage of $[c-R\sqrt{\varepsilon}, c+R\sqrt{\varepsilon}]$. which in view of Proposition 7 can be made as close to $\frac{\pi}{\sqrt{\alpha}}$ as we wish by choosing R large. At the same time $\sqrt{\varepsilon}\times$the time spent outside of this interval tends to 0 when we increase R. The set which we have to remove consists of points where $f_\varepsilon(x) \in [c - R\sqrt{\varepsilon}, c + R\sqrt{\varepsilon}]$ or $f_\varepsilon(x) \in [d, d + (1/f'(d))^{1/\sqrt{\varepsilon}}]$. □

Let $a_{l,\varepsilon} = f_\varepsilon^l a_0$, $b_{-k,\varepsilon} = f_\varepsilon^{-k} b_0$, $J_k^\varepsilon = [b_{-(k+1),\varepsilon}, b_{-k,\varepsilon}]$, and let $j(\varepsilon)$ be the first j then $\mathrm{length}(J_j^\varepsilon) \leq \varepsilon$. We now define $s(\varepsilon)$ by waiting until $a_{l,\varepsilon}$ hits $J_{j(\varepsilon)}^\varepsilon$ and rescaling between the image point between $[0,1]$ using $H_{J_{j(\varepsilon)}^\varepsilon}$. Let ν_ε be the absolutely continuous invariant measure for F_ε. Put

$$\hat{A}(x,s) = A(x) + A^+(fx) + A^*(f^{m(x)}x) + A^{**}(\tilde{F}_s)(x).$$

Given observables A, B set

$$D(A,B,F_*) =$$

$$\nu_*(AB) - \nu_*(A)\nu_*(B) + \sum_{j=1}^{\infty} \left[\nu_*(A(B\circ F_*^j)) + \nu_*(B(A\circ F_*^j)) - 2\nu_*(A)\nu_*(B)\right].$$

Lemma 10.

$$\nu_\varepsilon(\tilde{A}) \sim \nu_{s(\varepsilon)}(\hat{A}), \tag{14}$$
$$D(\tilde{A}, F_\varepsilon) \sim D(\hat{A}, \tilde{F}_{s(\varepsilon)}),$$
$$\sqrt{\varepsilon}\nu_\varepsilon(\tau_\varepsilon) \sim \nu_{s(\varepsilon)}(\hat{\tau}),$$
$$\varepsilon D(\tau_\varepsilon, F_\varepsilon) \sim D(\hat{\tau}, \tilde{F}_{s(\varepsilon)}),$$
$$\sqrt{\varepsilon} D(\tau_\varepsilon, \tilde{A}, F_\varepsilon) \sim D(\hat{\tau}, \hat{A}).$$

Proof. Let us prove (14). Define a new norm $|||\cdot|||$. Let $|||A||| = K$ if $||A||_{L^1} \le K$ and given ρ there is a function $A^{(\rho)}$ such that

$$||A - A^{(\rho)}||_{L^1} \le \varepsilon \text{ and } ||A^{(\rho)}||_{C^\gamma} \le \frac{K}{\rho^4}.$$

It is easy to see from Corollary 3 and Lemma 8 and their proofs that $|||\hat{A}(\cdot, s)|||$ and $|||\tilde{A}(\cdot, \varepsilon)|||$ are uniformly bounded. For example, to approximate $\hat{A}(\cdot, s)$ we set

$$U_\rho = [d, d+\rho] \bigcup [c-\rho, c+\rho], \tag{15}$$

and let $\hat{A}^{(\rho)}(x, s) = \hat{A}(x, s)$ for $x \notin U_{\rho^2}$ and define $\hat{A}^{(\rho)}(x, s)$ on the intervals (15) by interpolating linearly between their endpoints. Then Corollary 2 implies that

$$||\hat{A}(\cdot, s) - \hat{A}^{(\rho)}(\cdot, s)|| \le \text{Const } \rho^2 \ln \rho$$

and outside U_{ρ^2} we have

$$\left| \frac{d}{dx} A^{(\rho)}(\cdot, s) \right| \le \text{Const}(\rho^2)^{-2} = \text{Const } \rho^{-4}.$$

Let $K = \max(\sup |||\hat{A}|||, \sup |||\tilde{A}|||)$. By preceding estimates we know that both F_ε and $\tilde{F}_s$ are uniformly expanding and have bounded distortion.

Lemma 11. *Uniformly with respect to the parameters*

$$\left| \int A(F_*^n x) dx - \nu_*(A) \right| \le \text{Const}|||A|||\theta^n. \tag{16}$$

(Lemmas 5 and 6 give this for C^γ norm but $\hat{A}$ does not belong to C^γ and hence $||\tilde{A}||_{C^\gamma}$ are not uniformly bounded.)

Proof. Let F stand for either F_ε or $\tilde{F}_s$. Denote by $\mathcal{L}$ the adjoint of $A \to A \circ F$. Then

$$\int A(F^N x) dx = \int A^{(\rho)}(F^N x) dx + \int [A(F^N x) - A^{(\rho)}(F^N x)] dx =$$
$$\nu(A^{(\rho)}) + O(\theta^N |||A|||\rho^{-4}) + \int (\mathcal{L}^N 1)[A - A^{(\rho)}] dx =$$
$$\nu(A^{(\rho)}) + O(\theta^N |||A|||\rho^{-4}) + O(|||A|||\rho||\mathcal{L}^N 1||_{L^\infty}.$$

Now for any $B \in L^\infty$

$$\mathcal{L}^N B = \sum_I (B \circ F^{-N})(F_I^{-N})'$$

where F_I^{-N} denotes the branch of F^{-N} with range I. By the distortion estimate

$$||\mathcal{L}^N B||_{L^\infty} \le C_1 \sum_I |I|||B||_{L^\infty}$$

where the constant depend only on distortion constant for F. In particular, $\frac{d\nu}{dx}$ has density uniformly bounded by C_2 and hence

$$\nu(A^{(\rho)}) = \nu(A) + O(\rho)|||A|||.$$

This implies

$$\int A(F^N x)dx - \nu(A) = |||A|||O\left(\theta^N \rho^{-4} + \rho\right).$$

Optimizing in ρ we obtain the result needed. □

Choose δ_0. Take N such that

$$\left|\int B(F_*^N x)dx - \nu_*(B)\right| \le \frac{\delta_0}{6} \tag{17}$$

for all B with $|||B||| \le K$. In particular

$$\left|\int \tilde{A}(F_\varepsilon^N x, \varepsilon)dx - \nu_\varepsilon(\tilde{A}(\cdot, \varepsilon))\right| \le \frac{\delta_0}{6}$$

Next, given N, ε we construct a set $\Omega = \Omega(\varepsilon)$ such that

$$\mathrm{mes}(L - \Omega) \le \frac{\delta_0}{6}$$

$$\left|\int_{L-\Omega} \tilde{A}dx\right| \le \frac{\delta_0}{6}, \quad \left|\int_{L-\Omega} \hat{A}dx\right| \le \frac{\delta_0}{6} \tag{18}$$

and on Ω

$$\left|\tilde{A}(F_\varepsilon^N x, \varepsilon) - \tilde{A}(\tilde{F}_{s(\varepsilon)}^N x, \varepsilon)\right| \le \frac{\delta_0}{6}, \tag{19}$$

$$\left|\tilde{A}(\tilde{F}_{s(\varepsilon)}^N x, \varepsilon) - \hat{A}(\tilde{F}_{s(\varepsilon)}^N x, s(\varepsilon))\right| \le \frac{\delta_0}{6}, \tag{20}$$

Let $\Omega = L - \bigcup_{j=0}^{N+1} F_\varepsilon^{-j} U_\rho$. It then follows by bounded distortion that

$$\mathrm{mes}(\Omega) \le \mathrm{Const}\rho(N+2)$$

so that by decreasing ρ we can satisfy (17). Also, Corollary 3 and Lemma 8 show that $\tilde{A}$ and $\hat{A}$ are uniformly integrable, so by further decreasing ρ we can satisfy (18).

Next, the computations of Lemma 6 together with Proposition 7 imply that for a fixed ρ, the map $F_\varepsilon(x)$ is close to $\tilde{F}_{s(\varepsilon)}(x)$ if $x \notin U_\rho$. Now with N and ρ fixed, $(F_\varepsilon^N)'$ is uniformly bounded on Ω. It follows that for a small ε the map $F_{s(\varepsilon)}^N$ is uniformly close to F_ε^N. In particular, $F^j_{s(\varepsilon)}x \notin U_{\rho/2}$ for $j = 0, 1 \dots N$. Hence $(F^N_{s(\varepsilon)})'$ is uniformly bounded as well. So for a small enough ε (19) and (20) are satisfied. Next,

$$\begin{aligned}
\nu_\varepsilon(\tilde{A}) &= \int_L \tilde{A}(F^N_\varepsilon x, \varepsilon)dx + r_1 && \text{(by (17))} \\
&= \int_\Omega \tilde{A}(F^N_\varepsilon x, \varepsilon)dx + r_2 && \text{(by (18))} \\
&= \int_\Omega \tilde{A}(\tilde{F}^N_{s(\varepsilon)} x, \varepsilon)dx + r_3 && \text{(by (19))} \\
&= \int_\Omega \hat{A}(\tilde{F}^N_{s(\varepsilon)} x, s(\varepsilon))dx + r_4 && \text{(by (20))} \\
&= \int_L \hat{A}(\tilde{F}^N_{s(\varepsilon)} x, s(\varepsilon))dx + r_5 && \text{(by (18))} \\
&= \nu_{s(\varepsilon)}(\hat{A}) + r_6 && \text{(by (17))}
\end{aligned}$$

where $|r_j| \leq \delta_0 j/6$. This proves (14). Proofs of other asymptotics are similar using identities

$$D(A, B, F_\varepsilon) = \frac{1}{4}\left[D(A + B, F_\varepsilon) - D(A - B, F_\varepsilon)\right],$$

$$D_\varepsilon(A) = \lim_{N\to\infty} \frac{1}{N} \int \left[\sum_{j=0}^{N-1} A(F^j_\varepsilon x)\right]^2 dx.$$

□

We now use formulas

$$\nu_\varepsilon(A) = \frac{\nu_\varepsilon(\tilde{A})}{\nu_\varepsilon(\tau_\varepsilon)}, \tag{21}$$

$$D_\varepsilon(A) = \frac{D(\tilde{A}, F_\varepsilon) + \left[\frac{\nu(\tilde{A})}{\nu(\tau_\varepsilon)}\right]^2 D(\tau_\varepsilon, F_\varepsilon) + 2\frac{\nu_\varepsilon(\tilde{A})}{\nu_\varepsilon(\tau_\varepsilon)} D(\tau_\varepsilon, \tilde{A}, F_\varepsilon)}{\nu_\varepsilon(\tau_\varepsilon)}. \tag{22}$$

Lemma 10, (21) and (22) allow us to approximate $\mu_\varepsilon(A)$ and $D_\varepsilon(A)$ by expressions involving correlation functions for $F_{s(\varepsilon)}$. To complete the proof of Theorem 1 it remains to establish the weak periodicity of $s(\varepsilon)$.

Lemma 12.
$$\frac{ds}{d\varepsilon} \sim \frac{\pi}{\sqrt{\alpha\varepsilon^3}}.$$

Proof. We defined s by waiting until $a_{l,\varepsilon}$ hits $J^\varepsilon_{j(\varepsilon)}$. In fact for any k we can define $s^{(k)}(\varepsilon)$ by waiting until $a_{l,\varepsilon}$ hits J^ε_k and rescaling the image between 0 and 1. We claim that for any fixed c_1, c_2 we have

$$\frac{d}{d\varepsilon}\left[s^{(c_1/\sqrt{\varepsilon})} - s^{(c_2/\sqrt{\varepsilon})}\right] = O(\sqrt{\varepsilon}).$$

Indeed, by Lemma 2, if we replace k by $k+1$ for k of order $1/\sqrt{\varepsilon}$ then

$$\frac{d}{d\varepsilon}\left[s^{(k+1)} - s^{(k)}\right] = O(\varepsilon).$$

Now fix R and consider $\frac{d}{d\varepsilon}s^{(k)}$ with $k=k(R)=[\alpha/(R\sqrt{\varepsilon})]$, so that $b_{-k,\varepsilon}\sim R\sqrt{\varepsilon}$. Let $l(R,\varepsilon)$ be the first moment when $a_{l(R,\varepsilon),\varepsilon}>b_{-k,\varepsilon}$.

$$s^{(k(R))}(\varepsilon)=\frac{a_{l,\varepsilon}-b_{-k,\varepsilon}}{b_{-k,\varepsilon}-b_{-k+1,\varepsilon}}. \tag{23}$$

Now as before we can show that

$$\max_{m\leq k(R)}\sqrt{\varepsilon}\frac{d}{d\varepsilon}b_{-m,\varepsilon}\to 0 \text{ as } R\to\infty. \tag{24}$$

On the other hand, for a large R we can approximate $\sqrt{\varepsilon}\frac{da_{l(R,\varepsilon),\varepsilon}}{d\varepsilon}$ by $r(T)$ where r is the solution of

$$\dot{x}=\alpha x^2+1 \quad x\to-\infty, \quad t\to-\frac{\pi}{2\alpha}, \quad x(T)=R,$$
$$\dot{r}=2\alpha x(t)r+1, \quad r\to 0, \quad t\to-\frac{\pi}{2\alpha}.$$

Solving this equation we get

$$x(t)=\frac{1}{\sqrt{\alpha}}\tan(\sqrt{\alpha}t), \quad r(t)=\frac{\frac{\pi}{2\sqrt{\alpha}}+\frac{t}{2}+\frac{\sin(2\sqrt{\alpha}t)}{4\sqrt{\alpha}}}{\cos^2(\sqrt{\alpha}t)}.$$

For T near $\frac{\pi}{2\alpha}$ we have

$$x(T)\sim\frac{1}{\sqrt{\alpha}\cos(\sqrt{\alpha}T)}, \quad r(T)\sim\frac{\pi}{\sqrt{\alpha}\cos^2(\sqrt{\alpha}t)}.$$

Hence $r\sim\pi\sqrt{\alpha}x^2=\pi\sqrt{\alpha}R^2$. This gives

$$\frac{da_{l(R,\varepsilon),\varepsilon}}{d\varepsilon}\sim\frac{\pi\sqrt{\alpha}R^2}{\sqrt{\varepsilon}}. \tag{25}$$

On the other hand

$$b_{-k,\varepsilon}-b_{-k+1,\varepsilon}\sim\alpha R^2\varepsilon. \tag{26}$$

Combining (23)–(26) we obtain the statement of the lemma. □

Let us now explain why Lemma 12 implies the weak periodicity of $s(\varepsilon)$. The quantity $s(\varepsilon)$ has jumps at the jumps of $j(\varepsilon)$, but the magnitude of these jumps is $O(\varepsilon)$ by Lemma 2. An argument similar to the proof of Lemma 12 shows that the distance between the jumps is $O(\varepsilon^{-3/2})$. Hence the main contribution to the growth of s comes from $\int\frac{ds}{d\varepsilon}d\varepsilon$. This completes the proof of Theorem 1. □

References

[1] Afraimovich V., Liu, W. S. & Young, T. *Conventional multipliers for homoclinic orbits,* Nonlinearity **9** (1996) 115–136.

[2] Afraimovich V. & Young T. *Relative density of irrational rotation numbers in families of circle diffeomorphisms,* Erg. Th. & Dynam. Sys. **18** (1998) 1–16.

[3] Bakhtin V. I. *A direct method for constructing an invariant measure on a hyperbolic attractor,* Russian Acad. Sci. Izv. Math. **41** (1993) 207–227.

[4] Bakhtin V. I. *Random processes generated by a hyperbolic sequence of mappings I–II,* Russian Acad. Sci. Izv. Math. **44** (1995) 247–279 & 617–627.

[5] Bonetto F., Daems D. & Lebowitz J. *Properties of stationary nonequilibrium states in the thermostated periodic Lorenz gas* J. Stat. Phys. **101** (2000) 35–60.

[6] Bonetto F., Gentile G. & Mastropietro V. *Electric fields on a surface of constant negative curvature,* Erg. Th. & Dyn. Sys. **20** (2000) 681–696.

[7] Chernov N. & Dolgopyat D. *Regularity of diffusion matrix for dispersive billiards,* in preparation.

[8] Dolgopyat D. *On differentiability of SRB states for partially hyperbolic systems,* to appear in Inv. Math.

[9] Giampieri M. & Isola S. *A one parameter family of analytic Markov maps with an intermittency transition,* preprint.

[10] L. Jonker *Scaling of Arnold tongues for differential homeomorphisms of the circle,* Comm. Math. Phys. **129** (1991) 1–25.

[11] Katok A., Knieper G., Pollicott M. & Weiss H. *Differentiability of entropy for Anosov and geodesic flows,* Bull. AMS **22** (1990) 285–293.

[12] Katok A., Knieper G., Pollicott M. & Weiss H. *Differentiability and analyticity of topological entropy for Anosov and geodesic flows,* Invent. Math. **98** (1989), 581–597.

[13] Lyubich M. *The quadratic family as a qualitatively solvable model of chaos,* Notices AMS **47** (2000) 1042–1052.

[14] Ruelle D. *Differentiation of SRB states,* Comm. Math. Phys. **187** (1997) 227–241 and **234** (2003) 185–190.

[15] Ruelle D. *Smooth dynamics and new theoretical ideas in nonequilibrium statistical mechanics* J. Stat. Phys. **95** (1999) 393–468.

[16] Ruelle D. *Nonequilibrium statistical mechanics near equilibrium: computing higher-order terms,* Nonlinearity **11** (1998) 5–18.

[17] Ruelle D. *Perturbation theory for Lyapunov exponents of a toral map: extension of a result of Shub and Wilkinson,* Israel J. Math. **134** (2003) 345–361.

[18] Teplinsky A. & Khanin K. *Rigidity for circle diffeomorphisms with singularities,* preprint.

[19] Young L.-S. *Statistical properties of dynamical systems with some hyperbolicity,* Ann. of Math. **147** (1998) 585–650.