Prevalence of rapid mixing-II: topological prevalence.

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Abstract

We continue the study of mixing properties of *generic* hyperbolic flows started in [4]. Our main result is that generic suspension flow over subshift of finite type is exponentially mixing. This is a quantitative version of one of the results of [8].

1 Introduction.

Let S^t be a smooth flow on a manifold M preserving a measure μ. If A and B are $L^2(\mu)$ functions on M let

$$\rho_{A,B}(t) = \int_M A(S^t x)B(x)d\mu(x),$$

$$\bar{\rho}_{A,B}(t) = \rho_{A,B} - \mu(A)\mu(B).$$

Call S^t rapidly mixing if for $A, B \in C^\infty(M)$ $\bar{\rho}_{A,B}$ is in Schwartz class $\mathcal{S}(\mathbf{R})$ (i.e $\forall n_1, n_2\, |t^{n_1}(\partial_t^{n_2}\bar{\rho}_{A,B}(t))| \to 0$ as $t \to \infty$) and the map $\bar{\rho} : C^\infty(M) \times C^\infty(M) \to \mathcal{S}(\mathbf{R})$ is continuous. In a previous article ([4]) we discussed metric prevalence of rapid mixing. Namely, we considered an $n-$parameter family of flows having a hyperbolic invariant set. We proved that if a certain non-degeneracy condition holds then the set of parameters for which corresponding flows are rapidly mixing with respect to any Gibbs measure on the hyperbolic set is conull (and moreover the Hausdorff dimension of its complement is zero). The set of parameters for which our proof worked was a union of Cantor sets of positive measure. So the dependence on parameters of the constants defining rapid mixing is very irregular. Therefore there is a question if there are flows in a neighborhood of which these bounds hold uniformly (some partial results were obtained in [3], [4]). It seems reasonable that the following stronger statement holds

Conjecture 1. *For any $r > 1$ the set of exponentially mixing Axiom A flows contains an C^r–open and dense subset of the set of all Axiom A flows.*

In this note we give one evidence in favor of this conjecture by proving this bound in an easier setup of suspension flows over subshifts of finite type. (We refer the reader to [7] for background on subshifts of finite type and their suspensions.)

To state our result we need some notation. Let (Σ, σ) be a topologically mixing subshift of a finite type. We equip Σ with metric d_θ such that $d_\theta(\omega^1, \omega^2) = \theta^N$ where $N = \max\{k : \omega^1_j = \omega^2_j \text{ for } |j| < k\}$. Let $C_\theta(\Sigma)$ be the space of d_θ-Lipschitz functions. For $\tau \in C_\theta(\Sigma)$ let $\tau_n(\omega) = \sum_{j=0}^{n-1} \tau(\sigma^j x)$. Call τ *eventually positive* if there exists n such that $\tau_n > 0$. Clearly the set of eventually positive elements is open in $C_\theta(\Sigma)$ and hence it is a Baire space. If $\tau \in C_\theta(\Sigma)$ is an eventually positive function let $\Sigma^\tau = \Sigma \times \mathbf{R}/\{(\omega, s) \sim (\sigma\omega, s + \tau(\omega))\}$. (The assumption that τ is eventually positive (or eventually negative) is needed to guarantee that Σ^τ is a compact Hausdorff space.) Suspension flow S^t on Σ^τ is defined locally by $S^t(\omega, s) = (\omega, s + t)$. Let $\tilde{d}_\theta((\omega^1, s_1), (\omega^2, s_2)) = d_\theta(\omega^1, \omega^2) + |s_1 - s_2|$ and denote by $C_\theta(\Sigma^\tau)$ the space of $\tilde{d}_\theta$–Lipschitz functions. If $F \in C_\theta(\Sigma^\tau)$ denote by μ_F the Gibbs measure with potential F. Call $S^t(\tau)$ *exponentially mixing* if $\forall F\ \forall \theta'\ \exists C, \alpha$ such that $\forall A, B \in C_{\theta'}(\Sigma^\tau)$

$$|\mu_F((A \circ S^t)B) - \mu_F(A)\mu_F(B)| \leq Ce^{-\alpha t}\|A\|_{\theta'}\|B\|_{\theta'} \tag{1}$$

Actually it is enough to verify (1) for some θ' because if $\theta'' < \theta'$ then any element $A \in C_{\theta''}(\Sigma^\tau)$ can be ε–approximated in $L^2(\mu_F)$–norm by $A_\varepsilon \in C_{\theta''}(\Sigma^\tau)$ with $||A_\varepsilon||_{\theta'} \leq \mathrm{Const}(\theta', \theta'')\varepsilon^{-N(\theta', \theta'')}||A||_{\theta''}$. Our main result is the following

Theorem 1.1. *For all $0 < \theta < 1$ the set of τ such that $S^t(\tau)$ is exponentially mixing contains an open and dense subset of (eventually positive elements of) $C_\theta(\Sigma)$.*

It is easy to see from the proof of Theorem 1.1 that the constants C, α can be chosen uniformly in a neighborhood of (τ, F).

Remark. To prove this statement we demonstrate that certain twisted transfer operators do not have poles near imaginary axes (see Lemma 4.1). It is known that the same bound guarantees exponential error bound in the Prime Orbit Theorem (see [9] for details.) More precisely, let $\pi(\tau, T)$ be the number of closed orbits of $S^t(\tau)$ of period less than T. Denote by $h(\tau)$ the topological entropy of $S^t(\tau)$. Finally let $\mathrm{li}(t) = \int_2^t \frac{ds}{\ln s}$. The estimates of Section 4 together with the results of [9] imply that

For all $0 < \theta < 1$ *the set of* τ *such that there exists* $\beta(\tau) > 0$ *such that*

$$\pi(\tau, T) = \mathrm{li}(e^{h(\tau)T})(1 + O(e^{-\beta(\tau)T})), \quad T \to \infty$$

contains an open and dense subset of (eventually positive elements of) $C_\theta(\Sigma)$.

Remark. Let τ be some element of $C_\theta(\Sigma)$ such that the conclusion of Theorem 1.1 holds and let $\bar{\theta} > \theta$. Then $C_\theta \subset C_{\bar{\theta}}$ so it make sense to ask if small $C_{\bar{\theta}}$ perturbations of τ preserve exponential mixing. Unfortunately it is not the case. Indeed (see [7]) τ can be arbitrary well $C_{\bar{\theta}}$–approximated by locally constant functions $\tau^{(j)}$. By another small approximation we can achieve that there is M_j such that for any t_j in the range of $\tau^{(j)}$ $M_j t_j \in \mathbf{Z}$. Then $A(x, s) = \exp(2\pi i M_j s)$ is an eigenfunction for $S^t(\tau^{(j)})$. (In particular, exponential mixing is not open since $C_\theta \supset C_{\theta'}$ for $\theta > \theta'$.) This is the main reason why Theorem 1.1 can not be applied to obtain Conjecture 1. In fact, to any Axiom A flow we can associate a suspension over subshift of a finite type via symbolic dynamics. Now, loosely speaking, fixing θ corresponds to fixing the regularity of hyperbolic splitting but there is no reason to expect that the splitting would not become less regular after the perturbation (see [5]). Let us remark however that there is an open subset of contact Anosov flows satisfying bunching conditions of [6] there the splitting is actually C^1 and Conjecture 1 could be verified ([3]).

Thus openness is major problem in proving Conjecture 1. However density is also unknown. The problem is that the correspondence between the smooth flow and the symbolic system is not continuous so if we change τ inside $C_\theta(\Sigma)$ the corresponding Axiom A perturbation would be probably be discontinuous let alone smooth. One can hope that the situation may be better if non-wandering set is small (i.e. its Hausdorff dimension is close to one) but even in this case the problem seems to be open. (By contrast, if non-wandering set is large (e.g. locally connected), then different methods probably should be used. See [1], [2], [8].)

Let us describe the organization of the paper. First we present the set of good roof functions (strong non-integrability condition of Section 2). In Section 3 we show that this set is open and dense. In Section 4 we prove that that strong non-integrability implies exponential mixing. The proof is modeled on that from [3]. However in [3] we used heavily geometry of our phase space, whereas here we show that in fact our arguments are purely symbolic. Moreover our proof here is little bit simpler because in symbolic setting measure and metric structure of the phase space are nicely related. The lack of such a relationship in smooth case explains difficulties in extending the results of [3] about three–dimensional contact flows to higher dimensions.

So the new ingredient in the proof of Theorem 1.1 is denseness of strong non-integrability. This part is motivated by a paper of Parry and Pollicott ([8]). Among other things they showed that the set of mixing flows is open and dense. In Section 3 we refine their arguments to get strong non-integrability which implies exponential mixing via arguments of [3]. Let us also note that our set is much smaller then that from [8]. In fact it is not hard to prove that the latter set contains functions with arbitrary slow decay rates.

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2 Scheme of the proof.

Here we give the scheme of the proof of the main theorem. Proposition 2.1 and Lemma 2.2 are proven in Section 3 while the proof of Lemma 2.3 is given in Section 4. The proof of the main theorem consists of three steps. Let $C^+_\theta(\Sigma)$ be the set of functions depending only on the 'future' coordinates $\tau(\omega)=\tau(\omega_0,\omega_1,\dots\omega_n\dots)$.

(I) It is enough to prove our result with $C_\theta(\Sigma)$ replaced by $C^+_\theta(\Sigma)$. Indeed let $B_\theta(\Sigma)$ be the space of coboundaries $B_\theta(\Sigma)=\{\tau\in C_\theta(\Sigma)$ such that $\exists f\in C_\theta(\Sigma):\tau(\omega)=f(\omega)-f(\sigma\omega), f\in C_\theta(\Sigma)\}$, $B^+_\theta=B_\theta\bigcap C^+_\theta(\Sigma)$. $B_\theta(\Sigma)$ is closed in $C_\theta(\Sigma)$ by Livsic theorem (see [7]). If $\tau'-\tau''\in B_\theta$ then τ' is eventually positive iff τ'' is eventually positive. In this case $S^t(\tau')$ and $S^t(\tau'')$ are Lipschitz conjugated (by a change of variables $(\bar\omega,\bar t)=(\omega,t+f(\omega)))$

and Holder spaces are preserved by this conjugation. Thus we can speak of an element of C_θ/B_θ being exponentially mixing. Now, according to ([8], Proposition 4) there is an isomorphism $\psi : C_\theta/B_\theta \to C^+_{\sqrt{\theta}}/B^+_{\sqrt{\theta}}$ such that if $[\tau^*] = \psi[\tau]$ then $\tau^* - \tau \in B_{\sqrt{\theta}}$. From this it is easy to see that following statements are equivalent:

exponential mixing is generic in C_θ;
exponential mixing is generic in C_θ/B_θ;
exponential mixing is generic in $C^+_{\sqrt{\theta}}/B^+_{\sqrt{\theta}}$;
exponential mixing is generic in $C^+_{\sqrt{\theta}}$.

Remark. It is also clear from (II) that the condition we use to prove exponential mixing is formulated in terms of C^+_θ/B^+_θ rather then C_θ.

(II) Now we introduce the condition of strong non-integrability we use to obtain exponential mixing. To this end we recall the definition of *temporal distance function.* Write $W^s(\omega) = \{\varpi : \exists n \text{ with } \varpi_i = \omega_i \text{ for } i \geq n\}$, $W^u(\omega) = \{\varpi : \exists n \text{ with } \varpi_i = \omega_i \text{ for } i \leq n\}$. Consider a quadruple $\omega^1, \omega^2, \omega^3, \omega^4$ such that $\omega^2 \in W^u(\omega^1)$, $\omega^4 \in W^u(\omega^3)$, $\omega^3 \in W^s(\omega^1)$, $\omega^4 \in W^s(\omega^2)$. Denote

$$\varphi(\omega^1, \omega^2, \omega^3, \omega^4) = \sum_{n=-\infty}^{+\infty} [\tau(\sigma^n\omega^1) - \tau(\sigma^n\omega^2) - \tau(\sigma^n\omega^3) + \tau(\sigma^n\omega^4)]. \quad (2)$$

This series converges exponentially fast (cf. the proof of Proposition 2.1).

To explain geometric meaning of φ recall the notion of the local product structure. For $\omega \in \Sigma$ let ω_+, ω_- be the sequences $\{\omega_i\}_{i=0}^{\infty}$, $\{\omega_i\}_{i=-\infty}^{0}$. If $\omega_0^1 = \omega_0^2$ let $[\omega_1, \omega_2]$ denote the local product of ω^1 and ω^2 that is $[\omega^1, \omega^2]_- = \omega^1$, $[\omega^1, \omega^2]_+ = \omega^2$. For $(\omega, s) \in \Sigma^\tau$ define its strong stable and strong unstable sets as follows.

$$W^{ss}(\omega, s) = \{(\varpi, \bar{s}) : \tilde{d}_\theta(S^t(\tau)(\omega, s), (\varpi, \bar{s})) \to 0, \text{ as } t \to +\infty.\}.$$

$$W^{uu}(\omega, s) = \{(\varpi, \bar{s}) : \tilde{d}_\theta(S^t(\tau)(\omega, s), (\varpi, \bar{s})) \to 0, \text{ as } t \to -\infty.\}.$$

It is easy to see that

$$W^{ss}(\omega, s) = \{(\varpi, \bar{s}) : \varpi \in W^s(\omega) \text{ and } \bar{s} - s = \Delta_s(\omega, \varpi)\}$$

$$W^{uu}(\omega, s) = \{(\varpi, \bar{s}) : \varpi \in W^u(\omega) \text{ and } \bar{s} - s = \Delta_u(\omega, \varpi)\},$$

where

$$\Delta_s(\omega, \varpi) = \sum_{n=0}^{\infty} [\tau(\sigma_j\omega) - \tau(\sigma_j\varpi)],$$

$$\Delta_u(\omega,\varpi) = -\sum_{n=-\infty}^{0}[\tau(\sigma_j\omega) - \tau(\sigma_j\varpi)].$$

We can also consider local versions of these sets, that is $W^s_{loc}(\omega) = \{\varpi : \varpi_+ = \omega_+\}$, $W^u_{loc}(\omega) = \{\varpi : \varpi_- = \omega_-\}$, $W^{ss}_{loc}(\omega, s) = \{(\varpi, \bar{s}) \in W^{ss}(\omega, s) : \varpi \in W^s_{loc}\}$, $W^{uu}_{loc}(\omega, s) = \{(\varpi, \bar{s}) \in W^{uu}(\omega, s) : \varpi \in W^u_{loc}\}$. Now if $\omega' \in W^s(\omega'')$ ($\omega' \in W^u(\omega'')$) let $H_{\omega',\omega''}$ be the holonomy map from $\{\omega'\} \times \mathbf{R}$ to $\{\omega''\} \times \mathbf{R}$ along the strong stable (strong unstable) sets of the flow. Thus $H_{\omega',\omega''}(\omega', s) = \omega'' \times \mathbf{R} \bigcap W^{ss}_{loc}$ i.e.

$$H_{\omega',\omega''}(\omega', s) = (\omega'', s + \Delta_s(\omega', \omega''))$$

(respectively $H_{\omega',\omega''}(\omega', s) = \omega'' \times \mathbf{R} \bigcap W^{uu}_{loc}$ i.e.

$$H_{\omega',\omega''}(\omega', s) = (\omega'', s + \Delta_u(\omega', \omega''))).$$

Then

$$H_{\omega^3,\omega^1} \circ H_{\omega^2,\omega^3} \circ H_{\omega^4,\omega^2} \circ H_{\omega^1,\omega^4} : t \to t + \varphi(\omega^1, \omega^2, \omega^3, \omega^4).$$

It is clear from (2) that φ remains the same if we change τ by a coboundary, so, actually φ is defined on C^+_θ / B^+_θ. We use τ as a subscript if it is not clear which roof function is considered. The following bound is immediate. (See Section 3.)

Proposition 2.1. *If* $\omega^1_+ = \omega^3_+$, $\omega^2_+ = \omega^4_+$ *and* $d_\theta(\omega^1, \omega^2) \le \theta^N$, $d_\theta(\omega^3, \omega^4) \le \theta^N$ *then* $|\varphi(\omega^1, \omega^2, \omega^3, \omega^4)| \le \frac{2}{1-\theta}\|\tau\|\theta^N$.

Definition. Call $\tau \in C^+_\theta(\Sigma)$ *strongly non-integrable* if $\exists C, \delta, \varpi^1, \varpi^2 : \varpi^1_+ = \varpi^2_+$ and a neighborhood $U(\varpi^1)$ such that $\forall \omega^1 \in U$ such that

$$\omega^1_- = \varpi^1_- \;\; \exists \omega^2, \omega^3, \omega^4 : \;\; \omega^2_- = \varpi^1_-, \;\; \omega^3_- = \omega^4_- = \varpi^2_-, \;\; \omega^3_+ = \omega^1_+, \;\;\; \omega^4_+ = \omega^2_+ \tag{3}$$

$d_\theta(\omega^1, \omega^2) \le \theta^N$, $d_\theta(\omega^3, \omega^4) \le \theta^N$ and

$$|\varphi(\omega^1, \omega^2, \omega^3, \omega^4)| \ge \delta\theta^N. \tag{4}$$

Let us explain the meaning of (4). Although φ appears to be a function of four variables, it is actually determined by ω^1 and ω^4 since $\omega^2 = [\omega^1, \omega^4]$ and $\omega^2 = [\omega^4, \omega^1]$. Now (4) says that on a large subset of $\Sigma \times \Sigma$ φ is as irregular as possible. (In the above definition we ask that ω^1 and ω^4 lie on fixed local unstable sets, but it is done in order to simplify the proof of the denseness. We then show in Lemma 4.8 that this requirement can be disposed of.)

It is easy to see ([4]) that if $\varphi \equiv 0$ then S^t has a continuous eigenfunction (i.e. S^t is integrable). So strongly non-integrable systems are as far from integrable ones as possible (according to Proposition 2.1). This proposition also shows that strong non-integrability is an open property.

Lemma 2.2. *Strong non-integrability is dense in $C_\theta^+(\Sigma)$.*

The proof is given in the next section.

Remark. It is also possible to define strong non-integrability for elements of $C_\theta(\Sigma)$. In this case the upper bound is $|\varphi(\omega^1, \omega^2, \omega^3, \omega^4)| \leq \text{Const}\|\tau\|\theta^{\frac{N}{2}}$ and thus one should require $|\varphi(\omega^1, \omega^2, \omega^3, \omega^4)| \geq \delta\theta^{\frac{N}{2}}$ in place of (4). (The appearance of $\frac{N}{2}$ here explains the $\sqrt{\theta}$ in (I).)

(III) The last ingredient in the proof is the following.

Lemma 2.3. *Strong non-integrability implies exponential mixing.*

The proof appears in Section 4. Clearly, (I)-(III) prove Theorem 1.1.

3 Prevalence of strong non-integrability.

Proof of Proposition 2.1: Writing

$$\sum_{n=0}^{+\infty}[\tau(\omega^1) - \tau(\omega^2) - \tau(\omega^3) + \tau(\omega^4)] =$$

$$\sum_{n=0}^{+\infty}[\tau(\omega^1) - \tau(\omega^3) - \sum_{n=0}^{+\infty}[\tau(\omega^2) - \tau(\omega^4)]$$

we see that if $\omega^1_+ = \omega^3_+$, $\omega^2_+ = \omega^4_+$ then all positive terms in (2) vanish so we can write

$$\varphi(\omega^1, \omega^2, \omega^3, \omega^4) = \sum_{n=1}^{\infty}[\tau(\sigma^{-n}\omega^1) - \tau(\sigma^{-n}\omega^2)] - \sum_{n=1}^{\infty}[\tau(\sigma^{-n}\omega^3) - \tau(\sigma^{-n}\omega^4)].$$

The result follows since the $n-$th term in the both sums is bounded by $\|\tau\|\theta^{N+n}$. ■

Proof of Lemma 2.2: First we describe the choice of ϖ^1, ϖ^2 and U.

Since the number of periodic points of period n grows exponentially, for some m_0 there are two periodic points α^1 and α^2 of prime period m_0 with $\alpha^1_0 = \alpha^2_0$ and such that the orbits of α^1 and α^2 are different. Let $\alpha^1 = (w^1)^\infty$, $\alpha^2 = (w^2)^\infty$ be infinite concatenations of words w^1, w^2 of length m_0, that is $w^j = \alpha^j_0 \alpha^j_1 \dots \alpha^j_{m_0-1}$. Choose ϖ^1 and ϖ^2 so that $\varpi^1_- = \alpha^1_-$, $\varpi^2_- = \alpha^2_-$, and so that the words corresponding to the first m_0 symbols of ϖ^1_+ $(= \varpi^2_+)$ are different from the words corresponding to the first m_0 symbols of of $\sigma^j \alpha^1$ and $\sigma^j \alpha^2$ for $0 \le j < m_0$ (for large m_0 this is possible since the number of words of length m_0 starting with a given symbol a grows exponentially). Let $U = \{\omega : d_\theta(\omega, \varpi^1) \le \theta^{m_0}\}$.

Before proceeding further let us make a comment. It appears that for fixed N condition (4) involves an infinite number of inequalities. However, it is enough to verify a finite number of them as we now explain. Given a word $W = w_0 w_1 \dots w_{l-1}$ denote by $\mathcal{C}_W$ the cylinder $\mathcal{C}_W = \{\omega : \omega_j = w_j$ for $0 \le j \le l-1\}$. Let Σ_n be the set of words of length n. It is sufficient to prove that $\exists m$ such that $\forall N$ $\forall W \in \Sigma_{mN}$ such that $\mathcal{C}_W \subset \tilde{U} = p(U)$ $\exists \omega^1, \omega^2, \omega^3, \omega^4 \in \mathcal{C}_W$ satisfying (3) and such that

$$|\varphi(\omega^1, \omega^2, \omega^3, \omega^4)| \ge \delta \theta^{mN}. \tag{5}$$

Since Σ_{mN} is finite (5) contains finite number of inequalities for any fixed N. To show that (5) implies (4) consider $\tilde{\omega} \in U$ such that $\tilde{\omega}_- = \varpi^1_-$ and let $\hat{\omega} = [\varpi^2, \tilde{\omega}]$. Given N let $\tilde{N} = [\frac{N}{m}]$. Consider $\omega^1, \omega^2, \omega^3, \omega^4$ with $\omega^j_i = \tilde{\omega}_i$ for $0 \le j \le m(\tilde{N}+1)$ and $|\varphi(\omega^1, \omega^2, \omega^3, \omega^4)| \ge \delta \theta^{m(\tilde{N}+1)}$. We have

$$\varphi(\omega^1, \omega^2, \omega^3, \omega^4) = \varphi(\tilde{\omega}, \omega^2, \hat{\omega}, \omega^4) - \varphi(\tilde{\omega}, \omega^1, \hat{\omega}, \omega^3),$$

so at least one of the terms on the RHS is greater than $\frac{\delta}{2} \theta^{m(t(N+1)} \ge \tilde{\delta} \theta^N$ whereas $d_\theta(\tilde{\omega}, \omega^1) \le \theta^N$, $d_\theta(\tilde{\omega}, \omega^2) \le \theta^N$.

Now for each symbol b choose a sequence $\mathcal{W}^b \in \Sigma_m$ such that $b\mathcal{W}^b$ is admissible (here $b\mathcal{W}^b$ denotes concatenation of b and $\mathcal{W}^b$). For any $W \in \Sigma_{mN}$ let $V(W) = W\mathcal{W}^{W_{mN-1}}$. We will consider perturbations of the form

$$\tilde{\tau} = \tau + \sum_N \sum_{W \in \Sigma_{mN}} \varepsilon_W \theta^{m(N+1)} I_{w^1 V(W)} \tag{6}$$

where $I_{\bar{\omega}_0 \dots \bar{\omega}_n}$ is the indicator function of the set $\{\omega : \omega_j = \bar{\omega}_j \text{ for } 0 \le j \le n\}$. We show that given ϵ we can choose δ, m and $\varepsilon_W = \pm \epsilon$ so as to satisfy (5). Indeed fix N_0 and $W \in \Sigma_{mN_0}$. Consider $\omega^1, \omega^2, \omega^3, \omega^4$ such that (3) holds,

$\omega^i \in \mathcal{C}_W$, $\omega^1 \in \mathcal{C}_{V(W)}$, $\omega^2 \notin \mathcal{C}_{V(W)}$. Write $\varphi_{\tilde\tau} = \varphi_\tau + \varphi_{\tau^-_{N_0}} + \varphi_{\tau_{N_0}} + \varphi_{\tau^+_{N_0}}$, where $\tau^-_{N_0}, \tau_{N_0}$ and $\tau^+_{N_0}$ correspond to the summations over the terms with $N < N_0$, $N = N_0$ and $N > N_0$ in (6). Assume that ε_W are already chosen for $N < N_0$. We have $\varphi_{\tau_{N_0}}(\omega^1, \omega^2, \omega^3, \omega^4) = \varepsilon_W \theta^{m(N_0+1)}$. Indeed, due to our choice of ϖ^1, ϖ^2 and U only one negative term ($j = -m_0$) in (2) is different from 0 and all positive terms vanish as in Proposition 2.1. By the same argument

$$|\varphi_{\tau^+_{N_0}}(\omega^1, \omega^2, \omega^3, \omega^4)| \le \epsilon \sum_{j=N_0+2}^{\infty} \theta^{mj} = \epsilon \frac{\theta^{m(N_0+2)}}{1-\theta^m}. \tag{7}$$

Take $\delta = \frac{\epsilon\theta^m}{2}$. If m is large, $\delta > \epsilon\frac{\theta^{2m}}{1-\theta^m}$. Choose ε_W of the same sign as $(\varphi_\tau + \varphi_{\tau^-_{N_0}})(\omega^1, \omega^2, \omega^3, \omega^4)$ then

$$|(\varphi_\tau + \varphi_{\tau^-_{N_0}})(\omega^1, \omega^2, \omega^3, \omega^4) + \varepsilon_W \theta^{m(N_0+1)}| \ge \epsilon\theta^{m(N_0+1)} = 2\delta\theta^{mN_0}. \tag{8}$$

Finally, (7) and (8) imply that $|\varphi(\omega^1, \omega^2, \omega^3, \omega^4)| \ge \delta\theta^{mN_0}$ as needed. ■

4 Proof of Lemma 2.3.

Here we prove Lemma 2.3. So let $F \in C_\theta(\Sigma^\tau)$ and $\bar f = \int_0^{\tau(\omega)} F(\omega, s)ds$. Introduce the operator $\mathcal{L}_g : C^+_\theta(\Sigma) \to C^+_\theta(\Sigma)$ given by $(\mathcal{L}_g h)(\omega) = \sum_{\sigma\varpi=\omega} e^{g(\varpi)} h(\varpi)$. Let s_0 be the root of $Pr(\bar f - s_0\tau) = 0$, where Pr stands for topological pressure. It is proved in [3] that exponential mixing is the consequence of the following estimate.

Lemma 4.1. *$\exists C, p, \varepsilon, R > 0$ and $\lambda < 1$ such that for $|\Re s - s_0| < \varepsilon$ $|\Im s| > R$*

$$\|\mathcal{L}^n_{\bar f - s\tau} h\| \le C\lambda^n |b|^p \|h\|.$$

As in [3] we use Lemma 4.1 to prove exponential mixing for observables from the space $C_{\theta,2}(\Sigma^\tau)$ of functions two times differentiable in the direction of the flow and then approximate elements of $C_\theta(\Sigma^\tau)$ by those from $C_{\theta,2}(\Sigma^\tau)$. In this section we show how to verify Lemma 4.1 in our setting.

Let $s = s_0 + a - b$,

$$f^{(a)} = \bar f - (s_0 + a)\tau - \ln Pr(\bar f - (s_0 + a)\tau) + \ln h_a - \ln h \circ \sigma,$$

where h_a is the leading eigenvalue of $\mathcal{L}_{\bar{f}-(s_0+a)\tau}$. Introduce a norm

$$\|h\|_{(b)} = \max\left(\|h\|_0, \frac{L(h)}{|b|}\right),$$

where $L(h)$ is the Lipschitz constant of h

$$L(h) = \sup_{\omega_0^1=\omega_0^2} \frac{|h(\omega^1) - h(\omega^2)|}{d_\theta(\omega^1, \omega^2)}.$$

(Note that this definition is slightly different from the usual definition of the Lipschitz constant. This is done in order to simplify the formulae below.) Because of analyticity of $Pr(\bar{f} - (s_0 + a)\tau)$ and $\ln h_a$ in a and since $\mathcal{L}_g$ and $\mathcal{L}_{g+g'-g'\circ\sigma}$ are conjugated by multiplication by $e^{g'}$ it is enough to prove

Lemma 4.2. *$\exists \tilde{C}, \tilde{p}, \varepsilon, R > 0$ and $\tilde{\lambda} < 1$ such that for $|a - s_0| < \varepsilon$ $|b| > R$*

$$\|\mathcal{L}^n_{f^{(a)}-ib\tau} h\|_{(b)} \leq \tilde{C}\tilde{\lambda}^n |b|^{\tilde{p}} \|h\|_{(b)}.$$

Write $\mathcal{L}_{ab} = \mathcal{L}_{f^{(a)}+ib\tau}$.

Proposition 4.3. *If γ is a local branch of σ^{-n} (that is $\sigma^n \circ \gamma = id$ and $d_\theta(\gamma\omega^1, \gamma\omega^2) = \theta^n d_\theta(\omega^1, \omega^2)$) then*

$$|\tau_n(\omega^1) - \tau_n(\omega^2)| \leq \|\tau\| \frac{d_\theta(\omega^1, \omega^2)}{1-\theta}.$$

Proof:

$$|\tau_n(\omega^1) - \tau_n(\omega^2)| \leq \sum_{j=0}^{n-1} |\tau(\sigma^j\omega^1) - \tau(\sigma^j\omega^2)| \leq \sum_{j=0}^{n-1} \|\tau\| d_\theta(\omega^1, \omega^2)\theta^{n-j}. \blacksquare$$

This bound implies (cf. [7], Ch 4)

Proposition 4.4. *There exists a constant $K = K(f, \tau)$ such that*

$$||\mathcal{L}^n_{ab} h||_{(b)} \leq K(||h||_{L^2(\mu_a)} + \theta^n |b| ||h||_{(b)}).$$

Denote

$$q = \frac{4||\tau||}{1-\theta}. \tag{9}$$

Define $\mathcal{K}_A = \{H \geq 0 : L(\ln H) \leq A\}$. Let $h \in C^+_\theta(\Sigma)$. We say that H dominates h (writing $h \diamond H$) if $H \in \mathcal{K}_{q|b|}$, $|h(\omega)| \leq H(\omega)$ and for $d_\theta(\omega^1, \omega^2) \leq \frac{1}{b}$ the difference $|h(\omega^1) - h(\omega^2)| \leq q|b|H(\omega^1)d_\theta(\omega^1, \omega^2)$. We prove the following estimate

Proposition 4.5. *There are $\varepsilon, \bar{n}$ so that given s there is a finite number $\mathcal{N}_1(s), \mathcal{N}_2(s) \dots \mathcal{N}_{l(s)}(s)$ of linear operators on $C^+_\theta(\Sigma)$ such that*

(a) $\mathcal{N}_j(s)$ preserves $\mathcal{K}_{q|b|}$;

(b) for $H \in \mathcal{K}_{q|b|}$

$$\int (\mathcal{N}_j H)^2 d\mu_a \leq (1-\varepsilon) \int H^2 d\mu_a$$

where μ_a is the equilibrium state for $f^{(a)}$;

(c) If $h \diamond H$ then $\exists j = j(h, H)$ so that $(\mathcal{L}^{\bar{n}}_{ab} h) \diamond (\mathcal{N}_j H)$.

Let us check that Proposition 4.5 implies Lemma 4.2.

Proof of Lemma 4.2: Let $n = R \ln |b|$ where R is large enough, then

$$||\mathcal{L}^n_{ab} h||_{L^2(\mu_a)} \leq \frac{1}{b} ||h||_{(b)}. \tag{10}$$

Indeed, $q||h||_{(b)} 1$ dominates h and so $||h||_{L^2(\mu_a)} \leq$

$$||\mathcal{L}^{\tilde{n}}_{ab} \mathcal{N}_{i_m} \mathcal{N}_{i_{m-1}} \dots \mathcal{N}_{i_1} (q||h||_{(b)} 1)||_{L^2(\mu_a)} \leq$$
$$q||h||_{(b)} ||\mathcal{N}_{i_m} \mathcal{N}_{i_{m-1}} \dots \mathcal{N}_{i_1} 1||_{L^2(\mu_a)} \leq$$
$$q||h||_{(b)} (1-\varepsilon)^m,$$

where $n = \tilde{n} + n\bar{n}$. Combining (10) with Proposition 4.4 we get the following matrix inequality

$$\begin{pmatrix} ||\mathcal{L}^n_{ab} h||_{L^2(\mu_a)} \\ ||\mathcal{L}^n_{ab} h||_{(b)} \end{pmatrix} \leq \begin{pmatrix} 0 & \frac{1}{|b|} \\ K & \frac{K}{|b|} \end{pmatrix} \begin{pmatrix} ||h||_{L^2(\mu_a)} \\ ||h||_{(b)} \end{pmatrix}$$

Iterating we get

$$||\mathcal{L}^{2n}_{ab} h||_{(b)} \leq \frac{K}{|b|} (K||h||_{L^2(\mu_a)} + (1 + \frac{K}{|b|}) ||h||_{(b)}).$$

This proves Lemma 4.2 with $\tilde{\lambda}$ being any number greater than $\exp(-\frac{1}{2R})$ and $\tilde{p} = R \ln \lambda$.) ■

The proof of Proposition 4.5 occupies the rest of the paper.

We begin with describing $\mathcal{N}_j$. Denote

$$\varepsilon_0 = \frac{\theta(1-\theta)\delta}{10||\tau||}, \tag{11}$$

where δ comes from (4). Let ε_1 be a small number (more precisely, we require that

$$\varepsilon_1 < \frac{1}{4} \tag{12}$$

and $\forall \triangle ABC$ such that $\angle A \geq \frac{\varepsilon_0}{100}$ and $|AB| \geq \frac{|AC|}{16}$

$$|BC| \leq |AB| + (1-\varepsilon_1)|AC|.) \tag{13}$$

Let m be a natural number such that

$$\theta^m < \frac{\varepsilon_0}{10} \tag{14}$$

Let $\tilde{U} = pU$ where U is a the set where (4) holds. Let

$$\bar{n} = n_0 + n_1 \tag{15}$$

where n_0 is the smallest number such that

$$\sigma^{n_0}\tilde{U} = \Sigma^+ \tag{16}$$

and n_1 be the smallest number such that

$$\theta^{n_1} \leq \min(\frac{1}{8}, \frac{\varepsilon_0}{400}) \tag{17}$$

Denote $\mathcal{M}_a = \mathcal{L}_{a0}^{\bar{n}}$. Given b denote by $N = N(b)$ the smallest natural number such that $\theta^N \leq \frac{1-\theta}{4|b|||\tau||}$. Thus

$$\frac{(1-\theta)\theta}{4|b|||\tau||} < \theta^N \leq \frac{1-\theta}{4|b|||\tau||}. \tag{18}$$

If J is a subset of $\Sigma_{N+m+\bar{n}}$ write $\psi_J = \sum\limits_{W\in J} I_{\mathcal{C}_W}$. For $J \in \Sigma_{N+m+\bar{n}}$ and positive number ε define $\mathcal{N}^{(J,\varepsilon)}(H) = \mathcal{M}_a((1-\varepsilon\psi_J)H)$. Call J dense if $\forall W \in \Sigma_N \exists V \in J$ such that $V = W_1 W W_2$ where W_1 and W_2 are words of length $\bar{n}$ and m respectively.

Proposition 4.6. *(a)* $\mathcal{N}^{(J,\varepsilon_1)} : \mathcal{K}_{q|b|} \to \mathcal{K}_{q|b|}$;

(b) If $h \diamond H$ *then for* $d_\theta(\omega^1, \omega^2) < \frac{1}{|b|}$

$$|(\mathcal{L}_{ab}^{\bar{n}} h)(\omega^1) - (\mathcal{L}_{ab}^{\bar{n}} h)(\omega^2)| \leq q|b|(\mathcal{N}^{(J,\varepsilon_1)}H)(\omega^1) d_\theta(\omega^1, \omega^2);$$

(c) If J *is dense then there exists* $\varepsilon_2 = \varepsilon_2(f, m, \bar{n})$ *such that for all* $H \in \mathcal{K}_{q|b|}$

$$\int (\mathcal{N}^{(J,\varepsilon_1)}H)^2 \, d\mu_a \leq (1-\varepsilon_2) \int H^2 \, d\mu_a.$$

PROOF:To prove (a) note that $(1-\varepsilon_1\psi_J)\in\mathcal{K}_{\tilde{\delta}|b|}$ where $\tilde{\delta}\to 0$ as $\varepsilon_1\to 0$. Denote $\tilde{H}=(1-\varepsilon_1\psi_J)H$, then $\tilde{H}\in\mathcal{K}_{(q+\tilde{\delta})|b|}$ Now if $\omega_0^1=\omega_0^2$ then

$$(\mathcal{M}_a\tilde{H})(\omega^1)=\sum_{\sigma^{\bar{n}}\varpi^1=\omega^1} e^{f_{\bar{n}}^{(a)}(\varpi^1)}\tilde{H}(\varpi^1)\le$$

$$\sum_{\sigma^{\bar{n}}\varpi^2=\omega^2} e^{f_{\bar{n}}^{(a)}(\varpi^2)+\frac{||f^{(a)}||}{1-\theta}d_\theta(\omega^1,\omega^2)}\tilde{H}(\varpi^2)e^{\theta d_\theta(\omega^1,\omega^2)(\tilde{\delta}+q)|b|}=$$

$$(\mathcal{M}_a\tilde{H})(\omega^2)\exp\left(\frac{||f^{(a)}||}{1-\theta}+\theta(\tilde{\delta}+q)|b|\right).$$

where the inequality follows from Proposition 4.3. If b is large and $\tilde{\delta}$ is small

$$\frac{||f^{(a)}||}{1-\theta}+\theta(\tilde{\delta}+q)|b|\le q|b|;$$

$$(b)\quad(\mathcal{L}_{ab}^{\bar{n}}h)(\omega^1)-(\mathcal{L}_{ab}^{\bar{n}}h)(\omega^2)=\sum_{\sigma^{\bar{n}}\varpi=\omega}[e^{(f_{\bar{n}}^{(a)}+ib\tau_{\bar{n}})(\varpi_1)}h(\varpi_1)-e^{(f_{\bar{n}}^{(a)}+ib\tau_{\bar{n}})(\varpi_2)}h(\varpi_2)]=$$

$$\sum_{\sigma^{\bar{n}}\varpi=\omega}[e^{(f_{\bar{n}}^{(a)}+ib\tau_{\bar{n}})(\varpi_1)}(h(\varpi_1)-h(\varpi_2))]+\sum_{\sigma^{\bar{n}}\varpi=\omega}[(e^{(f_{\bar{n}}^{(a)}+ib\tau_{\bar{n}})(\varpi_1)}-e^{(f_{\bar{n}}^{(a)}+ib\tau_{\bar{n}})(\varpi_2)})h(\varpi_1)]$$
$$=(I)+(II).$$

Now

$$|(I)|\le q|b|(\mathcal{M}_aH)(\omega^1)d_\theta(\omega^1,\omega^2)\theta^{\bar{n}}\le q|b|(\mathcal{N}^{(J,\varepsilon_1)}H)(\omega^1)\frac{\theta^{\bar{n}}d_\theta(\omega^1,\omega^2)}{1-\varepsilon_1},$$

$$|(II)|\le\sum_{\sigma^{\bar{n}}\varpi=\omega}|(e^{(f_{\bar{n}}^{(a)}(\varpi^1)+ib\tau_{\bar{n}})(\varpi_2)}-e^{(f_{\bar{n}}^{(a)}(varpi^2)+ib\tau_{\bar{n}})(\varpi_2)}||h(\varpi_1)|+$$

$$\sum_{\sigma^{\bar{n}}\varpi=\omega}(e^{(f_{\bar{n}}^{(a)}(\varpi^1)}|e^{ib\tau_{\bar{n}})(\varpi_1)}-e^{ib\tau_{\bar{n}})(\varpi_2)})||h(\varpi_1)|=(II_a)+(II_b)$$

where

$$(II_a)\le\sum_{\sigma^{\bar{n}}\varpi=\omega}|(e^{f_{\bar{n}}^{(a)}(\varpi^1)}2\frac{||f^{(a)}||d(\omega^1,\omega^2)}{1-\theta}H(\varpi^1)\le$$

$$2\frac{d(\omega^1,\omega^2)||\tau||}{1-\theta}(\mathcal{M}_aH)(\omega^1)\le$$

$$2\frac{d(\omega^1,\omega^2)||\tau||}{1-\theta}\frac{(\mathcal{N}^{(J,\varepsilon_1)}H)(\omega^1)}{1-\varepsilon_1}$$

(the first inequality uses the fact that $d(\omega^1,\omega^2)$ is very small) and

$$(II_b) \le \frac{|b|||\tau||}{1-\theta}(\mathcal{M}_a H)(\omega^1) \le \frac{|b|||\tau||}{1-\theta}\frac{(\mathcal{N}^{(J,\varepsilon_1)}H)(\omega^1)}{(1-\varepsilon_1)}$$

Hence

$$(II) \le \frac{2(||f||+|b|||\tau||)}{(1-\theta)(1-\varepsilon_1)}(\mathcal{N}^{(J,\varepsilon_1)}H)(\omega^1)d(\omega^1,\omega^2)$$

and so

$$|(\mathcal{L}_{ab}^{\bar n}h)(\omega^1)-(\mathcal{L}_{ab}^{\bar n}h)(\omega^2)| \le |b|\frac{q\theta^{\bar n}+\frac{2}{1-\theta}(||\tau||+\frac{||f||}{|b|})}{1-\varepsilon_1}(\mathcal{N}^{(J,\varepsilon_1)}H)(\omega^1)d_\theta(\omega^1,\omega^2).$$

If $\bar n$ and b are large then (recall (12))

$$\frac{q\theta^{\bar n}+\frac{2}{1-\theta}(||\tau||+\frac{||f||}{|b|})}{1-\varepsilon_1} < q; \tag{19}$$

(One can check that

$$\theta^{\bar n} \le \frac{1}{8} \tag{20}$$

and $|b| > 100\frac{||f||}{||\tau||}$ suffices for (19) .)

(c) Let $W \in \Sigma_N$, $V = W_1WW_2 \in \Sigma_{N+m+\bar n}$, $V \in J$. Denote $\bar W = WW_2$. By Cauchy-Schwartz $\forall \omega \in \mathcal{C}_{\bar W}$ $(\mathcal{N}^{J,\varepsilon_1}H)(\omega) \le (1-\delta)(\mathcal{M}_a H)(\omega)$. Now (c) follows from (a) and the fact that there is a constant $c = c(q)$ such that for all $H \in \mathcal{K}_{q|b|}$,

$$\int_{\mathcal{C}_W} H^2 d\mu_a \le c\int_{\mathcal{C}(\bar W)} H^2 d\mu_a$$

(see [3], Lemma 12 for details). ■

Thus it remains to prove that if $h\diamond H$ then there exists a dense $J = J(h,H)$ so that $|(\mathcal{L}_{ab}^{\bar n}h)(\omega)| \le (\mathcal{N}^{(J,\varepsilon_1)}H)(\omega)$. Given two inverse branches γ_1 and γ_2 of $\sigma^{-\bar n}$ so that $\forall W \in \Sigma_N$ denote

$$\rho_1^\varepsilon(\omega) = \frac{|e^{(f_{\bar n}^{(a)}+ib\tau_{\bar n})(\gamma_1\omega)}h(\gamma_1\omega)+e^{(f_{\bar n}^{(a)}+ib\tau_{\bar n})(\gamma_2\omega)}h(\gamma_2\omega)|}{(1-\varepsilon)e^{f_{\bar n}^{(a)}(\gamma_1\omega)}H(\gamma_1\omega)+e^{f_{\bar n}^{(a)}(\gamma_2\omega)}H(\gamma_2\omega)},$$

$$\rho_2^\varepsilon(\omega) = \frac{|e^{(f_{\bar n}^{(a)}+ib\tau_{\bar n})(\gamma_1\omega)}h(\gamma_1\omega)+e^{(f_{\bar n}^{(a)}+ib\tau_{\bar n})(\gamma_2\omega)}h(\gamma_2\omega)|}{e^{f_{\bar n}^{(a)}(\gamma_1\omega)}H(\gamma_1\omega)+(1-\varepsilon)e^{f_{\bar n}^{(a)}(\gamma_2\omega)}H(\gamma_2\omega)}.$$

Lemma 4.7. *γ_1 and γ_2 can be chosen in such a way that $\forall W \in \Sigma_N$ $\exists j \in \{1, 2\}, V \in \Sigma_{N+m}$ such that $W \subset V$ and $\forall \omega \in \mathcal{C}_V$ $\rho_j^{\varepsilon_1}(\omega) \leq 1$.*

(Here we write $W \subset V$ to mean that W is the beginning of V). The proof of the lemma is given later.

Proof of Proposition 4.5: Let $\{\mathcal{N}_j\} = \{\mathcal{N}^{(J,\varepsilon_1)}\}$, where ε_1 is a fixed small number and J runs over all dense sets of $\Sigma_{N+m+\bar{n}}$. Then by Proposition 4.6 conditions (a) and (b) are satisfied so only (c) remains to be proved. Let h, H be given with $h \diamond H$. For each $W \in \Sigma_N$ choose one pair $\{j, V\}$ so that $V \in \Sigma_{N+m}$, $W \subset V$ and $\rho_j^{\varepsilon_1}(\omega) < 1$ for $\omega \in \mathcal{C}_V$. (Given W the set of such pairs is non-empty by Lemma 4.7.) Let $J = \{\gamma_j(W)V(W)\}$. Then J is dense. Now in view of Proposition 4.6(b) we only need to show that $|(\mathcal{L}_{ab}^{\bar{n}} h)(\omega)| \leq (\mathcal{N}^{(J,\varepsilon_1)} H)(\omega)$. Let $\omega \in \Sigma^+$ and let $W \in \Sigma_N$ be a word with $\omega \in \mathcal{C}_W$. If $\omega \notin \mathcal{C}_{V(W)}$ then $(\mathcal{N}^{(J,\varepsilon_1)} H)(\omega) = (\mathcal{M}_a H)(\omega)$ so there is nothing to prove. If $\omega \in \mathcal{C}_V$ and, say, $j = 1$ then

$$|(\mathcal{L}_{ab}^{\bar{n}} h)(\omega)| \leq$$

$$|e^{(f_{\bar{n}}^{(a)} + ib\tau_{\bar{n}})(\gamma_1\omega)} h(\gamma_1\omega) + e^{(f_{\bar{n}}^{(a)} + ib\tau_{\bar{n}})(\gamma_2\omega)} h(\gamma_2\omega)| + \sum_{\varpi \notin \{\gamma_1\omega, \gamma_2\omega\}} |e^{(f_{\bar{n}}^{(a)} + ib\tau_{\bar{n}})(\varpi)} h(\varpi)| \leq$$

$$|e^{f_{\bar{n}}^{(a)}(\gamma_1\omega} H(\gamma_1\omega)(1 - \varepsilon_1) + e^{f_{\bar{n}}^{(a)}(\gamma_2\omega)} H(\gamma_2\omega) + \sum_{\varpi \notin \{\gamma_1\omega, \gamma_2\omega\}} |e^{f_{\bar{n}}^{(a)})(\varpi)} H(\varpi)| \leq$$

$$(\mathcal{N}^{(J,\varepsilon_1)} H)(\omega). \blacksquare$$

To establish Lemma 4.7 we need two auxiliary estimates. The first one essentially proves Lemma 4.7 when $\text{Arg}(h)$ is constant while the second one allows us to control oscillations of $\text{Arg}(h)$.

Lemma 4.8. *There exist two inverse branches γ_1 and γ_2 of $\sigma^{-\bar{n}}$ so that $\forall W \in \Sigma_N$ $\forall V \in \Sigma_{N+m} : W \subset V$ $\exists \tilde{V} \in \Sigma_{N+m} : W \subset \tilde{V}$ such that if*

$$\Phi^{(n)}(\omega, \tilde{\omega}) = \tau_n(\gamma_1\omega) - \tau_n(\gamma_2\omega) - \tau_n(\gamma_1\tilde{\omega}) + \tau_n(\gamma_2\tilde{\omega})$$

then $\forall \omega \in \mathcal{C}_V, \tilde{\omega} \in \mathcal{C}_{\tilde{V}}$

$$\frac{\varepsilon_0}{|b|} \leq |\Phi^{(\bar{n})}(\omega, \tilde{\omega})| \leq \frac{1}{|b|}. \tag{21}$$

Remark. The upper bound is easy and holds true whether τ satisfies strong non-integrability or not. The key estimate is the lower one. The point of the upper bound is only to ensure that $\text{dist}(|b|\Phi^{(\bar{n})}(\omega^1, \omega^2), 2\pi\mathbf{Z}) \geq \varepsilon_0$.

PROOF:We prove the statement of the lemma for the case $W \subset \tilde{U}$ with $\bar{n}$ replaced by n_1 since if $\tilde{\gamma}_i$ work for $\tilde{U}$ then $\gamma_i = \tilde{\gamma}_i \circ \gamma$ work for Σ^+ where $\gamma : \Sigma \to \tilde{U}$ is an inverse branch of $\sigma^{\bar{n}_0}$. Let $V \subset W \subset \tilde{U}$ be given. Take some $\omega \in V$. By strong non-integrability $\exists \omega^1, \omega^2, \omega^3, \omega^4 : \omega^1_+ = \omega^3_+ = \omega$, $d_\theta(\omega^1, \omega^2) \leq \theta^N$, $d_\theta(\omega^3, \omega^4) \leq \theta^N$ and $|\varphi(\omega^1, \omega^2, \omega^3, \omega^4)| \geq \delta\theta^N$. Thus $\frac{5\varepsilon_0}{2|b|} \leq |\varphi(\omega^1, \omega^2, \omega^3, \omega^4)| \leq \frac{1}{2|b|}$ (the second inequality is by Proposition 2.1.) To define $\tilde{\gamma}_j$ we have to specify $(\tilde{\gamma}_j\omega)_i$ for $i < n_1$. Set $(\tilde{\gamma}_1\omega)_i = \omega^1_{i-n_1}$, $(\tilde{\gamma}_2\omega)_i = \omega^3_{i-n_1}$ for $i < n_1$. Then

$$\varphi^{(n_1)}(\omega^1, \omega^2, \omega^3, \omega^4) = \varphi(\omega^1, \omega^2, \omega^3, \omega^4) - \Phi^{(n_1)}(\omega^1_+, \omega^2_+)$$

satisfies $|\varphi^{(n_1)}(\omega^1, \omega^2, \omega^3, \omega^4)| \leq \frac{2||\tau||\theta^{N+n_1}}{1-\theta}$ (the proof is the same as in Proposition 2.1 but now the first non-zero term is $j = -n_1$). Thus if n_1 is so large that

$$\theta^{n_1} \leq \min(\frac{\varepsilon_0}{2}, \frac{1}{4}) \tag{22}$$

then

$$\frac{2\varepsilon_0}{|b|} \leq |\Phi^{(n_1)}(\omega^1_+, \omega^2_+)| \leq \frac{3}{4|b|}. \tag{23}$$

(since in this case

$$\frac{2||\tau||\theta^{N+n_1}}{1-\theta} \leq \min(\frac{\varepsilon_0}{2|b|}, \frac{1}{4|b|})$$

by (18).) Now if $\omega', \tilde{\omega}'$ satisfy $\omega'_i = \omega^1_i$, $\tilde{\omega}'_i = \omega^2_i$ for $0 \leq i \leq N+m$ then

$$|\Phi^{(n_1)}(\omega', \tilde{\omega}') - \Phi^{(n_1)}(\omega^1_+, \omega^2_+)| \leq |\tau_{n_1}(\tilde{\gamma}_1\omega^1_+) - \tau_{n_1}(\tilde{\gamma}_1\omega')| + |\tau_{n_1}(\tilde{\gamma}_2\omega^1_+) - \tau_{n_1}(\tilde{\gamma}_2\omega')| +$$

$$|\tau_{n_1}(\tilde{\gamma}_1\omega^2_+) - \tau_{n_1}(\tilde{\gamma}_1\tilde{\omega}')| + |\tau_{n_1}(\tilde{\gamma}_2\omega^2_+) - \tau_{n_1}(\tilde{\gamma}_2\tilde{\omega}')| \leq \frac{2||\tau||\theta^{N+m}}{1-\theta}$$

because each term is less than $\frac{2||\tau||\theta^{N+m}}{1-\theta}$ by Proposition 4.3. Now

$$\frac{2||\tau||\theta^{N+m}}{1-\theta} \leq \frac{\theta^m}{2|b|} \leq \frac{\varepsilon}{20|b|}$$

and so $\tilde{V} = \{\tilde{\omega} : \tilde{\omega}_i = \omega^2_i \text{ for } 0 \leq i \leq N+m\}$ is the required set. ■

Proposition 4.9. *Let $h \diamond H$ then*

(a) $\forall W \in \Sigma_N \forall \omega^1 \omega^2 \in \mathcal{C}_W$

$$\frac{1}{2} \leq \frac{H(\gamma_i(\omega^1))}{H(\gamma_i(\omega^2))} \leq 2 \tag{24}$$

(b) $\forall i \in \{1,2\}$ *either*

$$\forall \omega^1, \omega^2 \in \mathcal{C}_W \quad |h(\gamma_i \omega^1)| \leq \frac{3}{4} H(\gamma_i \omega^2) \tag{25}$$

or

$$\forall \omega^1, \omega^2 \in \mathcal{C}_W \quad |h(\gamma_i \omega^1)| \geq \frac{1}{4} H(\gamma_i \omega^2); \tag{26}$$

(c) If for some $i \in \{1,2\}$ *(26) holds then* $|\mathrm{Arg}(\frac{h(\gamma_i \omega)}{h(\gamma_i \tilde{\omega})})| \leq \frac{\varepsilon_0}{100}$.

Proof:(a) $H(\gamma_i \omega^1) \leq e^{q|b| d_\theta(\gamma_i \omega^1, \gamma_i \omega^2)} H(\gamma_i \omega^2) \leq e^{q|b|\theta^{N+\bar{n}}} H(\gamma_i \omega^2)$ and $e^{q|b|\theta^{N+\bar{n}}} < e^{\frac{1}{8}} < 2$ by (9), (17) and (18);

(b) If $\exists \omega^1, \omega^2$:

$$|h(\gamma_i \omega^1)| \leq \frac{1}{4} H(\gamma_i \omega^2) \tag{27}$$

then $\forall \tilde{\omega}^1, \tilde{\omega}^2$

$$\begin{gathered} |h(\gamma_i \tilde{\omega}^1)| \leq (\mathit{domination}) \\ |h(\gamma_i \omega^1)| + q|b|\theta^{N+\bar{n}} H(\gamma_i \omega^1) \leq \quad (\mathit{domination}) \\ |H(\gamma_i \omega^1)| + q|b|\theta^{N+\bar{n}} H(\gamma_i \omega^1) \leq \quad (27) \\ \frac{1}{4} H(\gamma_i \omega^2) + q|b|\theta^{N+\bar{n}} H(\gamma_i \omega^1) \leq \quad (a) \\ 2(\frac{1}{4} + q|b|\theta^{N+\bar{n}}) H(\gamma_i \tilde{\omega}^2) \leq \quad ((9) \ \ \mathit{and} \ \ (18)) \\ (\frac{1}{2} + 2\theta^{\bar{n}}) H(\gamma_i \tilde{\omega}^2). \end{gathered}$$

Thus if

$$\theta^{\bar{n}} < \frac{1}{8} \tag{28}$$

then

$$|h(\gamma_i \tilde{\omega}^1)| \leq \frac{3}{4} H(\gamma_i \tilde{\omega}_2)$$

as claimed;

(c) $h(\gamma_i\tilde{\omega}) = h(\gamma_i\omega) + \delta(\omega, \tilde{\omega})$, where

$$|\delta(\omega, \tilde{\omega})| \leq q|b|\theta^{N+\bar{n}} H(\gamma_i\omega) \leq 4q|b|\theta^{N+\bar{n}}|h(\gamma_i\omega)|$$

Thus

$$\left|\frac{h(\gamma_i\omega)}{h(\gamma_i\tilde{\omega})} - 1\right| \leq 4q|b|\theta^{N+\bar{n}} \leq 4\theta^{\bar{n}}.$$

Thus (c) follows if

$$\theta^{\bar{n}} \leq \frac{\varepsilon_0}{400}. \blacksquare \tag{29}$$

Proof of Lemma 4.7: If for some $j \in \{1, 2\}$ (25) holds there is nothing to prove. (Recall (12).) Thus we can assume that (26) is satisfied for $j \in \{1, 2\}$. Consider some $V' \in \Sigma_{N+m}$, $W \subset V'$. Assume that

$$\exists \omega^1, \omega^2 \in \mathcal{C}_{V'} \quad \rho_1^{\varepsilon_1}(\omega^1) > 1 \quad \textit{and} \quad \rho_2^{\varepsilon_1}(\omega^2) > 1. \tag{30}$$

We claim that in this case

$$|\mathrm{Arg}(e^{ib\tau_{\bar{n}}(\gamma_1\omega)} h(\gamma_1\omega)) - \mathrm{Arg}(e^{ib\tau_{\bar{n}}(\gamma_2\omega)} h(\gamma_2\omega))| \leq \frac{\varepsilon_0}{10}. \tag{31}$$

To prove (31) suppose that for some $\omega^0 \in V'$ $|h(\gamma_1\omega^0)| > |h(\gamma_2\omega^0)|$. It follows from (26) that for all $\omega \in V'$

$$\frac{1}{4} \leq \frac{h(\gamma_j\omega)}{h(\gamma_j\omega^0)} \leq 4.$$

Thus $|h(\gamma_1\omega)| > \frac{1}{16}|h(\gamma_2\omega)|$. Now $\rho_2^{\varepsilon_1}(\omega^2) > 1$ implies (recall (12))

$$\left|\mathrm{Arg}(e^{ib\tau_{\bar{n}}(\gamma_1\omega^2)} h(\gamma_1\omega^2)) - \mathrm{Arg}(e^{ib\tau_{\bar{n}}(\gamma_2\omega^2)} h(\gamma_2\omega^2))\right| \leq \frac{\varepsilon_0}{100}$$

Combining this with Proposition 4.9 (c) and Proposition 4.3 we get

$$|\mathrm{Arg}(e^{ib\tau_{\bar{n}}(\gamma_1\omega)} h(\gamma_1\omega)) - \mathrm{Arg}(e^{ib\tau_{\bar{n}}(\gamma_2\omega)} h(\gamma_2\omega))| \leq$$

$$\frac{\varepsilon_0}{100} + |b|\left|\tau_{\bar{n}}(\gamma_1\omega) - \tau_{\bar{n}}(\gamma_1\omega^2)\right| + |b|\left|\tau_{\bar{n}}(\gamma_2\omega) - \tau_{\bar{n}}(\gamma_2\omega^2)\right| +$$

$$\left|\mathrm{Arg}\left(\frac{h(\gamma_1\omega)}{h(\gamma_1\omega^2)}\right)\right| + \left|\mathrm{Arg}\left(\frac{h(\gamma_2\omega)}{h(\gamma_2\omega^2)}\right)\right| \leq$$

$$\frac{\varepsilon_0}{100} + \frac{2|b|||\tau||\theta^{N+m}}{1-\theta} + \frac{2\varepsilon_0}{100} \leq$$

$$\frac{2\varepsilon_0}{25}$$

which proves (31). Let now $V, \tilde{V}$ be a pair satisfying the conditions of Lemma 4.8. Suppose that (31) holds true both on $\mathcal{C}_V$ and on $\mathcal{C}_{\tilde{V}}$. Then $\forall \omega \in \mathcal{C}_V, \tilde{\omega} \in \mathcal{C}_{\tilde{V}}$

$$\left| b\Phi^{(\bar{n})}(\omega, \tilde{\omega}) + \text{Arg}\left(\frac{h(\gamma_1\omega)}{h(\gamma_1\tilde{\omega})}\right) - \text{Arg}\left(\frac{h(\gamma_2\omega)}{h(\gamma_2\tilde{\omega})}\right)\right| \leq \frac{\varepsilon_0}{5}$$

Combining this with Proposition 4.9(c) we obtain $|b||\Phi^{(\bar{n})}(\omega, \tilde{\omega})| \leq \frac{11}{50}\varepsilon_0$ which contradicts (21). Hence (30) is false either on V or on $\tilde{V}$ as claimed. ■

This concludes the proof of Lemma 4.1. Finally note that the requirements on $\bar{n}$ are given in inequalities (20), (22), (28) and (29) so that (15) suffices to satisfy them.

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