STABLE ERGODICITY FOR PARTIALLY HYPERBOLIC ATTRACTORS WITH NEGATIVE CENTRAL EXPONENTS

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Dedicated to G. A. Margulis on the occasion of his 60th birthday

Abstract. We establish stable ergodicity of diffeomorphisms with partially hyperbolic attractors whose Lyapunov exponents along the central direction are all negative with respect to invariant SRB-measures.

1. Introduction

Let f be a diffeomorphism of a compact smooth Riemannian manifold M. A compact invariant subset $\mathcal{L} \subset M$ is called an *attractor* for f if there exists an open neighborhood U of $\mathcal{L}$ such that $\overline{f(U)} \subset U$ and

$$\mathcal{L} = \bigcap_{n \geq 0} f^n(U).$$

U is said to be a *basin of attraction*. The maximal open set with this property is called the *topological basin of attraction* for $\mathcal{L}$. An invariant set $\mathcal{L} = \mathcal{L}_f$ is called *partially hyperbolic* if $f|\mathcal{L}$ is partially hyperbolic, i.e., the tangent bundle $T\mathcal{L}$ admits an invariant splitting

$$T\mathcal{L} = E^s \oplus E^c \oplus E^u$$

into respectively, *strongly stable, center*, and *strongly unstable subbundles* (see the next section for details). Finally, a compact invariant subset $\mathcal{L}$ is said to be *a partially hyperbolic attractor* if $\mathcal{L}$ is an attractor for f and $f|\mathcal{L}$ is partially hyperbolic. Note that the subbundle E^u is integrable; the leaves of its integral lamination W^u are called the *global strongly unstable manifolds*. A partially hyperbolic attractor is

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the union of the global strongly unstable manifolds of its points, i.e., $W^u(x) \subset \mathcal{L}$ for every $x \in \mathcal{L}$.

We are interested in studying ergodic properties of the map $f|\mathcal{L}$ with respect to some "natural" invariant measures. In [7], the conservative case was studied (where $\mathcal{L}$ is the whole manifold and f preserves a smooth measure) under the assumption that there are negative (respectively positive) central exponents, i.e., at least on a set of positive measure, all vectors in the central subbundle have negative (respectively positive) Lyapunov exponents.

In the present paper we are interested in the dissipative case: starting with any measure κ in a neighborhood U of $\mathcal{L}$, which is absolutely continuous with respect to the Riemannian volume m, we consider its evolutions under the map, i.e., the limit measures (with respect to the weak star topology) of the sequence of measures

$$\mu_n = \frac{1}{n}\sum_{k=0}^{n-1} f_*^k \kappa. \tag{1}$$

It is easy to see that any limit measure μ of the sequence μ_n is concentrated on $\mathcal{L}$. These limit measures have the following crucial property:

Proposition 1. *For almost every $x \in \mathcal{L}$ the conditional measure $\mu^u(x)$ generated by μ on the global strongly unstable manifold $W^u(x)$ is equivalent to the Riemannian volume $m^u(x)$ on $W^u(x)$ (see Section 3 for details).*

Any measure with the above property is called a *u-measure.*

Proposition 1 was proved in [16] for the case when the measure κ in (1) is volume, and was extended to the case of general smooth measures in [5] (see Lemma 11.12).

As an immediate application of this observation we obtain the following property of u-measures. Given an invariant measure μ on $\mathcal{L}$, define its *basin* by

$$B(\mu) = \{x \in M : A_n(\varphi)(x) \to \int_M \varphi \, d\mu \text{ as } n \to \infty \ \forall \text{ continuous } \varphi\};$$

here

$$A_n(\varphi)(x) = \frac{1}{n}\sum_{k=0}^{n-1} \varphi(f^k(x))$$

are the Birkhoff averages of φ.

Since the averages $A_n(\varphi)$ are uniformly bounded, pointwise convergence implies convergence in mean, so Proposition 1 has the following consequence.

Proposition 2 ([5], subsection 11.2.3)**.** *Any measure whose basin has positive volume is a u-measure.*

While Proposition 1 guarantees that any partially hyperbolic attractor has a u-measure, measures with basins of positive volume need not exist (just consider the partially hyperbolic attractor for the product of the identity map and a diffeomorphism with a hyperbolic attractor). However, the following partial converse to Proposition 2 holds true.

Proposition 3. *If there is a unique u-measure for f in $\mathcal{L}$, then its basin has full volume in the topological basin of attraction of $\mathcal{L}$.*

This statement is proven in [10] for the case when $\mathcal{L}$ is the whole manifold M, but the argument works in the general case as well.

We now describe another way to construct u-measures. Fix a point $x \in \mathcal{L}$ and consider a local strongly unstable leaf $V^u(x)$ through x (this is the connected component containing x of the intersection of $W^u(x)$ with a sufficiently small ball centered at x; see the next section for details). One can view the Riemannian volume $m^u(x)$ on $V^u(x)$ – the *leaf volume* – as a measure on the whole of $\mathcal{L}$. Consider the sequence of measures on $\mathcal{L}$

$$\nu_n = \frac{1}{n}\sum_{k=0}^{n-1} f_*^k m^u(x). \tag{2}$$

Clearly, any limit measure ν of the sequence ν_n is concentrated on $\mathcal{L}$. It is shown in [16] that any such measure is a u-measure. Indeed, one can prove that this remains true if we replace the local unstable manifold $V^u(x)$ with any local manifold passing through x and sufficiently close to $V^u(x)$ in the C^1 topology.

Moreover, for any ergodic u-measure ν and almost every $x \in \mathcal{L}$ the sequence of measures (2) converges to ν. Therefore, the class of all limit measures for sequences of type (2) coincides with the class of all u-measures, while the class of limit measures for sequences of type (1) may be smaller. Observe that in the case of (completely) hyperbolic attractors the classes of limit measures for sequences of types (1) and (2) coincide.

We also point out the following property of u-measures.

Proposition 4 ([5], Lemma 11.13)**.** *Every ergodic component of a u-measure is again a u-measure.*

The idea of constructing invariant measures by iterating unstable leaves goes back to pioneering works of Margulis (see English translation of his dissertation in [15]) and Sinai [18].

It is well known that any C^1 diffeomorphism g, which is sufficiently close to f in the C^1 topology, possesses a partially hyperbolic attractor $\mathcal{L}_g$, which lies in a small neighborhood of $\mathcal{L}_f$. Of course, $\mathcal{L}_f$ is the attractor $\mathcal{L}$ that we have studied above. The following statement shows that u-measures depend continuously on the perturbation.

Theorem 5. *Let f_n be a sequence of $C^{1+\alpha}$ diffeomorphisms converging to a diffeomorphism f in the $C^{1+\alpha}$ topology. Assume that f possesses a partially hyperbolic attractor $\mathcal{L}$. Then for large n, each map f_n possesses a partially hyperbolic attractor $\mathcal{L}_n$. If μ_n is a u-measure for f_n and the sequence of measures μ_n converges in the weak* topology to a measure μ then μ is a u-measure for f on $\mathcal{L}$.*

This result, although well known, is not published anywhere, so for completeness we give its proof in Section 4.

The problem of classifying u-measures (in particular, proving uniqueness or showing that there are only finitely many of them) plays a crucial role in various areas of mathematics including number theory, rigidity theory, and averaging theory.

In this paper we are interested in studying local ergodicity and stable ergodicity of dynamical systems possessing partially hyperbolic attractors. We begin with local ergodicity.

Let f be a $C^{1+\alpha}$ diffeomorphism of a compact smooth Riemannian manifold M possessing a partially hyperbolic attractor $\mathcal{L}$ and let μ be a u-measure on $\mathcal{L}$ for f. We say that f has *negative (positive) central exponents* (with respect to μ) if there exists an invariant subset $A \subset \mathcal{L}$ with $\mu(A) > 0$ such that the Lyapunov exponents $\chi(x, v) < 0$ (respectively, $\chi(x, v) > 0$) for every $x \in A$ and every vector $v \in E^c(x)$.

Theorem 6. *Assume that f has negative central exponents on an invariant set A of positive measure with respect to a u-measure μ for f. Then the following statements hold:*

1. *Every ergodic component of $f|A$ of positive μ-measure is open* (mod 0)*; in particular, the set A is open* (mod 0) *(that is there exists an open set U such that $\mu(A \triangle U) = 0$).*
2. *If for μ-almost every x the trajectory $\{f^n(x)\}$ is dense in* $\operatorname{supp}(\mu)$, *then f is ergodic with respect to μ.*

We provide the following criterion, which guarantees the density hypothesis in Statement (2) of the previous theorem.

Theorem 7. *Assume that for every $x \in \mathcal{L}$ the orbit of the global strongly unstable manifold $W^u(x)$ is dense in $\mathcal{L}$. Then for any u-measure μ on $\mathcal{L}$ and μ-almost every x the trajectory $\{f^n(x)\}$ is dense in $\mathcal{L}$.*

This result is an immediate corollary of the following more general statement. Given $\epsilon > 0$, we say that a set is *ϵ-dense* if its intersection with any ball of radius ϵ is not empty.

Theorem 8. *Let f be a C^1 diffeomorphism of a compact smooth Riemannian manifold M possessing a partially hyperbolic attractor $\mathcal{L}$. The following statements hold:*

(1) *Let $U \subset \mathcal{L}$ be an open set. Assume that for every $x \in \mathcal{L}$ there exists $n = n(x, U)$ such that $f^n(W^u(x)) \cap U \neq \emptyset$. Then for any u-measure μ on $\mathcal{L}$ and μ-almost every $x \in \mathcal{L}$ there is $m = m(x)$ such that $f^m(x) \in U$.*
(2) *For every $\delta > 0$ and every $\varepsilon \leq \delta$ the following holds: assume that for every $x \in \mathcal{L}$ the orbit of the global strongly unstable manifold $W^u(x)$ is ε-dense in $\mathcal{L}$. Then for any u-measure μ on $\mathcal{L}$ and μ-almost every x the trajectory $\{f^n(x)\}$ is δ-dense in $\mathcal{L}$.*
(3) *Assume that for every $x \in \mathcal{L}$ the orbit of the global strongly unstable manifold $W^u(x)$ is dense in $\mathcal{L}$. Then* $\operatorname{supp}(\mu) = \mathcal{L}$ *for every u-measure μ.*

Let us point out that the proof of Theorem 8, given in Section 4.3, shows that the requirement "for every $x \in \mathcal{L}$ the orbit of $W^u(x)$ is ε-dense in $\mathcal{L}$ (or everywhere dense in $\mathcal{L}$)" is equivalent to "for every $x \in \mathcal{L}$ the positive semi-orbit of $W^u(x)$ is ε-dense in $\mathcal{L}$ (or respectively, everywhere dense in $\mathcal{L}$)".

In the case of a hyperbolic attractor, topological transitivity of $f|\mathcal{L}$ guarantees that there is a unique u-measure for f on $\mathcal{L}$. In contrast, in the partially hyperbolic situation, even topological mixing is not enough to guarantee that there is a unique u-measure. Indeed, consider $F = f_1 \times f_2$, where f_1 is a topologically transitive Anosov diffeomorphism and f_2 a diffeomorphism close to the identity. Then any measure $\mu = \mu_1 \times \mu_2$, where μ_1 is the unique SRB measure for f_1 and μ_2 any f_2-invariant measure, is a u-measure for F. Thus, F has a unique u-measure if and only if f_2 is uniquely ergodic. On the other hand, F is topologically mixing if and only if f_2 is topologically mixing. However, the assumption in Theorem 7 is strong enough to guarantee that a u-measure with negative central exponents is unique.

Theorem 9. *Let f be a $C^{1+\alpha}$ diffeomorphism of a compact smooth Riemannian manifold M possessing a partially hyperbolic attractor $\mathcal{L}$. Assume that:*

(1) *there exists a u-measure μ for f with respect to which f has negative central exponents on an invariant subset $A \subset \mathcal{L}$ of positive μ-measure;*

(2) *for every $x \in \mathcal{L}$ the orbit of the global strongly unstable manifold $W^u(x)$ is dense in $\mathcal{L}$.*

Then μ is the only u-measure for f and f has negative central exponents at μ-almost every $x \in \mathcal{L}$. In particular, (f, μ) is ergodic, $\text{supp}(\mu) = \mathcal{L}$, *and the basin $B(\mu)$ has full volume in the topological basin of attraction of $\mathcal{L}$.*

Theorems 6, 7, and 9 are essentially contained in [6], but we provide somewhat different proofs that are better adapted to establishing our main Theorems 10 and 11, which are new.

We now consider the stable ergodicity problem. This is closely related to the question of uniqueness of u-measures, not just for f, but also for its small perturbations. A crucial ingredient is an estimate of the size of local stable and unstable manifolds, which depends only on their Lyapunov exponents along a typical trajectory (see Proposition 14). This implies that every ergodic component of a u-measure contains a ball whose radius depends only on the Lyapunov exponents along a typical trajectory in that component.

Let g be a small perturbation of f in the $C^{1+\alpha}$ topology. The main result of this paper is:

Theorem 10. *Let f be a $C^{1+\alpha}$ diffeomorphism possessing a partially hyperbolic attractor $\mathcal{L}_f$. Assume that there exists a unique u-measure μ for f with respect to which f has negative central exponents almost everywhere. Then any $C^{1+\alpha}$ diffeomorphism g, which is sufficiently close to f in the $C^{1+\alpha}$ topology, also has negative central exponents on a set that has positive measure with respect to a u-measure μ_g. This measure is the unique u-measure for g, $g|\mathcal{L}_g$ is ergodic with respect to μ_g and the basin $B(\mu_g)$ has full volume in the topological basin of attraction of $\mathcal{L}_g$.*

Combining this result with Theorem 9 we obtain the following statement.

Theorem 11. *Let f be a $C^{1+\alpha}$ diffeomorphism possessing a partially hyperbolic attractor $\mathcal{L}_f$. Assume that:*

(1) *there exists a u-measure μ for f with respect to which f has negative central exponents on an invariant subset $A \subset \mathcal{L}_f$ of positive μ-measure;*
(2) *for every $x \in \mathcal{L}_f$ the global strongly unstable manifold $W^u(x)$ is dense in $\mathcal{L}_f$.*

Then any $C^{1+\alpha}$ diffeomorphism g, which is sufficiently close to f in the $C^{1+\alpha}$ topology, also has negative central exponents on a set of full *measure with respect to a u-measure μ_g. This measure is the unique u-measure for g, $g|\mathcal{L}_g$ is ergodic with respect to μ_g, and the basin $B(\mu_g)$ has full volume in the topological basin of attraction of $\mathcal{L}_g$.*

The measure μ in Theorem 9 and measures μ_g in Theorems 10 and 11 are SRB measures for f (see [2] for the definition of SRB measures).

One can show that in Theorem 9 the map f is Bernoulli with respect to its unique u-measure μ. Indeed, in this case for every $n \geq 1$ the map f^n satisfies the conditions of Theorem 9 and hence is ergodic with respect to μ. It is well known that an ergodic diffeomorphism with nonzero Lyapunov exponents can fail to be Bernoulli only if one of its powers is nonergodic; see [2]. Similarly, one can show that in Theorems 10 and 11 the map f is Bernoulli with respect to its unique u-measure μ_g thus establishing the stably Bernoulli property of the diffeomorphism f satisfying conditions of these theorems.

We now consider partially hyperbolic attractors whose Lyapunov exponents in the central direction are all positive. Since the stable and unstable directions play different roles in dissipative systems, this case cannot be reduced to the case in which the central exponents are negative by simply replacing f by f^{-1} as was done in the conservative situation in [7]. It turns out that a straightforward version of Theorem 6 is not true. To see this, consider the direct product F of two volume preserving Anosov diffeomorphisms f_1 and f_2. Let E_i^s and E_i^u, $i = 1, 2$ be the corresponding stable and unstable subspaces. Assume that the expansion rates along E_1^u are stronger than those along E_2^u. Set

$$E^- = E_1^s \oplus E_2^s, \quad E^c = E_2^u, \quad E^+ = E_1^u.$$

Then $TM = E^- \oplus E^c \oplus E^+$ is a partially hyperbolic splitting for F. Consider an F-invariant measure $\mu = \mu_1 \times \mu_2$ where μ_1 is the volume that is preserved by f_1 and μ_2 is a nontrivial mixture of any two ergodic measures of full support for f_2. It is easy to see that μ is a u-measure for F with positive central exponents whose ergodic components are not open (mod 0).

Nevertheless, Alves, Bonatti and Viana obtained analogues of Theorems 6 and 9 under the stronger assumption that there is a set of positive volume in a neighborhood of the attractor with positive central exponents (see [1] for precise statements).

We believe that a stable ergodicity result analogous to Theorem 10 should hold for partially hyperbolic attractors with positive central exponents. We hope that the general approach used in this paper and [7] to study stable ergodicity will extend to the case of attractors with nonzero Lyapunov exponents of both signs. The first step of proving local ergodicity is already much more difficult even in the context of conservative systems. We believe, however, that if this step can be made, then it should be possible to follow the remaining steps of our approach.

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2. Examples

Most of the theorems in this paper have two kinds of assumptions: analytical (existence of u-measures with negative central exponents) and topological (e.g., density of unstable leaves). In this section we describe several examples of systems satisfying these assumptions.

2.1. Negative central exponents.

Our results clearly hold true for uniformly hyperbolic (Axiom A) attractors (for which $E^c = 0$). In particular, our approach gives a different proof of the uniqueness of SRB measures for topologically transitive Axiom A attractors.

There are three ways to construct non-Axiom A examples.

(1) Perturbations of Anosov systems. Take an Anosov diffeomorphism and make a perturbation, which is large enough in the smooth category, so that it destroys uniform contraction in the stable subspace E^s but, which is small enough in the L^1 sense, so that the exponents in the former stable direction are still negative. To this end, Bonatti and Viana [6] consider an Anosov diffeomorphism of $\mathbb{T}^3$ with one-dimensional unstable and two-dimensional stable directions. They construct a perturbation, which destroys the stable foliation near a fixed point via the Hopf bifurcation. They show that if the perturbation is confined to a sufficiently small neighborhood of the fixed point then the central exponents of any u-measure are negative for parameters near the bifurcation value.

(2) Small perturbations of systems with zero central exponents. Viana [20] gave the first robust non Axiom A examples of attractors with nonzero central exponents. Shub and Wilkinson [19] considered the direct product $F_0 = f \times \mathrm{id}$, where f is a linear Anosov diffeomorphism and the identity acts on the circle. The map F_0 preserves volume. Shub and Wilkinson showed that arbitrary close to F_0 (in the C^1 topology) there is a volume-preserving diffeomorphism F whose only central exponent is negative on the whole of M. For this map its central foliation is not absolutely continuous — the phenomenon known as “Fubini’s nightmare”.

The result in [19] was extended by Ruelle [17] who showed that for an open set of one-parameter families of (not necessarily volume preserving) maps F_ε through F_0, each map F_ε possesses a u-measure with negative central exponent (note also that f need not be linear).

It is shown in [12] that, in the class of skew products, negative central exponents appear for generic perturbations and that there is an open set of one-parameter families of skew products near $F_0 = f \times \mathrm{id}$ (f is an Anosov diffeomorphism and id is the identity map of any manifold) where the central exponents are negative with respect to any u-measure (it is required that for the averaged system the omega limit set consists of a finite number of fixed points, e.g., the averaged system can be gradient). The results in [12] extend (virtually with no changes) to the case where the first factor is an Axiom A attractor.

In [11] a one-parameter family f_ε is considered where f_0 is the time-1 map of the geodesic flow on the unit tangent bundle over a negatively curved surface. It is shown that in the volume preserving case, generically, either f_ε or f_ε^{-1} has negative central exponent for small ε and that there is an open set of nonconservative families where the central exponent is negative for any u-measure.

(3) Systems with zero central exponents subjected to rare kicks. Given diffeomorphisms f and g, let $F_n = f^n \circ g$. It is shown in [11] that if f is either a $\mathbb{T}^1$ extension of an Anosov diffeomorphism or the time-1 map of an Anosov flow and g is close to id, then for typical g and any sufficiently large n, either F_n or F_n^{-1} has negative central exponent with respect to any u-measure.

2.2. **Density of unstable leaves.** Bonatti and Diaz [3] have shown that there is an open set of transitive diffeomorphisms near $F_0 = f \times \mathrm{id}$ (f is an Anosov diffeomorphism and id is the identity map of any manifold) as well as near the time-1 map of a topologically transitive Anosov flow. This result was used in [4] to construct examples of partially hyperbolic systems with minimal unstable foliation (i.e., every unstable leaf is dense in the manifold itself). Namely, let f be a partially hyperbolic diffeomorphism with one-dimensional central direction. A submanifold with boundary $\mathcal{S}$ is called a u-section if it is transversal to unstable leaves, $f(\mathcal{S}) \subset \mathcal{S}$ and $\bigcap_{n>0} f^n(\mathcal{S})$ is a finite union of circles and segments tangent to the central direction. A u-section is called complete if it intersects every unstable leaf. Frequently (e.g., for the systems discussed above) a complete u-section can be constructed by taking a small neighborhood of a compact periodic central leaf $\mathcal{C}$ inside $W^s(\mathcal{C})$. It is shown in [4] that the set of diffeomorphisms with minimal unstable foliation contains an open and dense subset of stably transitive diffeomorphisms having a complete u-section.

The above construction gives an open set of systems with minimal unstable foliation for the above mentioned examples with one-dimensional central direction. The next result shows that such examples appear for large measure sets in a typical one-parameter family of perturbations. Let f_0 be either 1) a skew product with the map in the base being a topologically transitive Anosov diffeomorphism or 2) the time-1 map of an Anosov flow. If f is a small perturbation of f_0 then f is partially hyperbolic and by [13], the central distribution of f is integrable. Furthermore, the central leaves are compact in the first case and there are compact leaves in the second case.

Theorem 12. *Assume that there is a compact periodic central leaf $\mathcal{C}$ for f such that $f^n(\mathcal{C}) = \mathcal{C}$ and the restriction $f^n|\mathcal{C}$ is a minimal transformation. Then the unstable foliation for f is minimal.*

2.3. **Density for systems with negative central exponents.** The constructions in the previous subsection work for arbitrary partially hyperbolic systems. The next theorem, which is a slight generalization of a result from [9], works for systems with negative central exponents.

We say that f has the *uniform usu-accessibility property* if there exist numbers r_1, r_2 such that for all $x, y \in \mathcal{L}$,

$$\exists z, w \in \mathcal{L}: \quad z \in W^u(x), \quad w \in W^s(z), \quad y \in W^u(w)$$

$$d^u(x,z) < r_1, \quad d^s(z,w) < r_2, \quad d^u(w,y) < r_1, \tag{3}$$

where d^s and d^u are the distances in W^s and W^u respectively, induced by the Riemannian metric.

Theorem 13. *Assume that the central exponents are negative with respect to any u-measure and that f has the uniform usu-accessibility property. Then for any u-measure μ the forward semi-orbit of any unstable leaf is dense in* $\operatorname{supp}(\mu)$. *In particular, μ is the unique u-measure for f and $B(\mu)$ has full volume in the topological basin of attraction of $\mathcal{L}$.*

2.4. **Attractors.** In many of the examples discussed above, $\mathcal{L}$ is the whole phase space. Multiplying the maps in these examples by a map of an open disc with a contracting fixed point (or by the north-south map on a sphere) produces examples where the attractor is a proper subset and, of course, the same holds for small perturbations.

3. Preliminaries

See [14, 2, 8, 16] for more details.

A diffeomorphism f of a compact smooth Riemannian manifold M is called *(uniformly) partially hyperbolic* on a compact invariant subset $\mathcal{L} = \mathcal{L}_f$ if for every $x \in \mathcal{L}$ the tangent space at x admits an invariant splitting

$$T_xM = E^s(x) \oplus E^c(x) \oplus E^u(x)$$

into *strongly stable* $E^s(x) = E^s_f(x)$, *central* $E^c(x) = E^c_f(x)$, and *strongly unstable* $E^u(x) = E^u_f(x)$ subspaces. This means that the manifold M is endowed with a Lyapunov-Mather Riemannian metric such that the following is true: there exist numbers $0 < \lambda_s < \lambda'_c \le 1 \le \lambda''_c < \lambda_u$ such that for every $x \in \mathcal{L}$,

$$\begin{aligned} v \in E^s(x) &\Rightarrow \|d_xf(v)\| \le \lambda_s\|v\|, \\ v \in E^c(x) &\Rightarrow \lambda'_c\|v\| \le \|d_xf(v)\| \le \lambda''_c\|v\|, \\ v \in E^u(x) &\Rightarrow \lambda_u\|v\| \le \|d_xf(v)\|. \end{aligned}$$

In the case $\mathcal{L} = M$ the diffeomorphism f is called *partially hyperbolic.*

Let $\mathcal{L}$ be a partially hyperbolic set for f. Given $x \in \mathcal{L}$, one can construct *local strongly stable* and *local strongly unstable manifolds* at x. We denote them by $V^s(x)$ and $V^u(x)$ respectively. They can be characterized as follows: there is a neighborhood $U(x)$ of the point x and a constant $C > 0$ such that

$$V^u(x) = \{y \in U(x) : d(f^{-n}(x), f^{-n}(y)) \le C\lambda_u^{-n}\, d(x, y), \forall n \ge 0\},$$
$$V^s(x) = \{y \in U(x) : d(f^n(x), f^n(y) \le C\lambda_s^n\, d(x, y), \forall n \ge 0\}.$$

We define the *global strongly unstable manifold* at x by

$$W^u(x) = \bigcup_{n\ge 0} f^n(V^u(f^{-n}(x))).$$

Recall that a partition ξ of $\mathcal{L}$ is called a *foliation* if there exist $\delta > 0$, $q > 0$, and an integer $k > 0$ such that for each $x \in \mathcal{L}$:

(1) There exists a smooth immersed k-dimensional manifold $W(x)$ containing x for which $\xi(x) = W(x) \cap \mathcal{L}$ where $\xi(x)$ is the element of the partition ξ containing x. (The manifold $W(x)$ is called the *(global) leaf* of the foliation at x; the connected component of the intersection $W(x) \cap B(x, \delta)$ that contains x is called the *local leaf* at x and is denoted by $V(x)$; the number δ is called *the size* of $V(x)$.)
(2) There exists a continuous map $\phi_x : \mathcal{L} \cap B(x, q) \to C^1(D, M)$ (where D is the n-dimensional unit ball) such that $V(\phi_x(y))$ is the image of the map $\phi_x(y)\colon D \to M$ for each $y \in \mathcal{L} \cap B(x, q)$.

The global strongly stable and global strongly unstable manifolds form two transversal foliations of $\mathcal{L}$.

We denote by

$$\chi(x,v) = \limsup_{n\to\infty} \frac{1}{n} \log \|df^n v\|$$

the *Lyapunov exponent* of a nonzero vector v at $x \in \mathcal{L}$ and by $\chi^i_f(x)$ the values of the Lyapunov exponents at x. Note that the functions $\chi^i_f(x)$ are invariant. There exists a subset $\tilde{\mathcal{L}} \subset \mathcal{L}$ of full measure, which consists of *Lyapunov regular points* (see [2]). Among other things Lyapunov regularity of x means that

$$\chi(x,v) = \lim_{n\to\pm\infty} \frac{1}{n} \log \|df^n v\|$$

for all nonzero $v \in T_xM$.

An invariant measure μ supported on $\mathcal{L}$ is called *hyperbolic* if μ-almost every $x \in \mathcal{L}$ has the property that $\chi(x,v) \neq 0$ for all nonzero $v \in T_xM$. For every $x \in \tilde{\mathcal{L}}$ with non-zero exponents, the tangent space at x admits an invariant splitting

$$T_xM = E^-_f(x) \oplus E^+_f(x)$$

into *stable* and *unstable* subspaces. Let $\lambda_-(x) = e^{\chi_-(x)}$ and $\lambda_+(x) = e^{\chi_+(x)}$ where $\chi_-(x)$ and $\chi_+(x)$ are respectively the largest negative Lyapunov exponent and the smallest positive Lyapunov exponent at x. For each $\epsilon > 0$ there are Borel functions $C(x) > 0$ and $K(x) > 0$ such that

(1) for each $n > 0$,

$$\|df^n v\| \le C(x)\lambda_-(x)^n e^{\epsilon n}\|v\|, \quad v \in E^-(x),$$
$$\|df^{-n} v\| \le C(x)\lambda_+(x)^{-n} e^{\epsilon n}\|v\|, \quad v \in E^+(x);$$

(2)

$$\angle(E^-(x), E^+(x)) \ge K(x);$$

(3) for each $m \in \mathbb{Z}$,

$$C(f^m(x)) \le C(x)e^{\epsilon|m|}, \quad K(f^m(x)) \ge K(x)e^{-\epsilon|m|}.$$

For every $x \in \tilde{\mathcal{L}}$ one can construct *local stable* and *unstable* manifolds $V^-(x)$ and $V^+(x)$. They can be characterized as follows: there is a neighborhood $U(x)$ of the point x such that for any $n > 0$ and $y \in V^+(x)$,

$$d(f^{-n}(x), f^{-n}(y)) \le C(x)\lambda_+(x)^{-n}e^{\epsilon n}\, d(x,y),$$

while for $y \in V^-(x)$,

$$d(f^n(x), f^n(y) \le C(x)\lambda_-(x)^n e^{\epsilon n}\, d(x,y).$$

The sizes of the local stable and unstable manifolds vary with x in a measurable way. If $\delta(x)$ is the size of a local stable or unstable manifold at x, then for every $m \in \mathbb{Z}$,

$$\delta(f^m(x)) \geq \delta(x)e^{-\epsilon|m|}.$$

The families of local stable and unstable manifolds possess the *absolute continuity property.*

In this paper we deal with the case $\chi_f^i(x, E^c(x)) < 0$ for some $x \in \mathcal{L}$ (usually for almost every x with respect to an invariant measure μ on $\mathcal{L}$). For such x we have $E^-(x) = E^s(x) \oplus E^c(x)$. In particular, $V^+(x) = V^u(x)$. We will use the notation $V^{sc}(x)$ for the local stable manifold $V^-(x)$. Note that $V^+(x) \supset V^u(x)$ and $V^-(x) \supset V^s(x)$.

Let us stress that the sizes of the local strongly stable and strongly unstable manifolds are bounded from below and that the families of these local manifolds possess the absolute continuity property.

We denote by $m^u(x)$ the Riemannian volume on $V^u(x)$ induced by the Riemannian metric on $V^u(x)$ as a smooth submanifold in M. Given $x \in \mathcal{L}$ and sufficiently small $r > 0$, consider the partition ξ^u of $B(x,r) \bigcap \mathcal{L}$ ($B(x,r)$ is the ball centered at x of radius r) by local strongly unstable manifolds $V^u(y)$, $y \in B(x,r) \bigcap \mathcal{L}$. An invariant measure μ on $\mathcal{L}$ is called a *u-measure* if the conditional measure $\mu^u(x)$ induced by μ on the element $\xi^u(x)$ of the partition ξ^u containing x is equivalent to $m^u(x)$ for μ-almost every x.

As in [7] the following statement plays a crucial role in our analysis of u-measures. It gives a condition, which guarantees that, for any u-measure μ and μ-almost every point x, the size of the local stable manifold $V^-(f^n(x))$ is sufficiently large for some (indeed infinitely many) n. Consider the set $\mathbb{L}(f, r_0)$ of those points $x \in \Lambda$ for which the size of the local stable manifold $V^-(x, f)$ is at least r_0.

Proposition 14. *Let f be a $C^{1+\alpha}$ diffeomorphism possessing a partially hyperbolic attractor. Then for every $a > 0$ there exist $r_0 = r_0(a, f) > 0$, $p_0 = p_0(a, f) > 0$, which depend continuously on f in the $C^{1+\alpha}$ topology, such that the following statements hold:*

(a) *Let μ be a u-measure for f with respect to which f has negative central exponents on an invariant subset A of positive μ-measure. Assume that*

$$|\chi(x, v)| \geq a \tag{4}$$

for μ-almost every $x \in A$ and all nonzero $v \in T_xM$. Then for μ-almost every $x \in A$ there is an $n \geq 0$ such that $f^{-n}(x) \in \mathbb{L}(f, r_0)$;

(b) *Let μ be an ergodic measure such that* (4) *holds for μ-almost every $x \in A$ and all nonzero $v \in T_xM$. Then $\mu(\mathbb{L}(f, r_0)) \geq p_0$.*

Proof. Part (a) is proven in [7], Lemma 2 (the proof is essentially contained in [1]). In fact, the numbers n satisfying our requrements are the so-called *hyperbolic times* for f. Now Corollary 3.2 in [1] shows that if (4) holds then the hyperbolic times appear with the frequancy which is bounded from below by some number p_0 depending only on a and f. Thus μ-almost every point visits $\mathbb{L}(f, r_0)$ with frequency at least $p_0(a, f)$. Now part (b) follows from Birkhoff's ergodic theorem. □

4. Proofs

4.1. **Proof of Theorem 5.** The openness of partial hyperbolicity is established in [13]. We need to prove the continuity of u-measures. It suffices to show that for any $x \in \mathcal{L}$ and any ball $B(x, r)$ at x of sufficiently small radius r, such that $\mu(\partial B(x, r)) = 0$, the restriction $\mu|B(x, r)$ generates conditional measures on unstable leaves $V^u(y, f)$ for f, $y \in B(x, r) \cap \mathcal{L}$, which are absolutely continuous with respect to the leaf volume. Take a submanifold $T \subset B(x, r)$ transversal to $V^u(x, f)$. Then T is also transversal to the unstable leaves $V^u(x, f_n)$ for f_n for large n. We have that (see [16])

$$d\mu_n(y)|B(x, r) = \frac{1}{c_n(y)} d\mu_n^u(y)\, d\lambda_n(y), \quad y \in T,$$

where λ_n is a Borel measure on T and $\mu_n^u(y)$ is the Borel measure on $V^u(y, f_n)$ with density

$$\rho_n(z) = \prod_{i=0}^{\infty} \frac{\mathrm{Jac}(df_n^{-1}|E_n^u)(f_n^{-i}(y))}{\mathrm{Jac}(df_n^{-1}|E_n^u)(f_n^{-i}(z))} \tag{5}$$

with respect to the leaf volume $m_n^u(y)$, i.e., $d\mu_n^u(y)(z) = \rho_n(z) dm_n^u(y)$ (here E_n^u is the unstable subspace for f_n). Also,

$$c_n(y) = \int_{V^u(y, f_n)} \rho_n(z)\, dm_n^u(z).$$

Using the fact that the maps f_n are of class $C^{1+\alpha}$ and converge to f in the $C^{1+\alpha}$ topology, it is not difficult to show that $\rho_n(z) \to \rho(z)$ where

$$\rho(z) = \prod_{i=0}^{\infty} \frac{\mathrm{Jac}(df^{-1}|E^u)(f^{-i}(y))}{\mathrm{Jac}(df^{-1}|E^u)(f^{-i}(z))}. \tag{6}$$

Indeed, it suffices to observe that: 1) the infinite products (5) and (6) converge uniformly in i and z and 2) given $\beta > 0$, there exist

$\gamma = \gamma(\beta) > 0$ and $M = M(\beta) > 0$ such that for every $g \in C^{1+\alpha}$, which γ-close to f in the $C^{1+\alpha}$ topology, any $z \in \mathcal{L}$, $y \in V^u(z, g)$ and $m \geq M$,

$$\left| \rho(z) - \prod_{i=0}^{m} \frac{\mathrm{Jac}(dg^{-1}|E_g^u)(g^{-i}(y))}{\mathrm{Jac}(dg^{-1}|E_g^u)(g^{-i}(z))} \right| < \beta.$$

By passing to a subsequence we may assume that the sequence of measures λ_n converges to a measure λ. Since $\mu(\partial B(x, r)) = 0$, for any $\epsilon > 0$ there is $\delta > 0$ such that for all n,

$$\mu_n(\partial_\delta B(x, r)) < \epsilon,$$

where ∂_δ denotes the δ-neighborhood of the boundary. Hence, for any continuous function φ the sequence

$$c_n(y) \int_{V^u(y, f_n)} \varphi(z)\, d\mu_n^u(z) = c_n(y) \int_{V^u(y, f_n)} \varphi(z) \rho_n(z)\, dm_n^u(z)$$

is uniformly integrable in n. It follows that

$$\mu(B(x, r)) = \int_T c(y) \mu^u(y)\, d\lambda(y),$$

where $\mu^u(y)$ is the measure on $V^u(y)$ with density ρ with respect to the leaf volume. □

4.2. **Proof of Theorem 6.** Let us call a point z *Birkhoff regular* if the Birkhoff averages

$$\varphi^-(z) = \lim_{n \to -\infty} \frac{-1}{n} \sum_{k=n+1}^{0} \varphi(f^k(z)) \text{ and } \varphi^+(z) = \lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(z))$$

are defined and equal for every continuous function φ on M. Applying Birkhoff's ergodic theorem to a countable dense subset of the continuous functions shows that the set $\mathcal{B}$ of Birkhoff regular points has full measure in $\mathcal{L}$ with respect to μ.

Let $\tilde{\mathcal{B}} = \mathcal{B} \cap \tilde{\mathcal{L}}$, where $\tilde{\mathcal{L}}$ is the set of Lyapunov regular points defined in Section 3. Then $\mu(\mathcal{L} \setminus \tilde{\mathcal{B}}) = 0$. Furthermore μ-almost every $x \in \tilde{\mathcal{B}}$ has the property that μ^u-almost every point of $V^u(x)$ belongs to $\tilde{\mathcal{B}}$ (recall that μ^u is the conditional measure on $V^u(x)$ generated by the measure μ); since μ is a u-measure, this means that m^u-almost every point of $V^u(x)$ belongs to $\tilde{\mathcal{B}}$ (recall that m^u is the leaf volume on $V^u(x)$). Note that two Lyapunov regular points that are in the same unstable leaf must have the same Lyapunov exponents. It follows that if $x \in A \cap \tilde{\mathcal{B}}$, then $V^u(x) \cap \tilde{\mathcal{B}} \subset A$.

We will show that any point $x \in A \cap \tilde{\mathcal{B}}$ with the property that m^u-almost every point of $V^u(x)$ belongs to $\tilde{\mathcal{B}}$ has a neighborhood in $\mathcal{L}$ on which the backwards Birkhoff average φ^- is μ-almost everywhere constant for any continous function φ. We can choose a number $\delta > 0$ and a measurable set $Y \subset V^u(x) \cap \tilde{\mathcal{B}}$ such that $m^u(Y) > 0$ and $V^-(y)$ has size at least δ for every $y \in Y(x)$. We can now choose $r > 0$ small enough so that $V^-(y) \cap V^u(x') \neq \emptyset$ for any $y \in Y$ and any $x' \in B^-(x,r) \cap \mathcal{L}$, where $B^-(x,r)$ is the ball in $V^-(x)$ centered at x of radius r.

Consider the sets

$$B = \bigcup_{y \in Y} V^-(y) \tag{7}$$

and

$$C = \bigcup_{x' \in B^-(x,r) \cap \mathcal{L}} V^u(x'). \tag{8}$$

Let $\tilde{C}$ be the subset of C obtained by restricting the union in (8) to those unstable leaves $V^u(x')$ in which m^u-almost every point belongs to $\tilde{\mathcal{B}}$.

The set C is open in $\mathcal{L}$, and $\mu(C \setminus \tilde{C}) = 0$. By the absolute continuity property of the local stable manifolds, $m^u(B \cap V^u(x')) > 0$ for all $x' \in B^-(x,r) \cap \mathcal{L}$. If $V^u(x') \subset \tilde{C}$, then $B \cap V^u(x')$ must contain a point $z \in \tilde{\mathcal{B}}$. Let y be the point in Y such that $z \in V^-(y)$. Then for any $w \in V^u(x')$ we have

$$\varphi^-(w) = \varphi^-(z) = \varphi^+(z) = \varphi^+(y) = \varphi^-(y) = \varphi^-(x).$$

We see that φ^- is constant on $\tilde{C}$ and hence μ-almost everywhere constant on C.

To prove the second statement, we first observe that $f|A$ is topologically transitive and hence, by Statement 1 of the theorem, is ergodic. It therefore suffices to show that $A = \mathcal{L}$ (mod 0). Assume for a contradiction that $\mathcal{D} := \mathcal{L} \setminus \mathcal{A}$ has nonzero measure. Since A is open (mod 0), it follows from the hypothesis that almost every trajectory is dense, that we can choose $n \geq 1$ such that

$$\mu\{x \in \mathcal{D} : f^n(x) \in A\} > 0.$$

However, this contradicts the f-invariance of A (and of $\mathcal{D}$). □

4.3. **Proof of Theorem 8.** The second statement is an immediate corollary of the first statement. To prove the first statement assume by contradiction that there exists a u-measure ν such that the set Y of points whose positive semi-trajectories never visit U has positive ν-measure. We need the following lemma.

Lemma 1. *There exists $\eta > 0$ such that for every $x \in \mathcal{L}$ and $\delta > 0$ we have that*

$$\frac{m^u(B^u(x,\delta) \setminus Y)}{m^u(B^u(x,\delta))} > \eta,$$

where $B^u(x,\delta)$ is a δ-ball in the leaf $V^u(x)$ centered at x and m^u is the leaf volume in $V^u(x)$.

This lemma immediately implies the desired result. Indeed, since Y has positive ν-measure, by the definition of u-measure there is a point x such that $m^u(V^u(x) \setminus Y) > 0$ and in fact, x is a density point of $V^u(x) \setminus Y$ contradicting the lemma.

Proof of the lemma. Given $x \in \mathcal{L}$ and $\delta > 0$, set $A_{x,\delta} = B^u(x,\delta) \setminus Y$. Observe that there is $\gamma > 0$ such that for all x,

$$\frac{m^u(B^u(x,\delta))}{m^u(B^u(x,\delta(1+\gamma)))} \geq \frac{1}{2}.$$

Given $\Delta > 0$, we can choose $m \geq 1$ such that, for all $y \in \mathcal{L}$,

$$f^m(B^u(y,\delta\gamma/2)) \supset B^u(f^m(y),\Delta). \tag{9}$$

We can then choose a cover by Δ-balls,

$$f^m(B^u(x,\delta)) \subset \bigcup_i B^u(f^m(x_i),\Delta).$$

By (9), we obtain that

$$B^u(f^m(x_i),\Delta) \subset f^m(B^u(x,\delta(1+\gamma))).$$

In particular,

$$\frac{m^u(A_{x,\delta(1+\gamma)})}{m^u(B^u(x,\delta))} \geq \frac{m^u(\bigcup_i f^{-m}(A_{f^m(x_i),\Delta}))}{m^u(\bigcup_i f^{-m}(B^u(f^m(x_i),\Delta)))}.$$

Moreover, using the Besicovitch Covering Lemma, we can assume without loss of generality that for this cover each point lies in at most K balls, for some fixed constant $K > 0$. We then have a lower bound

$$\frac{m^u(\bigcup_i f^{-m}(A_{f^m(x_i),\Delta}))}{m^u(\bigcup_i f^{-m}(B^u(f^m(x_i),\Delta)))} \geq \frac{1}{K}\frac{\sum_i m^u(f^{-m}(A_{f^m(x_i),\Delta}))}{\sum_i m^u(f^{-m}(B^u(f^m(x_i),\Delta)))}.$$

Using standard bounded distortion estimates we can write

$$\frac{m^u(f^{-m}(A_{f^m(x_i),\Delta}))}{m^u(f^{-m}(B^u(f^m(x_i),\Delta)))} = \frac{\int_{A_{f^m(x_i),\Delta}} \mathrm{Jac}(df^{-m})dm^u}{\int_{B^u(f^m(x_i),\Delta)} \mathrm{Jac}(df^{-m})dm^u}$$

$$\geq c\frac{m^u(A_{x_i,\Delta})}{m^u(B^u(x_i,\Delta))},$$

where

$$c = \inf_{m\geq 0} \inf_{y_1,y_2\in B^u(x,\Delta)} \frac{\mathrm{Jac}(d_{y_1}f^{-m})}{\mathrm{Jac}(d_{y_2}f^{-m})} > 0$$

Observe that the set Y is compact and that the leaf volumes $m^u(y)$ vary continuously with $y \in Y$. It follows that we can choose $\Delta > 0$ such that

$$\rho := \min_{y\in\mathcal{L}} \left\{ \frac{m^u(A_{y,\Delta})}{m^u(B^u(y,\Delta))} \right\} > 0.$$

Combining all of the above inequalities we get

$$\frac{m^u(A_{x,\delta(1+\gamma)})}{m^u(B^u(x,\delta(1+\gamma)))} \geq \frac{c\rho}{2K}.$$

Since $\delta > 0$ can be chosen arbitrarily small, the proof of the lemma is complete. The last statement of the theorem is an immediate corollary of Lemma 1 and the fact that every u-measure is a limit of the sequence of measures (2). □

4.4. **Proof of Theorem 9.** Let μ be a u-measure for f with negative central exponents on a subset $A \subset \mathcal{L}$ of positive measure. By Theorems 6 and 7, f has negative central exponents μ-almost everywhere and is ergodic with respect to μ. Let now ν be a u-measure for f (we do not assume at this point that ν has negative central exponents on a set of positive ν-measure). Consider the sets Y, B, and C constructed in the proof of Theorem 6, see (7) and (8). By the hypotheses of the theorem, for every $z \in \mathcal{L}$ the intersection $W^u(f^n(z)) \cap V^-(y)$ is not empty for some $n \in \mathbb{Z}$ and for every $y \in V^u(x) \cap Y$. Moreover, by the absolute continuity property of local stable manifolds, for every $z \in \mathcal{L}$ the intersection $W^u(f^n(z)) \cap B$ has positive leaf volume. Since ν is a u-measure, it follows that f has negative central exponents on an invariant subset $A_\nu \subset \mathcal{L}$ of positive ν-measure. Applying again Theorems 6 and 7, we conclude that f has negative central exponents ν-almost everywhere and is ergodic with respect to ν. Note that f has negative central exponents almost everywhere with respect to the measure $\frac{1}{2}(\mu+\nu)$ and is ergodic with respect to this measure. This implies that $\mu = \nu$. The fact that the basin of μ has full volume follows from Proposition 3 and the fact that $\mathrm{supp}(\mu) = \mathcal{L}$ from Statement 3 of Theorem 8. □

4.5. **An auxiliary lemma.** The following result is needed to prove Theorems 10 and 13.

Lemma 2. *Let $\mathcal{L}$ be a partially hyperbolic attractor for f such that $f|\mathcal{L}$ has only finitely many ergodic u-measures $\mu_1, \mu_2, \dots, \mu_N$. Then* *the forward semi-orbit of every unstable manifold is dense in* $\mathrm{supp}(\mu_j)$ *for some* $j \in \{1, 2, \ldots, N\}$*.*

Proof. For any x the set

$$L(x) = \bigcap_{N=0}^{\infty} \overline{\bigcup_{n=N}^{\infty} f^n(W^u(x))}. \tag{10}$$

is closed and invariant. Thus $\mu_j(L(x))$ is either 0 or 1 for each j. However, $\mu_j(L(x))$ cannot be zero for all j since otherwise the sequence (2) would converge to a u-measure, which is singular with respect to every μ_j. Hence, $\mu_j(L(x)) = 1$ for some j and we conclude that $\mu_j(L(x) \bigcap B(\mu)) = 1$. Ergodicity of μ_j then implies that every point in $B(\mu_j)$ is dense in $\mathrm{supp}(\mu_j)$. ☐

4.6. **Proof of Theorem 10.** We need an auxiliary result.

Lemma 3. *There exists a* u*-measure* ν *for* g *and a subset* $A_g \subset \mathcal{L}_g$ *of positive* ν *measure on which* g *has negative central exponents. More precisely,*

$$\chi(x, v) < -a, \quad x \in A_g, \quad v \in E^c_g(x),$$

where $a = a(f) > 0$ *is a constant.*

Proof. By Proposition 4, f is ergodic with respect to its unique u-measure and all central exponents of f are negative almost everywhere with respect to this measure. Therefore, there exists $\alpha > 0$ such that for almost every $x \in \mathcal{L}_f$,

$$\lim_{n\to+\infty} \frac{1}{n} \ln \|df^n | E^c_f(x)\| < -\alpha.$$

Integrating over $\mathcal{L}_f$ we obtain

$$\lim_{n\to\infty} \frac{1}{n} \int_{\mathcal{L}_f} \ln \|df^n | E^c_f(x)\| \, d\mu(x) < -\alpha.$$

In particular, there exists $n_0 > 0$ such that

$$\frac{1}{n_0} \int_{\mathcal{L}_f} \ln \|df^{n_0} | E^c_f(x)\| \, d\mu(x) < -\frac{\alpha}{2}.$$

Without loss of generality we may assume that $n_0 = 1$, so that

$$\int_{\mathcal{L}_f} \ln \|df | E^c_f(x)\| \, d\mu(x) < -\frac{\alpha}{2}.$$

If a diffeomorphism g is sufficiently close to f in the C^1 topology, then by Theorem 5, for any u-measure ν on $\mathcal{L}_g$ we have

$$\int_{\mathcal{L}_g} \ln \|dg | E^c_g(x)\| \, d\nu(x) < -\frac{\alpha}{4}.$$

Take a u-measure ν for g on $\mathcal{L}_g$. It follows that there exists a subset A_g with $\nu(A_g) > 0$ such that for every $x \in A_g$

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \ln \|dg|E^c_g(g^j(x))\| \le -\frac{\alpha}{4}.$$

Hence,

$$\lim_{n\to+\infty} \frac{1}{n} \ln \|dg^n|E^c_g(x)\| \le -\frac{\alpha}{4}$$

for every $x \in A_g$ and the desired result follows. □

We proceed with the proof of the theorem. Let ν be a u-measure in Lemma 3. By Proposition 4, we may assume that ν is ergodic. Let r_0 be the number given by Proposition 14 and let $\delta = r_0(p, f)/2$.

Take a small ε. By Theorem 5, we can choose the perturbation g so close to f in the $C^{1+\alpha}$ topology that the ν-measure of the ε-neighborhood of $\mathrm{supp}(\mu)$ becomes as close to 1 as we wish. On the other hand, by Proposition 14, $\nu(\mathbb{L}(g, r_0/2)) \ge p_0(a, f)/2$. This implies that that there is a point $x \in \mathbb{L}(g, r_0/2)$ such that $d(x, \mathrm{supp}(\mu)) < \varepsilon$. Consider the sets Y, B and C constructed in the proof of Theorem 6 (see (7) and (8)) using the point x. We can always choose x such that almost all points in Y are Birkhoff generic for the measure ν. Finally, we assume that the number r in the construction of the sets B and C is such that $C \supset B(x, \varepsilon)$.

Let $\tilde{\nu}$ be another ergodic u-measure for g. We wish to show that $\tilde{\nu} = \nu$. Since C is an open set, applying Theorem 8 to the measure $\tilde{\nu}$, we obtain that $\tilde{\nu}$-almost every orbit visits C. Repeating the argument in the proof of Theorem 9, we find that $\tilde{\nu}$-almost every trajectory is Birkhoff generic with respect to ν and hence $\tilde{\nu} = \nu =: \mu_g$. In other words, μ_g is the only ergodic u-measure for g. By Proposition 4, there are no non-ergodic u-measures for g. The desired result follows now from Theorem 6. □

4.7. **Proof of Theorem 13.** We need an auxiliary result. Let P be the set of points having weak stable manifolds of size at least r_0.

Lemma 4. *There are numbers ρ_0 and r_0 such that for any x*

$$m^u(B^u(x, \rho_0) \cap P) > 0.$$

Proof. Given an ergodic u-measure μ, let $C = C(\mu)$ be the set defined by (8). Then the proof of Theorem 9 shows that the trajectories of $C(\mu)$ are disjoint for different measures μ. Since the volume of $C(\mu)$ is bounded from below, there are only finitely many u-measures (this is the argument in [6], see Theorem A). Now Lemma 2 tells us that there is a measure μ such that the forward semi-orbit of $B^u(x, \rho_0)$ is dense in supp(μ). In particular, it intersects $C(\mu)$ in a subset of positive leaf volume and the proof of Theorem 6 shows that this subset has positive measure intersection with P. $\square$

We now take any $x \in \mathcal{L}$ and an ergodic u-measure μ. Iterating the chain (3) forward if necessary, we may assume that r_2 is sufficiently small so that the Hausdorff distance between $B^u(z, \rho_0)$ and $B^u(w, \rho_0)$ is less than r_0. By the absolute continuity property of local stable manifolds, $W^u(x) = W^u(z)$ contains a positive set of points from $B(\mu)$. Since the forward semi-trajectories of points in $B(\mu)$ are dense in supp(μ), we have the density statement.

To get the uniqueness of μ we observe that if there were another ergodic u-measure ν, then by the first part of the theorem, the forward trajectory of $C(\nu)$ would intersect $C(\mu)$. $\square$

4.8. **Proof of Theorem 12.** Given $x \in \mathcal{L}$, consider the set $L(x)$ defined by (10). It is closed, invariant, and saturated by unstable leaves. By [13], the center unstable foliation of f is topologically conjugate to the center-unstable foliation of the unperturbed system and hence, it is minimal. It follows that $L(x) \bigcap \mathcal{C} \neq \emptyset$. Since this set is closed and invariant, it contains $\mathcal{C}$ and since it is saturated by the unstable leaves, it coincides with the whole manifold. $\square$

References

[1] J. Alves, C. Bonatti and M. Viana, *SRB measures for partially hyperbolic systems whose central direction is mostly expanding*, Inv. Math., **140** (2000), 351–398.

[2] L. Barreira and Ya. Pesin, *Lyapunov exponents and smooth ergodic theory*, Univ. Lect. Series, Amer. Math. Soc. **23**, 2001.

[3] C. Bonatti and L. J. Diaz, *Persistence of transitive diffeomorphims,* Ann. Math., 143 (1995), 367–396.

[4] C. Bonatti, L. J. Diaz and R. Ures, *Minimality of strong stable and unstable foliations for partially hyperbolic diffeomorphisms,* J. Inst. Math. Jussieu, 1 (2002) no. 4, 513–541.

[5] C. Bonatti, L. Diaz and M. Viana, *Dynamics beyond uniform hyperbolicity. A global geometric and probabilistic perspective,* Encyclopaedia of Mathematical Sciences, 102. Mathematical Physics, III. Springer-Verlag, Berlin, 2005.

[6] C. Bonatti and M. Viana, *SRB measures for partially hyperbolic systems whose central direction is mostly contracting*, Israel J. Math., **115** (2000), 157–193.

[7] K. Burns, D. Dolgopyat and Ya. Pesin, *Partially Hyperbolic Diffeomorphisms With Non-Zero Exponents*, J. Statist. Phys., **108** (2002), 927–942.

[8] K. Burns, C. Pugh, M. Shub and A. Wilkinson, *Recent results about stable ergodicity*, pp. 327–366 in *Smooth Ergodic Theory and Its Applications*,

A. Katok, R. de la Llave, Ya. Pesin and H. Weiss eds., Proc. Symp. Pure Math., Amer. Math. Soc., 2001.

[9] D. Dolgopyat, *On dynamics of mostly contracting diffeomorphisms*, Comm. in Math. Phys., 213 (2000), 181-201.

[10] D. Dolgopyat, *Limit Theorems for partially hyperbolic systems*, Transactions of the AMS, 356 (2004), 1637-1689.

[11] D. Dolgopyat, *On differentiability of SRB states for partially hyperbolic systems,* Invent. Math., 155 (2004), 389–449.

[12] D. Dolgopyat, *Averaging and invariant measures*, Moscow Math. J. **5** (2005) 537–576.

[13] M. Hirsch, C. Pugh and M. Shub, *Invariant Manifolds*, Springer Lecture Notes on Mathematics, **583**, Springer-Verlag, Berlin-New York, 1977.

[14] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Encyclopedia of Mathematics and its Applications, Cambridge University Press, London – New York, 54, 1995.

[15] G. Margulis, *On some aspects of the theory of Anosov systems*, Springer, 2004, p. 138.

[16] Ya. Pesin and Ya. Sinai, *Gibbs measures for partially hyperbolic attractors*, Ergod. Theory and Dyn. Syst., **2** (1982), 417–438.

[17] D. Ruelle, *Perturbation theory for Lyapunov exponents of a toral map: extension of a result of Shub and Wilkinson,* Israel J. Math., 134 (2003), 345–361.

[18] Y. Sinai, *Gibbs measures in ergodic theory*, Russ. Math. Surv., **27** (1972), 21–70.

[19] M. Shub and A. Wilkinson, *Pathological foliations and removable zero exponents*, Invent. Math., **139** (2000) no. 3, 495–508.

[20] M. Viana, *Multidimensional nonhyperbolic attractors*, Inst. Hautes Études Sci. Publ. Math. **85** (1997), 63–96.

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