HAUSDORFF DIMENSION IN STOCHASTIC DISPERSION.

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Abstract. We consider the evolution of a connected set in Euclidean space carried by a periodic incompressible stochastic flow. While for almost every realization of the random flow at time t most of the particles are at a distance of order $\sqrt{t}$ away from the origin [DKK1], there is an uncountable set of measure zero of points, which escape to infinity at the linear rate [CSS1]. In this paper we prove that this set of linear escape points has full Hausdorff dimension.

Dedicated to our teacher Yakov Sinai on occasion of his 65th birthday.

1. Introduction.

One of the greatest achievements in mathematics of the second half of the last century was creation of the theory of hyperbolic dynamical systems in works of Anosov, Bowen, Ruelle, Sinai, Smale and many others. The importance of this theory is not so much in that it allows one to get new information about a large class of ordinary differential equations but in that it provides a paradigm for understanding irregular behavior in a large class of natural phenomena. From the mathematical point of view it means that the theory should be useful in many branches of mathematics beyond the study of finite-dimensional dynamical systems. The aim of this note is to illustrate this on a simple example. Namely, we show how the theory of nonuniformly hyperbolic systems, i.e. systems with non-zero Lyapunov exponents, can explain ballistic behavior in a problem of passive transport in random media.

This paper concerns the long time behavior of a passive substance (say an oil spill) carried by a stochastic flow. Various aspects of such behavior have been a subject of a number of recent papers (see [Cm, CC, CGXM, CS, CSS1, CSS2, DKK1, DKK2, LS, SS, ZC1, ZC2] etc.) Consider an oil spill at the initial time concentrated in a domain Ω. Let Ω evolve in time along trajectories of the stochastic flow and Ω_t be its image at time t. The papers mentioned study the rate of stretching of the boundary $\partial\Omega_t$, growth of the diameter and the "shape" of Ω_t, distribution of mass of Ω_t, and many other related questions. In this

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paper we model the stochastic flow by a stochastic differential equation driven by a finite-dimensional Brownian motion $\{\theta(t) = (\theta_1(t), \dots, \theta_d(t)) \in \mathbb{R}^d\}_{t\geq 0}$

$$dx_t = X_0(x_t)dt + \sum_{k=1}^{d} X_k(x_t) \circ d\theta_k(t) \quad x \in \mathbb{R}^N \tag{1}$$

where $\{X_k\}_{k=0}^d$ are C^∞-smooth space periodic divergence free vector fields on $\mathbb{R}^N$. Alternatively one can regard this system as a flow on $\mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N$. Below we impose certain nondegeneracy assumptions on vector fields $\{X_k\}_{k=1}^d$ from [DKK1]. These assumptions hold on an dense open set of C^∞-smooth divergence free vector fields on $\mathbb{T}^N$ or satisfied generically.

An interesting feature of the flow (1) is the dichotomy between growth of the mass and shape of the spill Ω_t. On one hand, most of the points of the tracer Ω_t move at distance of order $\sqrt{t}$ at time t. More precisely, let ρ be a smooth metric on $\mathbb{T}^N$, naturally lifted to $\mathbb{R}^N$ and ν be a measure of a finite energy, i.e. for some positive p we have

$$\iint \frac{d\nu(x)d\nu(y)}{\rho^p(x,y)} < \infty.$$

In particular, ν can be the Lebesgue probability measure supported on an open set Ω, which also supports the initial oil spill. Let ν_t be its image under the flow (1) and $\bar{\nu}_t$ be rescaling of ν_t, defined by as follows: for a Borel set $\Omega \subset \mathbb{R}^N$ put $\bar{\nu}_t(\Omega) = \nu_t(\sqrt{t}\Omega)$.

Theorem 1. *(*[DKK1]*) For almost every realization of the Brownian motion $\{\theta(t)\}_{t\geq 0}$ the measure $\bar{\nu}_t$ weakly converges to a Gaussian measure on $\mathbb{R}^N$ as $t \to \infty$.*

Remark 1. Notice that this is the Central Limit Theorem with respect to randomness in initial conditions, not with respect to randomness of the Brownian motion $\{\theta(t)\}_{t\geq 0}$.

On the other hand, there are many points with *linear growth.* Fix a realization of the Brownian motion $\{\theta(t)\}_{t\geq 0}$. Let $\mathbf{L}_\theta$ denote the random set of points with a linear escape

$$\mathbf{L}_\theta = \left\{ x \in \mathbb{R}^N : \lim\inf_{t\to+\infty} \frac{|x_t|}{t} > 0 \right\}.$$

The following result is a special case of [SS] (see also [CSS1]).

Theorem 2. *Let S be a connected set containing at least two points. Then for almost every realization of the Brownian motion $\{\theta(t)\}_{t\geq 0}$ the set $\mathbf{L}_\theta \bigcap S$ is uncountable.*

In fact in dimension 2 there is a limiting shape of the rescaled contaminated area. Namely, let Ω be a bounded open set, Ω_t be its image under the flow (1) and $\mathcal{C}_t = \bigcup_{0 \le s \le t} \Omega_s$. In other words, we call a point x *contaminated by the time* t if there is a trajectory from our curve which has passed through x before time t.

The Shape Theorem. *([DKK2]) If $N = 2$, then there exists a convex compact set $\mathbb{B} \subset \mathbb{R}^2$ such that for almost every realization of the Brownian motion $\{\theta(t)\}_{t\ge 0}$ and any $\delta > 0$ there exists $T = T(\delta)$ such that for all $t > T$*

$$(1-\delta)\, \mathbb{B} \subset \frac{\mathcal{C}_t}{t} \subset (1+\delta)\, \mathbb{B}.$$

Remark 2. The "shape" $\mathbb{B} \subset \mathbb{R}^2$ is independent of the initial spill Ω. Moreover, an open set Ω can be replaced by a smooth curve γ for the Shape Theorem to hold true.

In view of Theorems 1, 2, and the Shape Theorem it is interesting to see how large is the set of points with linear growth. In this paper we first prove the following

Theorem 3. *Let γ be a smooth curve on $\mathbb{R}^2$. Then for almost every realization of the Brownian motion $\{\theta(t)\}_{t\ge 0}$ we have* $\mathrm{HD}(\mathbf{L}_\theta \bigcap \gamma) = 1$.

Then in Section 8 using this Theorem we derive the following main result of the paper

Theorem 4. *(**Main Result**) For almost every realization of the Brownian motion $\{\theta(t)\}_{t\ge 0}$ we have that points of the flow (1) with linear escape to infinity $\mathbf{L}_\theta$ form a dense set of full Hausdorff dimension* $\mathrm{HD}(\mathbf{L}_\theta) = N$.

By Theorem 1 for most points $x_0 = x$ in $\mathbb{R}^N$ its trajectory x_t is of order $\sqrt{t}$ away from the origin at time t. Also, the Law of Iterated Logarithm for functionals of diffusion processes and Fubini Theorem imply that the set of points $\mathbf{L}_\theta$ with linear escape has measure zero. This Corollary says that $\mathbf{L}_\theta$ is the "richest" possible set of measure zero in $\mathbb{R}^N$, namely, is of full Hausdorff dimension N.

2. Nondegeneracy assumptions.

In this section we formulate a set of assumptions on the vector fields, which in particular imply the Central Limit Theorem for measures, the estimates on the behavior of the characteristic function of a measure carried by the flow (see [DKK1]), and large deviations estimates (see [BS]). Such estimates are essential for the proof of our results. Recall that $X_0, X_1, \dots, X_d$ are assumed to be C^∞-smooth, periodic and divergence free.

(A) (*hypoellipticity for x_t*) For all $x \in \mathbb{R}^N$ we have

$$\mathrm{Lie}(X_1, \dots, X_d)(x) = \mathbb{R}^N.$$

Denote the diagonal in $\mathbb{T}^N \times \mathbb{T}^N$ by

$$\Delta = \{(x^1, x^2) \in \mathbb{R}^N \times \mathbb{R}^N : \ x^1 = x^2 \ (\mathrm{mod}\ 1)\}.$$

(B) (*hypoellipticity for the two–point motion*) The generator of the two–point motion $\{(x_t^1, x_t^2) : \ t > 0\}$ is nondegenerate away from the diagonal Δ, meaning that the Lie brackets made out of $(X_1(x^1), X_1(x^2)), \dots, (X_d(x^1), X_d(x^2))$ generate $\mathbb{R}^N \times \mathbb{R}^N$.

To formulate the next assumption we need additional notations. Let $Dx_t : T_{x_0}\mathbb{R}^N \to T_{x_t}\mathbb{R}^N$ be the linearization of x_t at t. We need the hypoellipticity of the process $\{(x_t, Dx_t) : \ t > 0\}$. Denote by TX_k the derivative of the vector field X_k thought as the map on $T\mathbb{R}^2$ and by $S\mathbb{R}^N = \{v \in T\mathbb{R}^N : \ |v| = 1\}$ the unit tangent bundle on $\mathbb{R}^N$. If we denote by $\tilde{X}_k(v)$ the projection of $TX_k(v)$ onto $T_v S\mathbb{R}^N$, then the stochastic flow (1) on $\mathbb{R}^N$ induces a stochastic flow on the unit tangent bundle $S\mathbb{R}^N$, defined by the following equation:

$$d\tilde{x}_t = \sum_{k=1}^{d} \tilde{X}_k(\tilde{x}_t) \circ d\theta_k(t) + \tilde{X}_0(\tilde{x}_t)dt.$$

With these notations we have condition

(C) (*hypoellipticity for* (x_t, Dx_t)) For all $v \in S\mathbb{R}^N$ we have

$$\mathrm{Lie}(\tilde{X}_1, \dots, \tilde{X}_d)(v) = T_v S\mathbb{R}^N \ .$$

For measure-preserving stochastic flows with conditions (C) Lyapunov exponents $\lambda_1, \dots, \lambda_N$ exist by *multiplicative ergodic theorem for stochastic flows* of diffeomorphisms (see [Cv], Thm. 2.1). Moreover, the sum of Lyapunov exponents $\sum_{j=1}^N \lambda_j$ should be zero (see e.g. [BS]). Under conditions (A)–(C) the leading Lyapunov exponent is positive

$$\lambda_1 = \lim_{t \to \infty} \frac{\log |d\varphi_t(x)(v)|}{t} > 0 \ , \tag{2}$$

where $d\varphi_t(x)$ is the linearization matrix of the flow (1) integrated from 0 to t at the point x. Indeed, Theorem 6.8 of [Bx] states that under condition (A) the maximal Lyapunov exponent λ_1 can be zero only if for almost every realization of the flow (1) one of the following two conditions is satisfied

(a) there is a Riemannian metric ρ' on $\mathbb{T}^N$, invariant with respect to the flow (1) or

(b) there is a direction field $v(x)$ on $\mathbb{T}^N$ invariant with respect to the flow (1).

However (a) contradicts condition (B). Indeed, (a) implies that all the Lie brackets of $\{(X_k(x^1), \ X_k(x^2))\}_{k=1}^d$ are tangent to the leaves of the foliation

$$\{(x^1, x^2) \in \mathbb{T}^N \times \mathbb{T}^N : \ \rho'(x^1, x^2) = \mathrm{Const}\ \}$$

and don't form the whole tangent space. On the other hand (b) contradicts condition (C), since (b) implies that all the Lie brackets are tangent to the graph of v. This positivity of λ_1 is crucial for our approach.

Remark 3. Let us mention an important difference between deterministic and stochastic dynamics. Most of the results dealing with statistical properties of deterministic systems assume that all Lyapunov exponents are non-zero. By contrast we need only one positive exponent. This is because in the random situation hypoellipticity condition (C) implies that growth rate of any deterministic vector is given by the largest exponent (see equation (19). This allows us to get our results without assuming that all the exponents are non-zero.

We further require that the flow has no deterministic drift, which is expressed by the following condition

(E) (*zero drift*)

$$\int_{\mathbb{T}^2} \left(\sum_{k=1}^{d} L_{X_k} X_k + X_0 \right)(x) dx = 0 \ ,$$

where $L_{X_k} X_k(x)$ is the derivative of X_k along X_k at the point x. Notice that $\sum_{k=1}^{d} L_{X_k} X_k + X_0$ is the deterministic components of the stochastic flow (1) rewritten in Ito's form.

The Central Limit Theorem for measures was formulated in [DKK1] under an additional assumption

$$\int_{\mathbb{T}^2} X_k(x) dx = 0 \ , \ k = 1, ..., d \ . \tag{3}$$

This assumption is not needed for the proof of Theorem 3 and as the result for the proof of the Main Theorem. However, in order to simplify the proof, i.e. use the results of [DKK1] without technical modifications, we shall assume (3) to hold.

3. Idea of the proof.

3.1. **A Model Example.** Below we define a random dynamical system on $\mathbb{R}$ which models the motion of the projection of the spill Ω_t onto a fixed line $l \subset \mathbb{R}^N$.

Introduce notations: $I(b;a) = [b - a/2, b + a/2]$ — the segment on $\mathbb{R}$ centered at b of length a; $s \in \{0,1\}^{\mathbb{Z}_+}$ a semiinfinite sequence of 0's and 1's, $s_k \in \{0,1\}^k$ a set of k numbers 0 or 1, $\{\{\theta_{s_k}(t)\}_{s_k \in \{0,1\}^k}\}_{k \in \mathbb{Z}_+}$ countable number of standard i.i.d. Brownian motions on $\mathbb{R}$ indexed by binary sequences. Let τ be positive.

The random dynamical system is defined as follows. Let $I^{\emptyset} = I(0;1)$. Then $\sigma_0^{\theta} : I^{\emptyset} \to \mathbb{R}$ stretches $I^{\emptyset}$ uniformly by 2 around its center and shifts it randomly by $\theta_\emptyset(\tau)$. Divide $\sigma_0^\theta(I^\emptyset)$ in two equal parts I^0 and I^1

$$\sigma_0^\theta(I^\emptyset) = I^0 \cup I^1 = I(\theta_\emptyset(\tau) - 1/2; 1) \cup I(\theta_\emptyset(\tau) + 1/2; 1). \tag{4}$$

Now σ_1^θ acts on each $\{I^i\}_{i=0,1}$ independently by stretching each I^i's uniformly by 2 around its center and shifting by $\theta_0(\tau)$ and $\theta_1(\tau)$ respectively.

$$\begin{aligned} &\sigma_1^\theta \circ \sigma_0^\theta(I^\emptyset) = (I^{00} \cap I^{01}) \cup (I^{10} \cup I^{11}) = \\ &I([\theta_\emptyset(\tau) - 1/2] + [\theta_0(\tau) - 1/2]; 1) \cup I([\theta_\emptyset(\tau) - 1/2] + [\theta_1(\tau) + 1/2]; 1) \\ \cup\ &I([\theta_\emptyset(\tau) + 1/2] + [\theta_0(\tau) - 1/2]; 1) \cup I([\theta_\emptyset(\tau) + 1/2] + [\theta_1(\tau) + 1/2]; 1), \end{aligned} \tag{5}$$

and so on.

Let $n \in \mathbb{Z}_+$. Then at the n-th stage "after time $n\tau$" the image of the initial unit interval $I^\emptyset = [-1/2, 1/2]$ consists of 2^n unit intervals. The preimage of each of those unit intervals is an interval of length 2^{-n} uniformly contracted. Let's give a different definition of the random dynamical system (4)-(5).

Consider an isomorphism of the dynamical system on the unit interval $I = I^\emptyset + 1/2 = [0, 1]$ given by $\phi : x \mapsto 2x \pmod 1$ and the one sided Bernoulli shift on two symbols, say 0 and 1. Such an isomorphism is given by $s : x \mapsto s(x) = \{s_k(x)\}_{k=0}^\infty \in \{0, 1\}^{\mathbb{Z}_+}$, where for each $k \in \mathbb{Z}_+$

$$\begin{cases} s_k(x) = 0 & \text{if} \quad \phi^n(x) < 1/2 \\ s_k(x) = 1 & \text{otherwise.} \end{cases} \tag{6}$$

Let $\eta_n(x) = \#\{k \le n : s_k(x) = 1\}$. Notice now that

$$\sigma_n^\theta \circ \sigma_{n-1}^\theta \circ \cdots \circ \sigma_0^\theta(x) = \sum_{k=0}^{n} \theta_{s_k}(\tau) + (\eta_{n+1}(x) - (n+1)/2)/2, \tag{7}$$

where $\theta_{s_k}(\tau)$'s are i.i.d. Brownian motions. Define $\eta_-(x) = \liminf_{n\to\infty} \eta_n(x)/n$. Then for almost all points $x \in I$ we have $\eta_-(x) = \lim_{n\to\infty} \eta_n(x)/n = 1/2$. Let us show however that there is full Hausdorff dimension set of points in the interval I such that frequency of 0's is less than frequency of 1's, i.e. $\mathrm{HD}\{x \in I : \eta(x) > 1/2\} = 1$. Since $\sum_{k=0}^n \theta_{s_k}(\tau)/n \to 0$ almost surely this would imply that the set of points in $I^\emptyset$ with a nonzero drift for the random dynamical system, defined by (4)-(5), has full Hausdorff dimension almost surely, but is of measure zero.

We shall justify the fact that $\mathrm{HD}\{x \in I : \eta(x) > 1/2\} = 1$.

3.2. **Points with a nonzero drift.** Fix an arbitrary small positive ε. The goal is to find a fractal set of points $I_\infty \subset I^\emptyset$ and a probability measure μ_∞ supported on I_∞ such that μ_∞-a.e. point $x \in I_\infty$ has a nonzero drift to the right, i.e. $\liminf_{n\to\infty} \sigma_n^\theta \circ \cdots \circ \sigma_0^\theta(x)/n > 0$. Moreover, $\mathrm{HD}(\mu_\infty)$ tends to 1 as ε tends to 0.

Construction of the set I_∞ and of the measure μ_∞ is inductive. I_∞ is defined as a countable intersection of a nested sequence of compact sets and μ_∞ is given as a weak limit of Lebesgue measures supported on those sets. We describe the base of the induction and the inductive steps.

• For $n = 1$ we have $\sigma_0^\theta(I^\emptyset)$ is a segment of length 2 or union of two segments I^0 and I^1 of length 1 each. Cut off the bottom ε-segment from each segment. This corresponds to cutting off ε-segments $[-1/2, -1/2 + \varepsilon]$ and $[0, \varepsilon]$ from $I^\emptyset$. Denote the surgery result by $I_0 \subset I^\emptyset$ and by μ_0 the Lebesgue probability measure supported on whole I_0. Notice that

$$\mu_0\{x \in I_0 : \sigma_1^\theta(x) = 1\} > \mu_0\{x \in I_0 : \sigma_1^\theta(x) = 0\} \tag{8}$$

creates a nonzero drift up, since frequency of 1's exceeds frequency of 0's.

• Suppose I_{n-1} and μ_{n-1} are constructed. To construct I_n and μ_n consider the image $\sigma_n^\theta(I_{n-1})$. It consists of 2^n segments of equal length close to 1. Cut off the bottom ε-segment from each. This corresponds to cutting off 2^n segments of length $2^{-n}\varepsilon$ from $I_{n-1} \subset I^\emptyset$. The result of the surgery is denoted by I_n and by μ_n we denote the Lebesgue probability measure supported on the whole I_n. Again the surgery increases probability of $s_n(x)$ being 1 over $s_n(x)$ being 0. Thus, this creates a positive drift.

The intersection $I_\infty = \cap_n I_n$ is a fractal set and the weak limit measure $\mu_\infty = \lim \mu_n$ has Hausdorff dimension approaching 1 as ε tends to zero. It follows from the construction that for μ_∞-almost every point $\eta_-(x) = \mu_0\{x \in I_0 : \sigma_1^\theta(x) = 1\} > 1/2$.

3.3. **Difficulties in extending of the Model Example to the case of the flow.** Let $\gamma \subset \mathbb{R}^N$ be a smooth curve, $l \in \mathbb{R}^N$ be a line, and $\pi_l : \mathbb{R}^N \to l$ be an orthogonal projection onto l. Suppose at the initial moment of time $\pi_l(\gamma) = I^\emptyset$ is the unit interval. If not, then rescale it and shift it to the origin.

The most subtle element in extending the Model Example is defining the stopping (stretching) time τ or deciding *when to stop* γ_t and *how to cut off* some parts of γ_t in order to create a nonzero drift as in (8). Such a stopping time needs to have several important features[1].

1. *Stretching property of the stopping time*: It is not difficult to show that if $|\pi_l(\gamma_0)| = 1$, then the stopping time

$$\tau_\gamma = \inf\{t \geq 0 : |\pi_l(\gamma_t)| = 2\} \tag{9}$$

has finite expectation and exponential moments uniformly bounded over all compact curves with projection of length 1 (see e.g. [CSS1]).

The analogy between this τ_γ and the model τ is clear. However, the geometry of γ_{τ_γ} in $\mathbb{R}^N$ might become quite complicated (γ_{τ_γ} might spin, bend, fold, and so on see computer simulations in [CC]) so it is not reasonable to stop all parts of the curve γ simultaneously and perform the surgery (cut off of "bottom" parts as in the passage preceding (8)). For this reason at the first stage of a

[1]In the Model Example τ is a constant

partition/cut off process we split γ_{τ_γ} not in *two* parts as in the Model Example, but in *a number* (may be countable) of random parts $\gamma = \cup_{j\in J}\gamma_j$ and each part γ_j will have *its own stopping time* τ_j.

2. *A countable Partition of* γ: We shall partition γ into at most countable number of segments $\gamma = \cup_{j\in J}\gamma_j$ (see Section 5). Each γ_j has its own stopping time τ_j so that the image $\varphi_{\tau_j}\gamma_j$ under the flow (1) is not too folded (see condition (b) of Theorem 5). Moreover, such a stopping time τ_j still has finite expectation and exponential moments (see condition (e) of Theorem 5).

Now if we have that the image $\varphi_{\tau_j}\gamma_j$ is "regular" it *does not reflect dynamics* on γ_j. In order to imitate the Model Example's cut off construction we need to stop $\varphi_t\gamma_j$ at the moment when $\varphi_t|_{\gamma_j}$ is more or less uniformly expanding on γ_j forward in time or $(\varphi_t)^{-1}|_{\varphi_t\gamma}$ is uniformly contracting on $\varphi_t\gamma$ backward in time (see conditions (a), (c), and (d) of Theorem 5). For example, if there is no backward contraction by dynamics of $\varphi_{\tau_j}^{-1}$ on $\varphi_{\tau_j}\gamma_j$, then if we cut off an ε-part of $\varphi_{\tau_j}\gamma_j$ its preimage in γ_j might not be small compare to length of γ_j. As we explain in more details below we need this smallness to estimate Hausdorff dimension of remaining points in $\gamma \supset \gamma_j$ after the surgery. The property of uniformness of distortion of a dynamical system is usually called:

3. *A bounded distortion property*: In the Model Example, Section 3.1 we have *uniform* backward contraction of intervals: at stage n (after time τn) by a factor 2^{-n}. So, when we cut off an ε-part of an interval at stage n, it corresponds to $2^{-n}\varepsilon$-part of the initial segment $I^{\emptyset}$. This remark makes an estimate of Hausdorff dimension of the set I_∞ or of the measure μ_∞ supported on I_∞ trivial, because the sets $\{I_n\}_{n\in\mathbb{Z}_+}$ have selfsimilar structure. Certainly, this is no longer true for the evolution of γ under the flow (1). Some parts of γ expanded by $\varphi_t|_\gamma$ expanded more than others. Condition (c) of Theorem 5 makes sure that there is a backward contraction in time and condition (d) of the same theorem says that rate of backward contraction Holder regularly depends on a point on a short interval γ_j. Thus, backward contraction is sufficiently uniform on γ_j's.

In the theory of deterministic dynamical systems with non-zero Lyapunov exponents the set of points satisfying uniform estimates for forward and backward expansion (as well as uniform estimates for angles between stable and unstable manifolds) are called *Pesin sets* and times when an orbit visits given Pesin set are called *hyperbolic times*. The existence of Pesin sets follows from abstract ergodic theory (see [P1]). Understanding the geometry of these sets in concrete examples is an important but often difficult task. In this paper we describe some properties of Pesin sets for stochastic flows. This description plays a key role in the proof of Theorem 3 and we also think it can be useful in many other questions about stochastic flows.

In particular let us mention that the estimates similar to ones given in Section 4 play important role in many other questions in the theory of deterministic systems such as periodic orbit estimates [K] and constructions of maximal measures [N], etc.

Our arguments in this paper are quite similar to [D1], [KM] even though the control of the geometry of images of curves is much more complicated in our case. Some interesting formulas for dimensions of nontypical points can be found in [BaSc]. We also refer the readers to the survey [Sz] and the book [P3] for more results about dimensions of dynamically defined sets.

The rest of the paper is organized as follows. In Section 4 in Theorem 5 we define a stopping time τ and prove that it has finite expectation and exponential moments. In Section 4.1 we investigate expansion properties of the flow (1) at the stopping time τ and complete the proof of Theorem 5. Recall that section 3.2 above was devoted to the construction of points with a nonzero drift. Namely, we need to construct a Cantor set I_∞ and a measure μ_∞ supported on I_∞ so that μ_∞-a.e. point has a nonzero drift. First, in Section 5 we present an algorithm of construction of a random Cantor set I inside the initial curve γ. Then, in Section 6 we define a probability measure μ supported on I with almost sure nonzero drift. Hausdorff dimension of such a measure is estimated in Section 7. Main Result (Theorem 4) is derived from Theorem 3 in Section 8. Auxiliary lemmas are in the Appendix at the end of the paper.

4. Hyperbolic moments. Control of the smoothness.

Introduce notations. Denote by φ_{t_1,t_2} a diffeomorphism of $\mathbb{T}^N$, obtained by solving (1) on the time interval $[t_1, t_2]$, and by φ_t the diffeomorphism $\varphi_{0,t}$. The flow (1) can also be thought as the product of independent diffeomorphisms $\{\varphi_{n,n+1} : \mathbb{T}^N \to \mathbb{T}^N\}_{n\in\mathbb{Z}_+}$.

Given positive numbers K and α we say that a curve γ is (K, α)-*smooth* if in the arclength parameterization the following inequality holds

$$\left|\frac{d\gamma}{ds}(s_1) - \frac{d\gamma}{ds}(s_2)\right| \le K\rho^\alpha(s_2, s_1) \text{ for each pair of points } s_1,\ s_2 \in \gamma.$$

In all the inequalities which appear below the distance ρ between the points on γ or its images $\varphi_t\gamma$'s is measured in the arclength metric induced on γ or $\varphi_t\gamma$ from the ambient space. In order to do not overload notations we omit dependence on γ or $\varphi_t\gamma$ when it is clear from the context which curve we use.

The goal of this section is to show that for a sufficiently small α and a sufficiently large K, starting with an arbitrary point x on a (K, α)-smooth curve γ, the part of image of this curve in a small neighborhood of the image of x is often smooth. More precisely, we prove the following statement. Let λ_1 be the largest Lyapunov exponent of the flow (1) which is positive see (2).

Theorem 5. *For any $0 < \lambda_1' < \lambda_1$ there exist sufficiently small $r > 0$, $\alpha \in (0,1)$, and sufficiently large $K > 0$ and $n_0 \in \mathbb{Z}_+$ with the following properties:*

For any (K,α)-smooth γ of length between $\frac{r}{100}$ and $100r$ and each point $x \in \gamma$ there is a stopping time $\tau = \tau(x)$, divisible by n_0, such that

(a) $\|d\varphi_\tau|T\gamma(x)\| > 100$ and length of the corresponding curve $l(\varphi_\tau\gamma) \geq r$;

Denote by $\bar{\gamma}_r$ a curve inside $\varphi_\tau\gamma$ of radius r with respect to induced in $\varphi_\tau\gamma$ length centered at $\varphi_\tau(x)$. Then

(b) $\bar{\gamma}_r$ is (K,α)-smooth

and for each pair of points $y_1, y_2 \in \bar{\gamma}_r$ the following holds

(c) for each integer $0 \leq k \leq \frac{\tau}{n_0}$ we have

$$\rho(\varphi_{\tau,\tau-kn_0}y_1, \varphi_{\tau,\tau-kn_0}y_2) \leq e^{-\lambda_1' kn_0}\rho(y_1,y_2);$$

(d) $|\ln \|d\varphi_\tau^{-1}|T\bar{\gamma}_r\|(y_1) - \ln \|d\varphi_\tau^{-1}|T\bar{\gamma}_r\|(y_2)| \leq \text{Const}\ \rho^\alpha(y_1,y_2)$;

Moreover, for such a stopping time $\tau(x)$ we have

(e) $\mathbb{E}\,\tau(x) \leq C_0$; $\mathbb{P}\{\tau(x) > T\} \leq C_1 e^{-C_2 T}$ for any $T > 0$,

All the above constants depend only on vector fields $\{X_k\}_{k=0}^d$ and λ_1, but independent of the curve γ.

Remark 4. Choosing integer n_0 is only for our convenience. Requirement that τ is divisible by n_0 will be used for construction of partition of γ in Section 5. This is also indication of flexibility in choice of both constants. The choice of constants 100, 1000, etc. in this paper is more or less arbitrary. Any constant greater than 1 would suffice.

Proof. The leading idea of the proof is that with probability close to 1 for a sufficiently large n_0 the diffeomorphism $\varphi_{t,t+n_0} : \mathbb{T}^N \to \mathbb{T}^N$ gets close to its asymptotic behavior. In particular, the norm of the linearization $\|d\varphi_{t,t+n_0}(x)\|$ as the matrix is $\sim \exp(\lambda_1 n_0 + o(n_0))$ as the top Lyapunov exponent predicts. Moreover, the linearization dominates higher order terms of $\varphi_{t,t+n_0}(x)$ and, therefore, determines local dynamics in a neighborhood of x. Thus, to some extend for large periods of time the flow (1) behaves similarly to uniformly hyperbolic system, for which properties of the Theorem are easy to verify. Now we start the proof.

First we construct a stopping time τ as a first moments satisfying a certain number of regularity inequalities (see (11)–(15)). This inequalities would include K, r, n_0 and some other parameters. Then we show that for any $\varepsilon > 0$ these parameters can be adjusted so that probability that the number of times each inequality is violated up to time T at least εT times decay exponentially in T. This would guarantee condition (e). Finally, we show that these inequalities imply conditions (a)–(d) and as the result prove the Theorem. In the Appendix we obtain large deviation estimates necessary for the proof below.

Our first goal is to control distortion of the unit tangent vector to images $\varphi_t\gamma$ of γ as time t evolves. Consider a collection of subsets of γ indexed by j

$$\mathcal{B}_{T,jn_0}(x)=\{y\in\gamma:\rho(\varphi_{n_0k}y,\varphi_{n_0k}x)\le re^{-\lambda_1'(T-n_0k)}\text{ for }0\le k\le j\},$$

where j varies from 1 to T/n_0. We would like to find an integer moment of time τ, divisible by n_0, such that

$$\varphi_{jn_0}\mathcal{B}_{\tau,\tau}(x)\text{ is }(Ke^{\epsilon(\tau-jn_0)},\alpha)-\text{smooth for all}\quad j=0,\dots,\frac{\tau}{n_0}. \tag{10}$$

The rest of this section is devoted to showing that the set of those T, divisible by n_0 for which (10) holds with $T=\tau$ has density close to 1 if K is sufficiently large. Given T denote by K_j the α-Holder norm of $\varphi_{jn_0}\mathcal{B}_{T,jn_0}(x)$. We would like to derive an inductive in j formula relating K_j and K_{j+1} so that in T/n_0 steps we get a required statement. Let z_1,z_2 be two points on $\varphi_{jn_0}\mathcal{B}_{T,jn_0}(x)$ and r is sufficiently small, then

$$\rho(\varphi_{jn_0,(j+1)n_0}z_1,\varphi_{jn_0,(j+1)n_0}z_2)\ge\frac{1}{2}\inf_{\varphi_{jn_0}\mathcal{B}_{T,jn_0}(x)}\|d\varphi_{jn_0,(j+1)n_0}|T\gamma\|\ \rho(z_1,z_2).$$

Let $d^2\varphi_{jn_0,(j+1)n_0}$ be the Hessian matrix consisting of second derivatives of the diffeomorphism $\varphi_{jn_0,(j+1)n_0}$. Assuming now that for each integer $j<\frac{T}{n_0}$ and some $R>0$ we have

$$\|d^2\varphi_{jn_0,(j+1)n_0}\|\le Re^{\epsilon(T-jn_0)} \tag{11}$$

and that condition (10) holds true up for each $j\le j^*$. Then we get

$$\rho(\varphi_{jn_0,(j+1)n_0}z_1,\varphi_{jn_0,(j+1)n_0}z_2)\ge \\ \frac{1}{2}\left(\|d\varphi_{jn_0,(j+1)n_0}|T\gamma\|(\varphi_{jn_0}x)-rRKe^{-(\lambda_1'n_0-2\epsilon)(T-j^*)}\right)\rho(z_1,z_2). \tag{12}$$

We would like to prove that (10) holds true for $j=j^*+1$. Assume also that for each $j<T/n_0$ we have

$$rRKe^{-(\lambda_1'n_0-2\epsilon)(T-jn_0)}\le\frac{\|d\varphi_{jn_0,(j+1)n_0}|T\gamma\|(\varphi_{jn_0}x)}{4} \tag{13}$$

then we get

$$\rho(\varphi_{j^*n_0,(j^*+1)n_0}z_1,\varphi_{j^*n_0,(j^*+1)n_0}z_2)\ge\frac{\|d\varphi_{j^*n_0,(j^*+1)n_0}|T\gamma\|(\varphi_{j^*n_0}x)}{4}\ \rho(z_1,z_2).$$

Let v_1 and v_2 be directions of the tangent vectors to $\varphi_{j^*n_0}\gamma$ at z_1 and z_2 respectively, then

$$\begin{aligned}\rho(d\varphi_{j^*n_0,(j^*+1)n_0}(z_1)v_1,d\varphi_{j^*n_0,(j^*+1)n_0}(z_2)v_2)\le\\ \rho(d\varphi_{j^*n_0,(j^*+1)n_0}(z_1)v_1,d\varphi_{j^*n_0,(j^*+1)n_0}(z_1)v_2)\\ +\rho(d\varphi_{j^*n_0,(j^*+1)n_0}(z_1)v_2,d\varphi_{j^*n_0,(j^*+1)n_0}(z_2)v_2).\end{aligned}$$

Denote the first and the second terms by I and $I\!I$ respectively. If (11) holds, then

$$I\!I \leq Re^{\epsilon(T-j^*n_0)}\rho(z_1, z_2).$$

Now since v_1 and v_2 are close $T(\varphi_{j^*n_0}\gamma)(x)$ and z_1 is close to $\varphi_{j^*n_0}x$ we have

$$\left[d\varphi_{j^*n_0,(j^*+1)n_0}(z_1)v_1 - d\varphi_{j^*n_0,(j^*+1)n_0}(z_1)v_2\right] \approx d_v d\varphi_{j^*n_0,(j^*+1)n_0}(x)(v_1 - v_2).$$

Let us give more precise estimates. Notice that if A is a linear map, then its action on the projective space satisfies

$$\|dA(v)\delta v\| = \frac{\|\Pi_{(Av)^\perp} A\delta v\|}{\|Av\|} \leq \frac{\|A\|}{\|Av\|},$$

where $\Pi_{(Av)^\perp}$ is the orthogonal projection onto the direction Av and δv is an element of $T_v T_x M$.

Therefore, we can assume that for a positive integer T/n_0 and each $j^* < T/n_0$ the following inequality is satisfied

$$\frac{\rho(Av_1, Av_2)}{\rho(v_1, v_2)} \leq 2\frac{\|d\varphi_{j^*n_0,(j^*+1)n_0}\|}{\|d\varphi_{j^*n_0,(j^*+1)n_0}|T\varphi_{j^*n_0}\gamma\|}, \tag{14}$$

for any linear map A such that $\|A - d\varphi_{j^*n_0,(j^*+1)n_0}\| \leq rRe^{(\lambda_1' n_0-\epsilon)(T-j^*n_0)}$ and for any pair (v_1, v_2) of tangent vectors such that $\rho(v_1, v_2) \leq Ke^{-(\lambda_1' n_0\alpha-\epsilon)(T-j^*n_0)}$.

Thus

$$I \leq 2K_{j^*}\ \rho^\alpha(z_1, z_2)\ \frac{\|d\varphi_{j^*n_0,(j^*+1)n_0}\|}{\|d\varphi_{j^*n_0,(j^*+1)n_0}|T\varphi_{j^*n_0}\gamma\|}.$$

Hence (11)–(14) imply that

$$K_{j^*+1} \leq \frac{8K_{j^*}\ \|d\varphi_{j^*n_0,(j^*+1)n_0}\|}{\|d\varphi_{j^*n_0,(j^*+1)n_0}|T\varphi_{j^*n_0}\gamma\|^{1+\alpha}} + \frac{2Re^{-(\lambda_1' n_0(1-\alpha)-\epsilon)(T-j^*n_0)}}{\|d\varphi_{j^*n_0,(j^*+1)n_0}|T\varphi_{j^*n_0}\gamma\|^\alpha}$$

If T is chosen so that

$$\|d\varphi_{j^*n_0,(j^*+1)n_0}|T\varphi_{j^*n_0}\gamma\| \geq \left(Re^{\epsilon(T-j^*n_0)}\right)^{-1}, \tag{15}$$

then the last inequality becomes

$$K_{j^*+1} \leq \frac{8K_{j^*}\|d\varphi_{j^*n_0,(j^*+1)n_0}\|}{\|d\varphi_{j^*n_0,(j^*+1)n_0}|T\varphi_{j^*n_0}\gamma\|^{1+\alpha}} + 2R^2e^{-(\lambda_1' n_0(1-\alpha)-2\epsilon)(T-j^*n_0)} \tag{16}$$

Let us summarize what we have learned so far.

Lemma 1. *For n_0 as above suppose that T is such that for every j such that $jn_0 \leq T$ estimates (11)–(15) hold true and also the solution of*

$$\bar{K}_{j+1} = \frac{4\bar{K}_j\|d\varphi_{jn_0,(j+1)n_0}\|}{\|d\varphi_{jn_0,(j+1)n_0}|T\varphi_{jn_0}\gamma\|^{1+\alpha}} + 2R^2, \quad \bar{K}_0 = \bar{K} \tag{17}$$

satisfies

$$\bar{K}_j \leq \bar{K}e^{(T-jn_0)\epsilon} \tag{18}$$

then inequality (10) holds.

Now we want to show that the set of points where either (11)–(15) or (18) fail has density less than εT except on a set of exponentially small probability. The result for (18) follows from Proposition 8 applied to $\ln \bar{K}_j$. To see that the conditions of this proposition are satisfied if α is sufficiently small it is enough to verify that $\ln \bar{K}_j$ has uniform drift to the left.

By Carverhill's extension of Oseledets' Theorem [Cv] for every point x on M and every unit vector v in T_xM

$$\frac{1}{n_0}\, \mathbb{E}\, \ln \|d\varphi_{n_0}(x)v\| \to \lambda_1 \tag{19}$$

uniformly as $n_0 \to \infty$ and [BS] provides exponential estimate for probabilities of large deviations. Since

$$\|d\varphi_{n_0}(x)\| \leq \sum_{j=1}^{N} \|d\varphi_{n_0}(x)v_j\|$$

where $\{v_j\}_{j=1}^N$ is any orthonormal frame, the above mentioned results of [BS] imply that

$$\frac{1}{n_0}\, \mathbb{E}\, \ln |\|d\varphi_{n_0}(x)\| \to \lambda_1 \text{ as } \quad n_0 \to \infty$$

with exponential bound for large deviations. Thus Proposition 8 from the Appendix applies to $\ln \bar{K}_j$ for a large enough n_0.

The fact that (11)–(15) fail rarely if n_0 is sufficiently large and r is sufficiently small follows from Lemma 10.

4.1. **Hyperbolic moments. Control of expansion.** We now define the stopping time τ as the first moment when (10), (11), and (15) are satisfied as well as

$$\|d\varphi_\tau|\ T\gamma\|(x) \geq 1000 \tag{20}$$

and for each positive integer $j \leq \tau/n_0$ and some constant $0 < \tilde{\lambda}_1 < \lambda_1$ we have

$$\|d\varphi_{\tau,\tau-jn_0}|\ T\varphi_\tau\gamma\| \leq e^{-\tilde{\lambda}_1 j n_0}. \tag{21}$$

Then the large deviations estimates of [BS] guarantee that property (20) has density close to 1.

Lemma 2. *For any $\varepsilon > 0$ and any $0 < \tilde{\lambda}_1 < \lambda_1$ there exists a positive integer n_0 such that with probability exponentially approaching to 1 the fraction of integers τ, divisible by n_0, with the linearization $d\varphi_{\tau,\tau-jn_0}|T\varphi_\tau\gamma$ contracting* *exponentially backward in time for all integer j between 0 and τ/n_0 tends to 1. More precisely,*

$$\mathbb{P}\left\{\frac{\#\{S \le L : \forall 0 \le j \le S,\ \tau = Sn_0 \quad \|d\varphi_{\tau,\tau-jn_0}|T\varphi_\tau\gamma\| \le e^{-\tilde{\lambda}_1 jn_0}\}}{L} \le 1-\varepsilon\right\}$$

decays exponentially in L.

Proof. We first show how to prove a weaker statement with "$\exists\varepsilon$" instead of "$\forall\varepsilon$" (which is enough to prove Theorem 5) and then explain briefly the changes needed to prove the sharp result.

Let τ_1 be the first moment such that for each integer $j \le \frac{\tau}{n_0}$

$$\|d\varphi_{\tau_1,\tau_1-jn_0}|T\varphi_\tau\gamma\| \le e^{-\tilde{\lambda}_1 jn_0}. \tag{22}$$

We claim that τ_1 has exponential tail. Indeed, let

$$Y_j = Y_j(\theta) = \Big(\|d\varphi_{jn_0}|T\gamma\|(x)\ e^{-(\tilde{\lambda}_1+\varepsilon)jn_0}\Big)^\theta, \quad Y_0 = 1,$$
$$\text{and} \quad Z_j = \|d\varphi_{jn_0}|T\gamma\|(x)\ e^{-\tilde{\lambda}_1 jn_0}, \quad Z_0 = 1.$$

Then [BS] shows that if n_0 is sufficiently large and ε, θ are sufficiently small, then Y_j is a submartingale. Thus the first moment $\hat{j}$ such that $Z_{\hat{j}} > 10$ has exponential tail. But there is at least one maximum $\bar{j}$ of Z_j between 0 and $\hat{j}$. Then $\bar{j}$ satisfies (22).

Now define τ_k inductively so that $\tau_{k+1} > \tau_k$ is the first moment such that for every $j \le \frac{\tau_{k+1}-\tau_k}{n_0}$

$$\|d\varphi_{\tau_{k+1},\tau_{k+1}-jn_0}|T\varphi_{\tau_{k+1}}\gamma\| \le e^{-\tilde{\lambda}_1 n_0 j}.$$

Then $\tau_{k+1}-\tau_k$ have exponential tails, so by Lemma 9 there exists c such that $\mathbb{P}\{\frac{\tau_k}{k} \ge C\}$ decays exponentially in k. However all τ_k satisfy (22). This proves the result with $\varepsilon = 1 - \frac{1}{C}$. To get the optimal result one should note that $\mathbb{P}\{\tau_1 = n_0\} \to 1$ as $n_0 \to \infty$ and apply the arguments of Lemma 9. We leave the details to the reader. □

Now we want to verify conditions (b), (c), and (d) of Theorem 5 with $\bar{\gamma}_r$ replaced by $\hat{\gamma} = \varphi_\tau B_{\tau,\tau}(x)$. Once we prove this we get from (c) that the main restriction on $B_{\tau,\tau}(x)$ is for $k = \tau$ so that $\bar{\gamma}_r = \hat{\gamma}$ and then (a) will also be true. Now (b) is true by Lemma 1. We will establish (c) and (d) by induction.

Namely we suppose that (c) is true for $k \geq k_0$. Then for every $y \in \hat\gamma$

$$\left|\ln \|d\varphi_{\tau,\tau-(k_0+1)n_0}|T\hat\gamma\|(x) - \ln \|d\varphi_{\tau,\tau-(k_0+1)n_0}|T\hat\gamma\|(y)\right| \leq$$

$$\sum_{m=0}^{k_0} \left|\ln \|d\varphi_{\tau-mn_0,\tau-(m+1)n_0}|T\hat\gamma\|(x) - \ln \|d\varphi_{\tau-mn_0,\tau-(m+1)n_0}|T\hat\gamma\|(y)\right| \leq$$

$$\sum_{m=0}^{k_0} \left|\ln \|d\varphi_{\tau-mn_0,\tau-(m+1)n_0}|T\hat\gamma\|(x) - \ln \|d\varphi_{\tau-mn_0,\tau-(m+1)n_0}|T\hat\gamma\|(y)\right| +$$

$$\sum_{m=0}^{k_0} \left|\ln \|d\varphi_{\tau-mn_0,\tau-(m+1)n_0}|T\hat\gamma\|(y) - \ln \|d\varphi_{\tau-mn_0,\tau-(m+1)n_0}|T\hat\gamma\|(y)\right|.$$

Denote the left term by I and the right term by $I\!I$ respectively. Now by (10) and (11)

$$I \leq \sum_{m=0}^{k_0} Re^{\epsilon m} K e^{\epsilon m} \rho^\alpha(\varphi_{\tau,\tau-mn_0}x, \varphi_{\tau,\tau-mn_0}y) \leq \sum_{m=0}^{k_0} KRe^{-(\lambda_1'\alpha n_0 - 2\epsilon)m} \rho^\alpha(x,y) \leq \text{Const } r^\alpha. \tag{23}$$

On the other hand

$$I\!I \leq \sum_{m=0}^{k_0} \frac{\|d^2\varphi_{\tau-mn_0,\tau-(m+1)n_0}\|}{\|d\varphi_{\tau-mn_0,\tau-(m+1)n_0}\|} \rho(\varphi_{\tau,\tau-mn_0}x, \varphi_{\tau,\tau-mn_0}y) \leq \sum_{m=0}^{k_0} R^2 e^{-(\lambda_1' n_0 - 2\epsilon)m} \rho(x,y) \leq \text{Const } r. \tag{24}$$

Hence (11) and (15)

$$\left|\ln \|d\varphi_{\tau,\tau-(k_0+1)n_0}|T\hat\gamma\|(x) - \ln \|d\varphi_{\tau,\tau-(k_0+1)n_0}|T\hat\gamma\|(y)\right| \leq C(R)\rho(y_1,y_2). \tag{25}$$

Thus for all y

$$\|d\varphi_{\tau-(k_0+1)n_0,\tau}|T\varphi_{\tau-(k_0+1)n_0}\gamma\|(y) \geq \exp\left(\tilde\lambda_1 k n_0 - C(R)r\right) \geq \exp\left(\lambda_1' k n_0\right) \tag{26}$$

if $\tilde\lambda_1 - \lambda_1' \geq C(R)r$. (26) implies that (c) is valid for $k_0 - 1$. Thus, we obtain (c) for all k. Now repeating the proof of (25) with x and y replaced by y_1 and y_2 (and using (26) instead of (21)) we obtain (d). This completes the proof of Theorem 5. □

Remark 5. The term *hyperbolic time* was introduced in [A] but the notion itself was used before, e.g. in [P1, P2, J, Y]. Considerations of this section are similar to [ABV, D2] but the additional difficulty is that in those papers the analogue of (10) was true by the general theory of partially hyperbolic systems

[HPS] whereas here additional arguments in spirit of [P1, P2] were needed to establish it.

One interesting question is how large can α be so that Theorem 5 still holds. We note that α appears in (16) twice. So we want α to be as large as possible to control the first part and we want α to be small to control the second term. In general, the optimal choice of α should depend on the ratio of leading exponents. We refer to [CL, L, PSW, JPL] for the discussion of this question.

5. Construction of the partition.

We are now ready to describe a partition $\gamma = \bigcup_{j\in\mathbb{Z}_+} \gamma(j)$. It will be defined inductively. Each of $\gamma(j)$'s is a finite union of intervals. As j tends to infinity size of intervals tends to zero and they fill up γ. To simplify the notation we assume that Theorem 5 is true with $n_0 = 1$. This can be achieved by rescaling the time. Fix an orientation from left to right on γ.

Suppose $\gamma(1), \gamma(2), \dots, \gamma(m)$ are already defined in an $\mathcal{F}_m$-measurable way. Let

$$K_{m+1} = \{x \in \gamma : \ \tau(x) = m+1\}. \tag{27}$$

By definition K_{m+1} is a finite union of intervals. Let $U_{m+1} = \varphi_{m+1} K_{m+1}$. We call *an obstacle* any point on the boundary of either K_{m+1}, $\bigcup_{j=1}^m \gamma(j)$ or γ. Fix r satisfying Theorem 5. Let C be a connected component of U_{m+1} and a and b be its left and right endpoints with respect to left-right orientation induced by φ_{m+1}. If distance from b to the closest image of an obstacle to the right on $\varphi_{m+1}(\gamma)$ is less than $\frac{r}{2}$ and b' is this image, then put $\tilde{b} = b'$. Otherwise let $\tilde{b}$ be a point at distance $\frac{r}{100}$ from b. Define $\tilde{a}$ similarly. Consider the set $W_{m+1} = \bigcup_C \tilde{a}\tilde{b}$. Divide W_{m+1} into the segments of lengths between $\frac{r}{100}$ and $\frac{r}{50}$ and denote this partition by V_{m+1}. Now we define partition of a subset of $\gamma \setminus \bigcup_{j=1}^m \gamma(j)$ by pulling back along φ_{m+1}^{-1} the partition V_{m+1}

$$\gamma(m+1) = \varphi_{m+1}^{-1} V_{m+1}. \tag{28}$$

To justify that this algorithm produces a partition which covers all of K_{m+1} we need to check that length of each component is at least $\frac{r}{100}$. To do this we argue by contradiction. Otherwise, there would be two obstacles x', x'' neither of which is from K_{m+1} such that $\rho(\varphi_{m+1}x', \varphi_{m+1}x'') \le \frac{r}{100}$ and a point from U_{m+1} between them. At least one of the obstacles would have to come from $\bigcup_{j=1}^m \gamma(j)$. Let x' be such an obstacle. Since both points are close to U_m for each $n \le m+1$ we have

$$\rho(\varphi_n x', \varphi_n x'') \le \frac{r}{100}\, e^{-\lambda_1'(m-n)}.$$

But in this case the interval $[x', x'']$ in γ with endpoints x' and x'' would be added to our partition at a previous step of the algorithm.

Denote by $\gamma = \bigcup_{j\in\mathbb{Z}_+} \gamma_j$ the partition which is made out of the partition $\gamma = \bigcup_{j\in\mathbb{Z}_+} \gamma(j)$ by renumerating intervals of this partition in length decreasing order. Let us summarize the outcome.

Proposition 6. *We can partition $\gamma = \bigcup_{j\in\mathbb{Z}_+} \gamma_j$ in such a way that*
(a) there exists a positive integer n_j such that $\|d\varphi_{n_j}|T\gamma\| \geq 100$ and length $l(\varphi_{n_j}\gamma_j) \geq \frac{r}{100}$;
(b) for each positive integer $m \leq n_j$ and lengths of the corresponding curves we have $l(\varphi_m\gamma_j) \leq l(\varphi_{n_j}\gamma_j)e^{-\lambda_1'(n_j-m)}$;
(c) $\big|\ln\|d\varphi_{n_j}|T\gamma\|(x') - \ln\|d\varphi_{n_j}|T\gamma\|(x'')\big| \leq$ Const $\rho^\alpha(\varphi_{n_j}x', \varphi_{n_j}x'')$ for every pair $x', x'' \in \gamma_j$;
(d) for some $\alpha > 0$ and each pair $x', x'' \in \gamma_j$ we have $|v(x', n_j) - v(x'', n_j)| \leq$ Const $\rho^\alpha(\varphi_{n_j}x', \varphi_{n_j}x'')$, where $v(x, n)$ denote the unit tangent vector to $\varphi_n\gamma$ at $\varphi_n(x)$;
(e) Let $j(x)$ be such that $x \in \gamma_{j(x)}$. Then $\mathbb{E}\ n_{j(x)} \leq$ Const and $\mathbb{P}\{n_{j(x)} > T\} \leq C_1e^{-C_2T}$ for some positive C_1, C_2 and any $T > 0$;

This Proposition is designed to allow application of Theorem 5 so that we can use regularity and geometric properties of γ_j's at stopping times τ_j's.

6. Construction of the measure with almost sure nonzero drift.

Now we construct a random Cantor set $I \subset \gamma$ and a probability measure μ supported on I such that μ–almost all points have a nonzero drift. This construction goes along the same line with the construction in Section 3.2 of the Cantor set I_∞ in the unit interval and a probability measure μ_∞ on I such that μ_∞–almost all points have nonzero drift.

Choose a direction $\vec{e} \in \mathbb{R}^N$. Let θ be a small parameter which we let to zero in the next section. We say that a curve is *$\vec{e}$-monotone* if its projection to $\vec{e}$ is monotone. Now we describe construction of a Cantor set $I \subset \gamma$ and a probability measure μ on I by induction. This Cantor set I at k-th step of induction consists of countable number of segments numerated by k-tuples of positive integers.

Denote k-tuples $(j_1, \cdots, j_k) \in \mathbb{Z}_+^k$ and $(n_1, \cdots, n_k) \in \mathbb{Z}_+^k$ by J_k and N_k respectively. Let $|N_k| = \sum_{j=1}^k n_j$.

The first step of induction goes as follows. Let γ_j, n_j be the sequence of pairs: a curve and an integer, described in Proposition 6. Let θ be a small positive number. If $\varphi_{n_j}\gamma_j$ is $\vec{e}$-monotone put $\sigma(j)$ equal $\varphi_{n_j}\gamma_j$ without the segment of length θr, which we cut off from the $\vec{e}$-bottom point of $\varphi_{n_j}\gamma_j$. Otherwise $\sigma(j) = \varphi_{n_j}\gamma_j$ with no cut off. Let $\gamma(J_1) = \varphi_{n_j}^{-1}\sigma(j)$ and $N_1(J_1) = n_j$ for $J_1 = j$

Suppose a collection of disjoint segments $\{\gamma(J_k)\}_{J_k\in\mathbb{Z}_+^k} \subset \gamma$ is defined as above and multiindices N_k (resp. J_k) are defined as the corresponding set of hyperbolic times multiindexed by J_k segments. Then

$$I_k = \cup_{J_k \in \mathbb{Z}_+^k} \gamma(J_k) \subset I_{k-1} \subset \cdots \subset I_1 \subset \gamma \tag{29}$$

is the k-th order of construction of the random Cantor set I (cf. with an open set I_k from Section 3.2).

The $(k+1)$-st step goes as follows. Pick a segment $\gamma(J_k)$ of partition (29). Consider the partition of the curve

$$\varphi_{|N_{J_k}|}\gamma(J_k) = \bigcup_{j_{k+1} \in \mathbb{Z}_+} \tilde{\gamma}(J_k, j_{k+1}) \tag{30}$$

defined in Section 5 and let $n_{J_k,j_{k+1}}$ be the corresponding hyperbolic times for $\tilde{\gamma}(J_k, j_{k+1})$ from Proposition 6. For brevity denote $|N_k(J_k)|$ by $n^{(k)}$ and $|N_k(J_k)| + n_{(J_k,j_{k+1})}$ by $n^{(k+1)}$. If the curve $\varphi_{n^{(k)},n^{(k+1)}}\tilde{\gamma}_{(J_k,j_{k+1})}$ is $\vec{e}$-monotone we let $\sigma(J_k, j_{k+1})$ be $\varphi_{n^{(k)},n^{(k+1)}}\tilde{\gamma}_{(J_k,j_{k+1})}$ with cut off of the segment of length θ starting from the $\vec{e}$-bottom. Otherwise, $\sigma(J_k, j_{k+1})$ equal $\varphi_{n^{(k)},n^{(k+1)}}\tilde{\gamma}(J_k, j_{k+1})$ with no cut off. Then a segment

$$\gamma(J_k, j_{k+1}) = \varphi^{-1}_{n^{(k+1)}}\sigma(J_k, j_{k+1}) \tag{31}$$

with $j_{k+1} \in Z_+$ this defines the $(k+1)$-st order partition $\{\gamma(J_{k+1})\}_{J_{k+1} \in \mathbb{Z}_+^{k+1}} \subset \gamma$ and the k-order set $I_{k+1} = \cup_{J_{k+1} \in \mathbb{Z}_+^{k+1}} \gamma(J_{k+1}) \subset \gamma$.

We now describe a sequence of measures μ_k's on $I_k \subset \gamma$ with $k \in \mathbb{Z}_+$ respectively. Let μ_0 be the arclength on γ. Suppose μ_k is already defined on I_k. Consider $\{\gamma(J_{k+1})\}_{J_{k+1} \in \mathbb{Z}_+^{k+1}}$. If $\varphi_{n^{(k+1)}}\gamma(J_{k+1})$ is *not* $\vec{e}$-monotone we let $\mu_{k+1}|_{\gamma(J_{k+1})} = \mu_k|_{\gamma(J_{k+1})}$. Otherwise, $\mu_{k+1}|_{\gamma(J_{k+1})} = \rho_{jk}\mu_k|_{\gamma(J_{k+1})}$, where ρ_{jk} is a normalizing constant.

Lemma 3. *Let k be an integer. If r is sufficiently small and $\gamma \subset \mathbb{R}^N$ is (K, α)-smooth as in Theorem 5, then if we consider partition of γ up to order $k+1$, then for each multiindex $J_k \in \mathbb{Z}_+^k$ the corresponding k-th order curve $\gamma(J_k) \subset \gamma$ satisfy the property: for any positive integer j_{k+1} the $(k+1)$-st order curve $\gamma(J_{k+1}) \subset \gamma(J_k)$ has $\vec{e}$-monotone with positive probability, i.e.*

$$\mathbb{P}\{\varphi_{n^{(k+1)}}\gamma(J_{k+1}) \ \ is \ \ \vec{e} - monotone \mid \ \mathcal{F}_{n^{(k)},n^{(k+1)}}\} > c$$

for some positive c and c is uniform for all (K, α)-smooth curves.

Proof. Pick a point $x \in \gamma(J_{k+1})$. By assumption (D) of hypoellipticity on the unit tangent bundle SM for the flow (1) probability that the angle between $\vec{e}$ and $T\varphi_{n^{k+1}}\gamma(x)$ makes less than $1°$ is positive. By definition $\varphi_{n^{(k+1)}}\gamma(J_{k+1})$ is (K, α)-smooth. Thus if r is small enough, then the tangent vectors to $\varphi_{n^{k+1}}\gamma$ are close to $T\varphi_{n^{k+1}}\gamma(x)$ with large probability, where x is a point on $\gamma(J_{k+1})$. This completes the proof. $\square$

Recall that $\theta > 0$ is a fraction of $\varphi_{n_j}\gamma_j$ we cut off from $\varphi_{n_j}\gamma_j$ on the j-th step, provided $\varphi_{n_j}\gamma_j$ is $\vec{e}$-monotone. Let $\mu = \mu(\theta)$ denote the weak limit of μ_k's

$$\mu = \lim_{k\to\infty} \mu_k.$$

Lemma 4. *For almost every realization of the Brownian motion* $\{\theta(t)\}_{t\geq 0}$ *and* μ*–almost every* x

$$\liminf_{t\to\infty} \frac{\langle x_t, e\rangle}{t} > 0.$$

Proof. The first step is to show that for any s for almost all realizations of the Brownian motion $\{\theta(t)\}_{t\geq 0}$

$$\liminf_{t\to\infty} \frac{\langle x^{(k)}_{n^{(k)}}, e\rangle}{n^{(k)}} > 0 \tag{32}$$

Applying Proposition 6 (e) and Lemma 9 we get that there exists a constant $C > 0$ such that

$$\limsup_{k\to\infty} \frac{n^{(k)}}{k} < C$$

almost surely. Therefore, to prove (32) it suffices to show that

$$\liminf_{k\to\infty} \frac{\langle x^{(k)}_{n^{(k)}}, e\rangle}{k} > 0 \tag{33}$$

However by Lemma 3 there exists c such that $\mathbb{E}\langle x^{(k+1)}_{n^{(k+1)}} - x^{(k)}_{n^{(k)}}, e\rangle > c$ uniformly in k, s. (This is because $\mathbb{E}\langle x^{(k)}_{n^{(k+1)}} - x^{(k)}_{n^{(k)}}, e\rangle = 0$ and

$$\langle x^{(k)}_{n^{(k+1)}}, e\rangle - \langle x^{(k)}_{n^{(k)}}, e\rangle \ \geq 0$$

with strict inequality having positive probability by Lemma 3.) Hence (33) follows by Lemma 9. Therefore (32) is established. □

Now we apply the following estimate.

Lemma 5. *(*[CSS2]*, Theorem 1) Let*

$$\Phi_{s,t} = \sup_{s\leq\tau\leq t} |x_\tau - x_s|, \quad \tilde{\Phi}_{s,t} = \frac{\Phi_{s,t}}{\max(1, t-s)}$$

then there exists a constant C *such that for all* s *and* t

$$\mathbb{E}\left(\exp\left\{\frac{\tilde{\Phi}^2_{s,t}}{\max(1, \ln^3 \tilde{\Phi}_{s,t})}\right\}\right) < C.$$

Combining this lemma with Proposition 6 (e) we obtain that there are positive constants α and D such that

$$\mathbb{E}\left(\exp\left\{\alpha \sup_{n^{(k)}<\tau<n^{(k+1)}} |x_\tau(s) - x_{n^{(k)}}|\right\}\right) < D.$$

Using Borel-Cantelli's lemma we derive from this that almost surely

$$\limsup_{k\to\infty} \frac{\sup_{n^{(k)}<\tau<n^{(k+1)}} |x_\tau(s) - x_{n^{(k)}}|}{\ln k} < +\infty.$$

Therefore for any s and for almost all realizations of the Brownian motion $\{\theta(t)\}_{t\geq 0}$ we have

$$\liminf_{\tau\to\infty} \frac{\langle x_\tau(s), e\rangle}{\tau} > 0.$$

By Fubini Theorem we have that for almost every realization of the Brownian motion $\{\theta(t)\}_{t\geq 0}$ the set

$$\left\{ s: \quad \liminf_{\tau\to\infty} \frac{\langle x_\tau(s), e\rangle}{\tau} > 0 \right\}$$

has full measure. □

7. Hausdorff dimension of μ.

In this section we complete the proof of Theorem 3 by establishing the following fact. Recall that $\theta > 0$ is a fraction of $\varphi_{n_j}\gamma_j$ we cut off from $\varphi_{n_j}\gamma_j$ on the j-th step, provided $\varphi_{n_j}\gamma_j$ is $\vec{e}$-monotone. Consider the measure μ we constructed in the previous Section.

Proposition 7. *With notations above we have that as $\theta \to 0$ Hausdorff dimension of the measure $\mu = \mu(\theta)$ tends to 1: $\mathrm{HD}(\mu(\theta)) \to 1$.*

Let us recall the following standard principle.

Lemma 6 (Mass distribution principle)**.** *Let S be a compact subset of a Euclidean (or metric) space such that there exists a probability measure ν such that $\nu(S) = 1$ and for each x we have $\nu(\mathcal{B}(x, r)) \leq Cr^s$ for some positive C and s. Then $\mathrm{HD}(S) \geq s$.*

Proposition 7 is a direct consequence of the following statements.

Lemma 7. *Let γ be a smooth curve in $\mathbb{R}^N$. Suppose there exist a nested sequence of partitions*

$$\gamma \supset \bigcup_{J_1\in\mathbb{Z}^1_+} \gamma(J_1) \supset \dots \supset \bigcup_{J_k\in\mathbb{Z}^k_+} \gamma(J_k) \supset \dots \tag{34}$$

and probability measures $\mu_0, \mu_1, \dots, \mu_k, \dots$ supported on $\gamma, \bigcup_{J_1\in\mathbb{Z}^1_+} \gamma(J_1), \dots, \bigcup_{J_k\in\mathbb{Z}^k_+} \gamma(J_k), \dots$ respectively such that μ_0 is the normalized arclength on γ and so on μ_k is the normalized arclength on $\bigcup_{J_k\in\mathbb{Z}^k_+} \gamma(J_k)$. Then if we have

(b) for all $J_k \in \mathbb{Z}^k_+$ length of the corresponding interval $\gamma(J_k)$ is bounded by $l(\gamma(J_k)) \leq 100^{-k}$;

(b) for each $l > k$ we have $\mu_l(\gamma(J_k)) = \mu_k(\gamma(J_k))$;

(c) $\frac{d\mu_{k+1}}{d\mu_k}(x) \le \rho_k(x)$ *for every point* $x \in \bigcup_{J_k \in \mathbb{Z}_+^k} \gamma(J_k)$*, where* $\rho_k < 1+\delta$*.*

Let $\mu = \lim_{k\to\infty} \mu_k$ *in the sense of weak limit. Then* $\mathrm{HD}(\mu) \ge d(\delta)$*, where* $d(\delta) \to 1$ *as* $\delta \to 0$*.*

Lemma 8. *For each* $\delta > 0$ *there exists* $\theta > 0$ *such that the densities of* $\frac{d\mu_{k+1}}{d\mu_k}(x)$ *used to define measures* μ_{k+1} *knowing* μ_k *satisfy condition (c) of Lemma 7.*

Proof of Lemma 7. We prove that for any segment I we have $\mu(I) \le \mathrm{Const}\, |I|^{1-\beta}$, where $\beta \to 0$ as $\delta \to 0$. Let $k(I) = \frac{|\ln |I||}{\ln 100}$ and a and b be the endpoints of I. Let $\tilde{a}$ be the left endpoint of the k-th partition containing a and $\tilde{b}$ be the right endpoint of the k-th partition containing b. Then

$$\mu(I) \le \mu([\tilde{a},\tilde{b}]) = \mu_k([\tilde{a},\tilde{b}]) \le (1+\delta)^k \mu_0([\tilde{a},\tilde{b}]) \le 3(1+\delta)^k |I| \le 3|I|^{1-\beta} \tag{35}$$

where $\beta = \frac{\ln(1+\delta)}{\ln 100}$. Thus, $\beta \to 0$ as $\delta \to 0$. Application of the mass distribution principle implies that $\mathrm{HD}(\mu) \ge 1-\beta$. □

Proof of Lemma 8. Recall the notation of Section 6. We need to show that

$$\sup_{J_{k+1}} \frac{|\varphi^{-1}_{n^{(k+1)}} \sigma(J_{k+1})|}{|\varphi^{-1}_{n^{(k)}} \gamma(J_{k+1})|} \to 1, \quad \theta \to 0 \tag{36}$$

By construction

$$\frac{|\varphi_{n^{(k+1)},n^{(k)}} \sigma(J_{k+1})|}{|\gamma(J_{k+1})|} \ge 1 - \left(\frac{\theta}{r/100}\right).$$

Hence to prove (36) it is enough to show that there is a constant C independent of j, k, l such that for any interval $I \subset \gamma(J_{k+1})$

$$\frac{|\varphi^{-1}_{n_j^{(k)}} I|}{|\varphi^{-1}_{n_j^{(k)}} \gamma(J_{k+1})|} \le C \frac{|I|}{|\gamma(J_{k+1})|}.$$

To do so it is enough to show that there is a constant $\bar{C}$ such that for every pair $y_1, y_2 \in \gamma(J_{k+1})$

$$\frac{\|d\varphi^{-1}_{n_j^{(k)}} | T\gamma(J_{k+1})\|(y_1)}{\|d\varphi^{-1}_{n_j^{(k)}} | T\gamma(J_{k+1})\|(y_2)} \le \bar{C}.$$

But by Proposition 6 there are constants C_1, C_2, and C_3 such that

$$\left|\ln \|d\varphi^{-1}_{n_j^{(k)}}|T\gamma(J_{k+1})\|(y_1) - \ln \|d\varphi^{-1}_{n_j^{(k)}}|T\gamma(J_{k+1})\|(y_2)\right| \leq$$

$$\sum_{m=1}^{k} \Big|\ln \|d\varphi_{n^{(m)},n^{(m-1)}}|\ T\varphi_{n^{(m)}}\|(\varphi_{n^{(k)},n^{(m)}} y_1)$$

$$- \ln \|d\varphi_{n^{(m)},n^{(m-1)}}|T\ \varphi_{n^{(m)}}\|(\varphi_{n^{(k)},n^{(m)}}(y_2))\Big| \leq$$

$$C_1 \sum_m \rho^\alpha(\varphi_{n^{(k)},n^{(m)}} y_1, \varphi_{n^{(k)},n^{(m)}}(y_2)) \leq C_2 \sum_m 100^{(m-k)\alpha} \rho^\alpha(y_1, y_2) \leq C_3\ r^\alpha.$$

This completes the proof. □

8. Proof of Theorem 4.

Let $\mathcal{G}$ denote the foliation of $\mathbb{T}^N$ by curves

$$\{x^1 = c^1, x^2 = c^2 \ldots x^{N-1} = c^{N-1}\}.$$

By (35) for each $\beta > 0$ and each leaf γ_c of $\mathcal{G}$ almost surely there exists a measure μ_c on γ_c such that $\mu_c(I) \leq 3|I|^{1-\beta}$ and $\mu_c(\mathbf{L}_\theta) = 1$. Let $\mu = \int \mu_c dc$. Then by Fubini Theorem almost surely for any cube $\mathcal{C}$ of side r we have $\mu(\mathcal{C}) \leq 3r^{N-\beta}$ and $\mu(\mathbf{L}_\theta) = 1$. The application of the mass distribution principle completes the proof. □

Appendix A. Large deviations.

Here we collect some estimates used throughout the proof of Theorem 3.

Lemma 9. *Let $\mathcal{F}_j$ be a filtration of σ-algebras and $\{\xi_j,\}$ be a sequence of $\mathcal{F}_j$-measurable random variables such that*
(a) there exist C_1, λ such that for every $|s| \leq \lambda$ we have $\mathbb{E}(e^{s\xi_{j+1}}|\mathcal{F}_j) \leq C_1$;
(b) there exists C_2 such that $\mathbb{E}(\xi_{j+1}|\mathcal{F}_j) \leq C_2$.

Then for each $\epsilon > 0$ the probability

$$\mathbb{P}\left\{\sum_{j=0}^{N-1} \xi_j \geq (C_2 + \epsilon)N\right\}$$

decays exponentially in N.

Proof. Consider

$$\Phi_n(s) = \exp\left\{\left(\sum_{j=0}^{n-1} \xi_j - \left(C_2 + \frac{\epsilon}{2}\right) n\right) s\right\}.$$

Then (a) and (b) imply that $\Phi_n(s)$ is a supermartingale if s is sufficiently small. Hence $\mathbb{E}\Phi_n(s) \leq \mathbb{E}\Phi_0(s) = 1$, and so

$$\mathbb{E}\ \exp\left\{\left(\sum_{j=0}^{n-1}\xi_j - (C_2+\epsilon)n\right)s\right\} \leq \exp\left(-\frac{n\epsilon s}{2}\right)\ ,$$

which proves the lemma. □

Lemma 10. *Let $\mathcal{F}_j$ be a filtration of σ-algebras and $\{\xi_j,\}$ be a sequence of $\mathcal{F}_j$-measurable random variables such that there exists constant C_1 such that*

$$\mathbb{E}(\xi_{j+1}|\mathcal{F}_j) \leq C \tag{37}$$

then for every $\epsilon, \varepsilon > 0$ there is $R > 0$ such that

$$\mathbb{P}\left\{\frac{\#\{n \leq N : \xi_j \leq Re^{\epsilon(n-j)} \text{ for all } 0 \leq j < n\}}{N} \leq 1-\varepsilon\right\}$$

tends to zero exponentially fast in N.

Proof. We say that a pair (j,n) is R-bad if

$$\xi_j > Re^{\epsilon(n-j)}.$$

By (37)

$$\mathbb{P}\{(j,n) \text{ is } R\text{-bad}\} \leq \frac{C}{R}e^{-\epsilon(n-j)}. \tag{38}$$

Now given k let $B_R(k)$ be the number of $n > k$ such that (k,n) is R-bad. By (38)

$$\mathbb{E}(B_R(k+1)|\mathcal{F}_k) \leq \frac{Ce^{-\epsilon}}{R(1-e^{-\epsilon})} \to 0$$

as $R \to \infty$. Thus by Lemma 9 there exists R such that

$$\mathbb{P}\left\{\sum_{k=1}^{N} B_R(k) \geq \varepsilon N\right\}$$

decays exponentially in N. This completes the proof of the lemma. □

Proposition 8. *Let x_j be (in general, non-homogeneous) random walk on $\mathbb{Z}$. Suppose that there exist constants C_1, C_2, C_3 such that*
(a) there exist m such that for every $x_j > m$ we have $\mathbb{E}(x_{j+1}-x_j|x_j) \leq -C_1$;
(b) for every x_j and every $\zeta < C_2$ we have $\mathbb{E}(e^{\zeta(x_{j+1}-x_j)}|x_j) \leq C_3$.

Fix $\delta > 0$. Let $F(M)$ denote the set of j such that for all $k < j$

$$x_k \leq \max(x_j, m) + M + \delta(j-k).$$

Then for every $\varepsilon > 0$ there exists $M > 0$ such that

$$\mathbb{P}\left\{\frac{\#\{F(M)\bigcap[1,N]\}}{N} \leq 1-\varepsilon\right\}$$

decays exponentially in N.

Proof. Let $\tau_1 < \tau_2 \cdots < \tau_k < \dots$ be the consecutive returns of x_j to $\{x \leq m\}$. Let

$$t_j = \tau_{j+1} - \tau_j, \quad X_j = \max_{\tau_{j-1} < l < \tau_j} x_l.$$

Lemma 11. *t_j and X_j have exponential tails.*

Proof. It suffices to prove it for t_1 and X_1 and the assumption that $x_0 \leq m$. Clearly it suffices to condition on $x_1 > m$ since otherwise $t_1 = 1$, $X_1 \leq m$. Then (b) implies that for small $\varepsilon_1, \varepsilon_2$

$$y_j = e^{\varepsilon_1 x_j + \varepsilon_2 j} 1_{\{j \leq \tau_1\}}$$

is a supermartingale. Thus

$$\mathbb{E}\, y_j \leq \mathbb{E}\, y_1 \leq C_4. \tag{39}$$

On the other hand

$$\mathbb{E}\, y_j \geq \mathbb{P}\{\tau_1 > j\}\ e^{\varepsilon_1 m + \varepsilon_2 j}.$$

Hence

$$\mathbb{P}\{\tau_1 > j\} \leq C_5 e^{-\varepsilon_2 j}$$

where $C_5 = C_4 e^{-\varepsilon_1 m}$. Now

$$\mathbb{E}\, e^{\varepsilon_1 X_1} \leq \mathbb{E}\left(\sum_{j=1}^{\tau_1} e^{\varepsilon_1 x_j}\right) \leq \sum_{j=1}^{\infty} \mathbb{E} e^{\varepsilon_1 x_j} \sum_{k=j}^{\infty} \mathbb{P}\{\tau_1 > k\} \leq C_4 \sum_j \frac{C_5 e^{-\varepsilon_2 j}}{1 - e^{-\varepsilon_2 j}} < \infty.$$

This completes the proof. □

The rest of the proof of Proposition 8 is similar to the proof of Lemma 10. We say that the pair (k, j) is bad if

$$x_k > \max(x_j, m) + M + \delta(j - k).$$

If (k, j) is bad then

$$j - k < \frac{x_k - m + M}{\delta}.$$

Let $B_l(M)$ be the number of bad pairs (k, j) such that $\tau_{l-1} < k < \tau_l$. By the previous lemma $\mathbb{E}\, B_l(M) < \infty$ and so by dominated convergence theorem $\mathbb{E}\, B_l(M) \to 0$ as $M \to \infty$. Hence by Lemma 9 the number of bad pairs such that $k < \tau_N$ is less than εN except on a set of exponentially small probability. Since $\tau_N \geq N$ the proposition follows. □

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