DIFFUSIVE MOTION AND RECURRENCE ON AN IDEALIZED GALTON BOARD

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Abstract. We study a mechanical model known as Galton board – a particle rolling on a tilted plane under gravitation and bouncing off a periodic array of rigid obstacles (pegs). This model is also identical to a periodic Lorentz gas where an electron is driven by a uniform electric field. Previous heuristic and experimental studies have suggested that the particle's speed $v(t)$ should grow as $t^{1/3}$ and its coordinate $x(t)$ as $t^{2/3}$. We derive these facts mathematically and find exact limit distributions for the rescaled velocity $t^{-1/3}v(t)$ and position $t^{-2/3}x(t)$. In addition, quite surprisingly, our analysis shows that the particle's motion is recurrent, i.e. the particle comes back to the top of the board with probability one.

1. Introduction

Galton board is one of the simplest mechanical devices exhibiting stochastic behavior. It consists of a vertical (or inclined) board with interleaved rows of pegs. A ball thrown into the Galton board moves under gravitation and bounces off the pegs on its way down.

In this paper we deal with an idealized infinite Galton board; our ball is a point particle of unit mass moving according to equations $\dot{\mathbf{q}} = \mathbf{v}$ and $\dot{\mathbf{v}} = \mathbf{g} = \text{const}$ and colliding elastically with immobile convex obstacles of infinite mass (scatterers), which are positioned periodically on the board and satisfy the 'finite horizon' condition to prevent 'ballistic' (collision-free) motion. We neglect friction and the spin of the ball.

This model is identical to a 2D periodic Lorentz gas [7], which illustrates the transport of electrons in metals in a spatially homogeneous electric field. Without external field (i.e., when $\mathbf{g} = 0$), the periodic Lorentz gas reduces to a billiard system on its fundamental domain

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(a torus minus scatterers). This is a dispersing billiard (Sinai billiard) [13]; it preserves a Liouville (equilibrium) measure and has strong ergodic and statistical properties. The position $\mathbf{q}(t)$ of the Lorentz particle at time t evolves as a 2D Brownian motion [1, 5], in particular, $\mathbf{q}(t)/\sqrt{t} \to \mathcal{N}(0, \mathbf{D})$, where $\mathbf{D}$ is a positive definite diffusion matrix determined by the geometry of scatterers.

Under a constant external field, which we denote by $\mathbf{g} = (g, 0)$, where $g > 0$, the moving particle is allowed to accelerate indefinitely, thus the system does not even have a stationary measure. The system preserves the total energy

$$E = \tfrac{1}{2}[v(t)]^2 - gx(t) = \text{const}, \tag{1}$$

where $v(t)$ denotes the particle's speed and $x(t)$ its displacement in the direction of the field. Thus the farther the particle travels, the faster it moves. On the other hand, higher speed leads to a stronger scattering effect, thus increasing the chances that the particle bounces back and hence temporarily decelerates (this is similar to Fermi, or diffusive shock acceleration [6, 14]).

It turns out that the backscattering effect slows down the particle's drift in the x direction so much that its average displacement $\langle x(t)\rangle$ at time t will only grow as t^a with some $a < 1$. It was estimated [8, 9, 10, 11] by heuristic and approximative arguments, as well as computer simulation, that the displacement of the particle typically grows as $t^{2/3}$. Due to the conservation of energy, its speed then grows as $t^{1/3}$.

We prove these conjectures rigorously, and we precisely describe the limit distributions for the rescaled velocity $t^{-1/3}v(t)$ and the rescaled position $t^{-2/3}x(t)$. We also show that this mechanical model, after a proper rescaling of space and time, is governed by a certain set of stochastic differential equations. This provides a complete solution to the classical Galton problem.

In addition we find, quite surprisingly, that the particle's motion is recurrent; precisely there are thresholds $C_v > 0$ and $C_x > 0$ such that with probability one

$$\liminf_{t\to\infty} v(t) \le C_v \qquad \text{and} \qquad \liminf_{t\to\infty} x(t) \le C_x. \tag{2}$$

In other words, the particle slows down and effectively returns to the top of the board infinitely many times!

The recurrence in our model leads to a complication in our analysis: if the ball bounces back up too far, its speed may drop to almost zero, and its motion will not be chaotic enough for us to control it by our methods. Thus we need to prevent such returns, which we can do in two different ways. First we can assume that our Galton board has 'open top' through which the ball simply escapes. Alternatively, we can close the top with a lid reflecting the ball back down every time it hits the lid on its way up.

We assume that the ball starts on the line $x = 0$ with its initial velocity $\mathbf{v}(0)$ pointing in the (general) x-direction, and its initial speed $v(0)$ must be high enough (then, in the closed board, it will stay high). The initial state of the ball is chosen via a smooth probability measure.

2. **Velocity process**

We prove two major facts:

(A) In the open board the ball escapes through $x = 0$ with probability one.

(B) In the closed board, there is a constant $c > 0$ such that $c\,t^{-1/3}v(t)$ converges, as $t \to \infty$, to a random variable with density

$$\frac{3z}{\Gamma(2/3)} \exp\left[-z^3\right], \qquad z \geq 0. \tag{3}$$

Accordingly, $2gc^2t^{-2/3}x(t)$ converges to a random variable with density

$$\frac{3}{2\Gamma(2/3)} \exp\left[-z^{3/2}\right], \qquad z \geq 0. \tag{4}$$

In addition, the coordinate $x(t)$ returns to zero infinitely many times with probability one.

The last statement means that the Galton particle evolves in a recurrent manner – its *excursions* into the depth of the Galton board alternate with *retreats* to the starting line $x = 0$. As time goes on, the particle makes longer and longer excursions that extend farther and farther into the board (because the average coordinate $\langle x(t) \rangle$ must grow as $t^{2/3}$), but every excursion is followed by a retreat of the particle back onto the starting line.

To derive our results we approximate the Galton dynamics in which the kinetic energy $K = v^2/2$ is large and may grow indefinitely with an isokinetic system where a particle moves at a fixed speed, but its trajectory will be close enough to that of the Galton particle. To this end we first rescale time $t \to t/\sqrt{\varepsilon}$, where ε will be chosen so that the average time between collisions is $\mathcal{O}(1)$. This rescaling brings our system to the form where the kinetic energy $\tilde{K} = \varepsilon K$ is of order one, but the force is weak $\mathbf{g} \to \varepsilon \mathbf{g}$; so we get a slow-fast dynamics, with a slow variable $\tilde{K}$ and a pair of fast variables $X = (q, \boldsymbol{\omega})$, where $\boldsymbol{\omega} = \mathbf{v}/v$ denotes the particle direction. In these variables, the rescaled equations of motion read

$$\dot{q} = \sqrt{2\tilde{K}}\boldsymbol{\omega}, \quad \dot{\boldsymbol{\omega}} = \frac{\varepsilon}{\sqrt{2\tilde{K}}}\big[\mathbf{g} - \langle \mathbf{g}, \boldsymbol{\omega}\rangle\,\boldsymbol{\omega}\big] + \mathcal{O}(\varepsilon^2) \tag{5}$$

$$\dot{\tilde{K}} = \varepsilon\sqrt{2\tilde{K}}\langle \mathbf{g}, \boldsymbol{\omega}\rangle.$$

Now we approximate (5) by an isokinetic system

$$\dot{q} = \sqrt{2K}\boldsymbol{\omega}, \quad \dot{\boldsymbol{\omega}} = \frac{\varepsilon}{\sqrt{2K}}\big[\mathbf{g} - \langle \mathbf{g}, \boldsymbol{\omega}\rangle\,\boldsymbol{\omega}\big], \quad \dot{K} = 0. \tag{6}$$

The advantage of this approximation is that the dynamics on any energy surfaces can be reduced to that on the unit speed surface. Namely, the solution to (6) with initial condition $(q_0, \boldsymbol{\omega}_0, K_0)$ takes the form

$$K(t) = K_0, \quad (q, \boldsymbol{\omega})(t, \varepsilon, q_0, \boldsymbol{\omega}_0, K_0) = (\hat{q}, \hat{\boldsymbol{\omega}})(t\sqrt{2K_0}, \varepsilon/2K_0, q_0, \boldsymbol{\omega}_0)$$

where $(\hat{q}, \hat{\boldsymbol{\omega}})(t, \varepsilon, q_0, \boldsymbol{\omega}_0)$ denotes the solution of

$$\dot{\hat{q}} = \hat{\boldsymbol{\omega}}, \quad \dot{\hat{\boldsymbol{\omega}}} = \varepsilon\left[\mathbf{g} - \langle \mathbf{g}, \hat{\boldsymbol{\omega}}\rangle\,\hat{\boldsymbol{\omega}}\right]. \tag{7}$$

with initial condition $(q_0, \boldsymbol{\omega}_0)$. Equations (7) describe a particle moving on a periodic Lorentz table under a constant external field $\varepsilon\mathbf{g}$ at unit speed with a Gaussian thermostat; this model was introduced in [9] and studied in [2, 3, 4]. It is known that the dynamics (7) preserves an ergodic SRB measure (steady state) μ_ε and satisfies Central Limit Theorem: if A is a smooth observable then

$$\int_0^T A(\hat{q}(t), \hat{\boldsymbol{\omega}}(t))\,dt = T\mu_\varepsilon(A) + \sqrt{T}\,\sigma_\varepsilon(A)\mathcal{Z} + o(\sqrt{T}) \tag{8}$$

where $\mathcal{Z} = \mathcal{N}(0,1)$ is a standard normal random variable and $\mu_\varepsilon(A)$ and $\sigma_\varepsilon(A)$ are asymptotic drift and standard deviation. An important role in our analysis is played by *Ohm's Law* proved in [3]:

$$\mu_\varepsilon(\hat{\boldsymbol{\omega}}) = \tfrac{1}{2}\,\varepsilon\mathbf{D}\mathbf{g} + o(\varepsilon) \tag{9}$$

where again $\mathbf{D} = \sigma_0^2(\hat{\boldsymbol{\omega}})$. The analysis of [2, 3, 4] relies heavily on the fact that (7) is a small perturbation of the Sinai billiard corresponding to $\varepsilon = 0$. In particular we shall use the continuous dependence of the diffusion matrix on the force strength:

$$\sigma_\varepsilon(\hat{\boldsymbol{\omega}}) = \sigma_0(\hat{\boldsymbol{\omega}}) + o(1). \tag{10}$$

Facts (A) and (B) follow from a more general result:

(C) Let $\bar{K} \geq 0$. Suppose the initial state $(X(0), \tilde{K}(0))$ of our particle (in the closed Galton board) is chosen according to a smooth probability measure such that $\tilde{K}(0) = \bar{K}$, then the (rescaled) kinetic energy $\tilde{K}(\tau\varepsilon^{-2})$, where $0 < \tau < 1$ is 'slow time', weakly converges, as $\varepsilon \to 0$, to an Ito diffusion process $\mathcal{K}(\tau)$ on $[0, \infty)$ satisfying stochastic differential equation (SDE)

$$d\mathcal{K} = \frac{\sigma^2}{2\sqrt{2\mathcal{K}}}\, d\tau + (2\mathcal{K})^{1/4}\sigma\, dW_\tau, \quad \mathcal{K}(0) = \bar{K} \tag{11}$$

where W_τ is the standard Brownian motion and $\sigma^2 = \langle \mathbf{g}, \mathbf{D}\mathbf{g} \rangle$.

Eq. (11) has a singularity at 0, which can be eliminated by changing variable $\mathcal{Q} = \mathcal{K}^{3/2}$, after which standard theorems [12, Section IX.3] guarantee the existence and uniqueness of $\mathcal{Q}$, and hence of $\mathcal{K}$. Actually, $\mathcal{Q}$ is known as a square Bessel process of index $-1/3$, see [12]. For the reader's convenience, we derive (A) and (B) from (C) in Appendix.

A crucial property of Eq. (11) is its self-similarity: it remains invariant under the transformation $t \to ct$, $\mathcal{K} \to c^{2/3}\mathcal{K}$. As a result, not only the rescaled kinetic energy $\tilde{K}$, but the original one K is well approximated by (11); in fact one can study the evolution of $K(t)$ for $0 < t < T$ by substituting $\varepsilon = T^{-2/3}$ in (C).

We now derive (C) from (5)–(10). Let $T = \delta\varepsilon^{-2}$ with a small $\delta > 0$; then approximations (5)–(7) give

$$\tilde{K}(T) - \tilde{K}(0) \approx \varepsilon\sqrt{2\bar{K}} \int_0^T \langle \mathbf{g}, \boldsymbol{\omega} \rangle\, dt \approx \varepsilon \int_0^{\hat{T}} \langle \mathbf{g}, \hat{\boldsymbol{\omega}} \rangle\, dt,$$

where $\hat{T} = T\sqrt{2\bar{K}}$. Using (8), (9) and (10) we obtain

$$\tilde{K}(T) - \tilde{K}(0) \approx \frac{\langle \mathbf{g}, \mathbf{D}\mathbf{g} \rangle \delta}{2\sqrt{2\bar{K}}} + (2\bar{K})^{1/4}\sqrt{\delta}\, \langle \mathbf{g}, \sigma_0(\hat{\boldsymbol{\omega}})\mathcal{Z}^{(2)} \rangle,$$

where $\mathcal{Z}^{(2)}$ denotes a normal 2-vector; and note that $\langle \mathbf{g}, \sigma_0(\hat{\boldsymbol{\omega}})\mathcal{Z}^{(2)} \rangle = \langle \mathbf{g}, \mathbf{D}\mathbf{g} \rangle^{1/2}\mathcal{Z}$. Likewise, if we divide a longer time interval $(0, \tau\varepsilon^{-2})$ into segments of size $\delta\varepsilon^{-2}$, we obtain

$$\tilde{K}_{j+1} - \tilde{K}_j \approx \frac{\sigma^2\delta}{2\sqrt{2\tilde{K}_j}} + (2\tilde{K}_j)^{1/4}\sigma\sqrt{\delta}\mathcal{Z}_j \tag{12}$$

where $\tilde{K}_j = \tilde{K}(j\delta\varepsilon^{-2})$ and $\mathcal{Z}_j$ are independent (due to strong chaotic properties of (7)); and (12) is a discrete approximation to (11).

3. Coordinate process

Here we determine the limit distribution for the y coordinate of the Galton particle. Let $\mathbf{h}$ be a unit vector in the y direction. For simplicity, assume that our fundamental domain is symmetric about the x axis, so that the Lorentz gas diffusion matrix $\mathbf{D}$ is diagonal, i.e. $\langle \mathbf{h}, \mathbf{Dg}\rangle = 0$. Let $\sigma_y^2 = \langle \mathbf{h}, \mathbf{Dh}\rangle$. For the rescaled system (5), we have $d\tilde{y}/dt = \varepsilon\langle \mathbf{v}, \mathbf{h}\rangle$, where $\tilde{y} = \sqrt{\varepsilon} y$. Now the same argument as in the previous section shows that $\tilde{y}$ can be approximated by the solution of SDE

$$d\mathcal{Y}(\tau) = (2\mathcal{K})^{1/4}\sigma_y\, d\tilde{W}_\tau + \frac{\langle \mathbf{h}, \mathbf{Dg}\rangle}{2\sqrt{2\mathcal{K}}}\, d\tau = (2\mathcal{K})^{1/4}\sigma_y\, d\tilde{W}_\tau \tag{13}$$

with $\mathcal{Y}(0) = 0$, here $\tilde{W}_\tau$ stands for a standard 1D Brownian motion independent from W_τ (thus (11) naturally decouples from (13)).

For any fixed trajectory of $\mathcal{K}(\tau)$ the conditional distribution of $\mathcal{Y}(\tau)$ is such that its increments are independent and

$$\mathcal{Y}(\tau+\Delta) - \mathcal{Y}(\tau) = \mathcal{N}\big(0, \sigma_y^2\sqrt{2\mathcal{K}(\tau)}\,\Delta + o(\Delta)\big),$$

therefore $\mathcal{Y}(\tau)$ is (conditionally) a Gaussian random variable with zero mean and variance $\sigma_y^2\int_0^\tau \sqrt{2\mathcal{K}}(\zeta)\, d\zeta$. Thus $\mathcal{Y}(\tau)/\big(\int_0^\tau \sqrt{2\mathcal{K}(\zeta)}\, d\zeta\big)^{1/2}$ is $\mathcal{N}(0, \sigma_y^2)$, independently of $\mathcal{K}(\tau)$.

As a result, $t^{-2/3}y(t)$ converges (in distribution) to a product of two independent random variables $Y_1 Y_2$ where $Y_1 = \mathcal{N}(0, \sigma_y^2)$ and $Y_2 = \big(\int_0^1 \sqrt{2\mathcal{K}(\zeta)}\, d\zeta\big)^{1/2}$ and $\mathcal{K}$ is the solution of (11) starting at 0. We see that $y(t) \sim t^{2/3}$.

Lastly, we estimate the expected number of times the particle collides with a given scatterer. In order to hit a scatterer during a time interval $[n, n+1]$, the particle needs to be at a distance $\mathcal{O}(1)$ from it at time n; and this event has probability $p_n \sim \mathcal{O}(n^{-4/3})$, since the distributions of both x and y coordinates have standard deviation of order $n^{2/3}$. Since $\sum p_n < \infty$, the expected total number of returns to any scatterer is finite. This indicates that the coordinate process is not recurrent.

4. Three dimensional process

The arguments presented here should also work in higher dimensions, but currently we are unable to give rigorous proofs due to overcomplicated structure of singularities. We conclude our paper with expected results in dimension 3. Since in 3D the analogues of (11) and (13) can be solved explicitly, the results are even easier to formulate.

For simplicity we assume that the fundamental domain is symmetric across the coordinate planes, so that the corresponding Lorentz gas diffusion matrix is again diagonal. Let W_1, W_2, W_3, and W_4 be some independent 1D Brownian motion processes. Then we expect the following:

- The velocity process is recurrent.
- The coordinate process is not recurrent.
- There are constants $c_1, c_2, c_3 > 0$ such that the rescaled coordinate vector $t^{-2/3}(c_1 x(t), c_2 y(t), c_3 z(t))$ converges in distribution to $\big((W_1^2(1) + W_2^2(1))^{2/3}, \Lambda W_3(1), \Lambda W_4(1)\big)$ where

$$\Lambda = \left[\int_0^1 \left(W_1^2(s) + W_2^2(s)\right)^{1/3} ds\right]^{1/2}.$$

Appendix

We use elements of Ito calculus [12]. An Ito diffusion process in $\mathbb{R}$ is defined by a stochastic differential equation (SDE)

$$dX = a(X,t)\,dt + b(X,t)\,dW_t, \qquad X(0) = X_0, \tag{14}$$

where $a(X,t)$ is the drift coefficient and $b(X,t)$ the diffusion coefficient. Equation (14) has a unique solution, which is a time-homogeneous Markov process with continuous paths. If a and b do not depend on t, the Fokker-Plank equation for this process reads

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\left[\frac{\partial}{\partial x}\right]^2 \left(b^2 \rho\right) - \frac{\partial}{\partial x}\left(a\rho\right). \tag{15}$$

Consider another process $Y = \lambda(X,t)$, where λ is a smooth function. The Ito formula asserts that

$$dY = \left[\lambda' a + \tfrac{1}{2}\lambda'' b^2 + \dot{\lambda}\right] dt + \lambda' b\,dW_t, \tag{16}$$

where the prime stands for space derivative and the dot for time derivative. Thus Y is also an Ito diffusion process.

Another useful tool is changing time variable: introducing new time $dt = \kappa(X,t)\,ds$ transforms (14) into

$$dX = a\kappa\,ds + b\sqrt{\kappa}\,dW_s.$$

Now combining (11) and (16) shows that the process $\mathcal{W} = \sqrt{\mathcal{K}}$ satisfies

$$d\mathcal{W} = \frac{\sigma}{2^{3/4}\mathcal{W}^{1/2}}\,dW_\tau$$

and changing time by $d\eta = \frac{\sigma^2}{2^{3/2}\mathcal{U}}\, d\tau$ gives $d\mathcal{W} = dW_\eta$, i.e. $\mathcal{W}(\eta)$ is a standard 1D Brownian Motion. The latter is a recurrent process, hence so is our $\mathcal{K}$, which proves the fact (A).

Next, the process $\mathcal{R} = \tau^{-2/3}\mathcal{K}$ satisfies SDE

$$d\mathcal{R} = \left[\frac{1}{2\sqrt{2\mathcal{R}}} - \frac{2\mathcal{R}}{3}\right] \frac{d\tau}{\tau} - \frac{(2\mathcal{R})^{1/4}}{\sqrt{\tau}}\, dW_\tau.$$

Changing time via $d\zeta = d\tau/\tau$ gives

$$d\mathcal{R} = \left[\frac{1}{2\sqrt{2\mathcal{R}}} - \frac{2\mathcal{R}}{3}\right] d\zeta - (2\mathcal{R})^{1/4}\, dW_\zeta.$$

The Fokker-Plank equation for $\mathcal{R}$ reads, see (15),

$$\frac{\partial \rho}{\partial \zeta} = \left[\frac{\partial}{\partial r}\right]^2 \left(\sqrt{2r}\rho\right) - \frac{\partial}{\partial r}\left(\left[\frac{1}{2\sqrt{2r}} - \frac{2r}{3}\right]\rho\right).$$

It is clear that any time independent integrable solution of this equation must satisfy

$$\frac{\partial}{\partial r}\left(\sqrt{2r}\rho\right) = \left[\frac{1}{2\sqrt{2r}} - \frac{2r}{3}\right]\rho,$$

thus the asymptotic density of $\mathcal{K}$ is (4). Lastly, (3) follows from (1).

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