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FROM A WAVELET AUDITORY MODEL TO DEFINITIONS OF THE FOURIER TRANSFORM

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ABSTRACT. A wavelet auditory model (WAM) is formulated. The implementation of WAM for speech compression depends on an irregular sampling theorem and an analysis of time-scale data. The time-scale plane for WAM is analogous to Gabor's dissection of the information plane by means of the uncertainty principle inequality. Generalizations of this inequality lead to other dissections of the information plane; and their proofs depend on weighted Fourier transform norm inequalities. These inequalities give rise to definitions of the Fourier transform on weighted Lebesgue spaces; the definitions are sometimes necessarily different than the usual one because of the behavior of the weights.

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1. INTRODUCTION

We shall take an excursion *back to the future*. Our starting point is an important problem in speech data compression. Because of the success of the human auditory system, we have formulated an auditory model based on previously studied auditory models along with new components related to signal reconstruction, nonlinearities, and filter design. The compression algorithm we develop involves a hands-on irregular sampling formula and trade-off issues related to the uncertainty principle. The concomitant problems are difficult; and, for the sake of simplicity and structure, we shall raise the level of abstraction and ultimately deal with fairly sophisticated questions concerning the definition of the Fourier transform.

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Our auditory model is really a wavelet auditory model (WAM). Its design and implementation is the subject of *Section 2*. *Section 3* is our brief transition from difficult applications with WAM to the essential notions of sampling and uncertainty implicit in these applications and that we shall discuss mathematically in the remaining sections. Thus, the irregular sampling component of WAM motivates us to formulate a rather general irregular sampling formula in *Section 4*. In *Section 5* we analyze the mathematical structure of uncertainty principle inequalities, and describe a weighted uncertainty relationship, where ultimately the weights can be thought of as filters. In the process, we see the role of weighted Fourier transform norm inequalities for establishing uncertainty principle inequalities. Finally, in *Section 6*, these weighted Fourier transform norm inequalities lead us to some subtle questions about the definition of the Fourier transform. Our answer to these questions about the venerable, classical workaday Fourier transform depend on modern methods and recent ideas in harmonic analysis.

2. Wavelet auditory model

We shall use a model of the human auditory system and give a mathematical description of it. Our goal is to deal with compression problems in a new and effective way.

2.1. Setting. In the auditory system an acoustic signal f produces a pattern of displacements W of the basilar membrane at different locations for different frequencies. Displacements for high frequencies occur at the basal end; and for low frequencies they occur at the wider apical end inside the spiral, e.g., [G]. The signal f causes a traveling wave on the basilar membrane; and the basilar membrane records frequency responses between 200 and 20,000 Hz. (Telephone speech bandwidth deals with the range 300-4000 Hz., and modern speech research deals with the range 300-10,000 Hz.) The cochlea analyzes sound in terms of these traveling waves much like a parallel bank of filters – in this case a band with 30,000 channels. The impulse responses of these filters along the length of the cochlea are related by dilation. Consequently, their transfer functions are invariant except for a frequency translation along the approximately logarithmic axis of the cochlea, e.g., [S1].

The time-frequency information contained in the displacements has the form

$$W(t,s) \equiv (f * D_s g)(t), \quad t \in \mathbf{R}, s > 0, \tag{2.1}$$

where $D_s g(t) \equiv s^{1/2} g(st)$. W is the *continuous wavelet transform* of f, where g is the impulse response for a "cochlear" filter. The *Fourier transform* of f is defined as

$$\hat{f}(\gamma) \equiv \int f(t) e^{-2\pi i t \gamma}\, dt,$$

where integration is over $\mathbf{R}$, and $\gamma \in \hat{\mathbf{R}}(= \mathbf{R})$. Clearly,

$$(D_s g)^\wedge(\gamma) = D_{s^{-1}} \hat{g}(\gamma).$$

The shape of $\hat{g}$ is critical for the effectiveness of the auditory model. Generally, g should be a causal filter and $\hat{g}$ should be "shark-fin" shaped, e.g., [B1], [BT2]. The design problems for such filters are dealt with in [BT1].

The displacements W of (2.1) are the output of the cochlear filter bank $\{(D_s g)^\wedge\}$. In the case of properly designed filters, the high frequency edges of the cochlear filters act as abrupt "scale" delimiters. Thus, a sinusoidal stimulus will propogate up to the appropriate scale and die out beyond it. The auditory system does not receive the wavelet transform directly, but, rather, a substantially modified version of it. In fact, in the next step of the human auditory process, the output of each cochlear filter is effectively high-passed by the velocity coupling between the cochlear membrane and the cilia of the hair cell transducers that initiate the electrical nervous activity by a shearing action on the tectorial membrane. Thus, the mechanical motion of the basilar membrane is converted to a receptor potential in the inner hair cells. It is reasonable to approximate this stage by a time derivative, obtaining the output $\partial_t W$. Because of the structure of the cochlear filter bank, we choose $s_k = a^k, k \in \mathbb{Z}$, for a given $a > 1$. The extrema of the wavelet transform $W(t, s_k)$ become the zero-crossings of the new function $\partial_t W$; and the output of auditory process at this stage is

$$\forall k \in \mathbb{Z}, \quad Z_k \equiv \{t(n; s_k) : \partial_t W(t(n; s_k), s_k) = 0\}.$$

Next, instantaneous sigmoidal non-linearities are applied, e.g., [B1], [MY], [S2]; but for the present discussion of compression it is not essential to analyze them, although they involve some inherently interesting mathematical problems when dealing with representation.

The human auditory nerve patterns determined by $W, \partial W$, and Z_k are now processed by the brain in ways that are not completely understood. One processing model, the lateral inhibitory network (LIN), has been closely studied with a view to extracting the spectral pattern of the acoustic stimulus [MY], [YWS]; and we shall implement it in our algorithm. Scientifically, it reasonably reflects proximate frequency channel behavior, and mathematically it is realtively simple.

For a given acoustic signal f, constant $a > 1$, and properly designed causal filter g, we generate Z_k and the set

$$\{\partial_s \partial_t W(t(n; s_k), s_k) : t(n; s_k) \in \mathbb{Z}_k\} \tag{2.2}$$

for each $k \in \mathbb{Z}$. The scaling partial ∂_s reflects LIN, e.g., [B1].

2.2. The WAM problem and solution. Let g be a properly designed finite energy causal filter and let $a > 1$. Suppose an unknown acoustic signal f has generated the set (2.2), and that the receiver has knowledge of (2.2) or of some subset. WAM designates "wavelet auditory model" and (2.2) is *WAM data*. Since WAM data is an irregular array in the $t - s$ plane it is natural for the receiver to reconstruct f by irregular sampling formulas. This is our *WAM problem*; and our proposed solution was made in [B1] by means of irregular sampling formulas we developed wilth Heller, e.g., [BH], [B2]. The fact that the problem can be solved theoretically by means of such formulas leaves open the problem of effective implementation. Thus, we want to solve the WAM problem at the practical level of reconstructing real speech data by means of WAM data or "compressed" subsets of WAM data. We now sketch this latter solution at a technical level between the original theoretical program and the actual computer implementation, e.g., [BT1], [BT2].

2.3. Discretization. WAM data (2.2) can be approximated by a factor or normalization of

$$\partial_t W(t(n;s_k),s_{k+1}) - \partial_t W(t(n;s_k),s_k); \tag{2.3}$$

and the second derivative in (2.3) vanishes since $t(n;s_k) \in Z_k$. Thus, we can approximate and calculate as follows:

$$\begin{aligned} \partial_s\partial_t W(t(n;s_k),s_k) &\approx \partial_t W(t(n;s_k),s_{k+1}) \\ &= \partial_t(f * D_{s_{k+1}}g)(t(n;s_k)) = (f * \partial_t D_{s_{k+1}}g)(t(n;s_k)) \\ &= \langle f, -\tau_{t(n;s_k)}(\partial_t D_{s_{k+1}}\tilde{g})\rangle, \end{aligned} \tag{2.4}$$

where $\langle \dots, \dots \rangle$ is the usual inner product in a Hilbert space H considered as a subspace of $L^2(\mathbb{R})$, and where $\tilde{g}$ is the involution of g.

2.4. Frame decompositions. Because of the WAM problem and the form of WAM data, the calculation (2.4) prompts us to consider frame properties of the functions

$$\psi_{m,n}(t) \equiv \tau_{t(n,s_m)}(\partial_t D_{s_{m+1}}\tilde{g})(t),$$

where $m \in \mathbb{Z}$, and for each m there are generally countably many n. In fact, it is reasonable to take H as a Paley-Wiener space

$$PW_\Omega \equiv \{f \in L^2(\mathbb{R}) : \text{ supp } \hat{f} \subseteq [-\Omega,\Omega]\},$$

which guarantees at most countably many n for each m. We refer to [D], [HW], [B2] for the theory of frames.

An easy calculation shows that

$$\psi_{m,n}(t) = s_{m+1} D_{s_{m+1}} \tau_{t_n}(\partial_t \tilde{g})(t), \tag{2.5}$$

where $t_n \equiv t(n;s_m)$. It is natural to ask to what extent $\partial\tilde{g}$ is a "wavelet". In fact, the vanishing moment property,

$$\int \partial_t \tilde{g}(t)\,dt = 0,$$

which is critical for wavelets ψ, e.g., [Me], is valid if $\tilde{g}, \partial_t\tilde{g} \in L^1(\mathbb{R})$, and $\partial_t g$ exists everywhere, e.g., [B3, page 151].

As far as calculating whether the translates and dilates $\{\psi_{m,n}\}$ form a wavelet frame, we calculate

$$\sum_{m,n} |\langle f, \psi_{m,n}\rangle|^2 = \sum_m \sum_n s_{m+1}^2 |\langle (D_{s_{m+1}}\hat{f})\overline{(\partial\tilde{g})^\wedge}, e_{-t_n}\rangle|^2, \tag{2.6}$$

where $e_t(\gamma) \equiv e^{2\pi i t \gamma}$. It is important to note, and a priori not good information vis a vis decompositions for large spaces, that $\{t(n; s_m)\}$ is dependent on f. On the other hand, our goal is only to reconstruct f. In any case, we assume that

$$\{e_{-t(n;s_m)} : n\} \tag{2.7}$$

is a frame for H with frame bounds A_m and B_m, for each fixed $m \in \mathbb{Z}$. Because of (2.6), we see that $\{\psi_{m,n}\}$ is a frame for H if there are constants A and B so that

$$A \leq \sum_m A_m s_{m+1}^3 |D_{s_{m+1}^{-1}} (\partial \tilde{g})^\wedge(\lambda)|^2 \text{ a.e.}$$

and

$$\sum_m B_m s_{m+1}^3 |D_{s_{m+1}^{-1}} (\partial \tilde{g})^\wedge(\lambda)|^2 \leq B \text{ a.e.}$$

There are interesting mathematical questions to be answered centering around frame properties of (2.7) and estimating constants A_m and B_m, cf., the more thorough calculation in [BT1].

2.5. Iterative reconstruction method. We suppose $\{\psi_{m,n}\}$ is a frame for H with frame bounds A and B, and we define the topological isomorphism $S : H \to H$ as $Sh \equiv \sum \langle h, \psi_{m,n} \rangle \psi_{m,n}$, recalling that $\{\psi_{m,n}\}$ depends on f as well as the known filter g and constant $a > 1$.

We define the Bessel map $L : \ell^2(\mathbb{Z}^2) \to H$ as $Lh = \{\langle h, \psi_{m,n} \rangle\}$, noting that $S = L^* L$, where L^* is the adjoint of L.

It is well-known that if $\|Id - \lambda S\| < 1$ then

$$h = \sum_{k=0}^{\infty} (Id - \lambda S)^k (\lambda S) h. \tag{2.8}$$

We can take $\lambda \equiv 2/(A+B)$ since

$$\|Id - \frac{2}{A+B} S\| \leq \frac{B-A}{A+B} < 1.$$

An induction argument shows that

$$\lambda \sum_{k=0}^{\infty} L^*(Id - \lambda L L^*)^k L h = \lambda \sum_{k=0}^{\infty} (Id - \lambda L^* L)^k L^* L h. \tag{2.9}$$

Combining (2.8) and (2.9) we have

$$h = \lambda L^* \left(\sum_{k=0}^{\infty} (Id - \lambda L L^*)^k \right) L h. \tag{2.10}$$

Now recall that for the WAM problem, being given WAM data (2.2) is equivalent to knowing Lf. As such we set up the following iterative scheme to compute f when we are given Lf.

Set $f_0 \equiv 0$ and $c_0 \equiv Lf$. Define $h_n \equiv L^* c_n, c_{n+1} \equiv c_n - \lambda L h_n$, and $f_{n+1} \equiv f_n + h_n$. An induction argument shows that

$$\forall n, \quad f_{n+1} = L^* \left(\sum_{k=0}^{n} (Id - \lambda L L^*)^k \right) c_0. \tag{2.11}$$

The right side of (2.11) is computable, and because of (2.10), we know that $\lim \lambda f_n = f$. Thus, f_{n+1} in (2.11) is an approximation of f; and we have solved the WAM problem at this level of implementation. These methods are fully developed in [BT1], [BT2], and we have made applications to compression and signal identification (of signals in noise) problems in these references.

3. WAM: Sampling and uncertainty

The approximation (2.11) is really an irregular sampling formula, and it is natural to seek effective irregular sampling formulas for other applications besides WAM. Thus, we are lead to establishing a theory of irregular sampling.

Implementation of our sampling formulas can only be applicable if various trade-offs can be successfully made. The trade-offs with which we are dealing in WAM concern thresholding for compression problems, the size of the dilation constant a, and the bandwidth and complexity of the filter g, e.g., [BT1], [BT2]. Such trade-offs are the germ of the uncertainty principle; and we are lead to establishing a theory of uncertainty principle inequalities.

At a computational level, there is also a relationship between sampling and uncertainty intertwined in computing W. In this case, f and g must be sampled in such a way that W can be computed with *accuracy* and *speed*. Accuracy is a function of proper sampling and "robust" decomposition formulas. Requirements of speed can deter from accuracy analogous to the model of the classical uncertainty principle inequality in terms of position and momentum. This relationship between sampling and uncertainty is the subject of [B2], which in turn was inspired by profound insights on the subject by Gabor [Ga].

4. Sampling

We begin by stating the classical sampling theorem:

4.1. Theorem. *Let $T, \Omega > 0$ be constants for which $2T\Omega \leq 1$. Then*

$$\forall f \in PW_\Omega, \quad f = T \sum f(nT) \frac{\sin 2\pi\Omega(t - nT)}{\pi(t - nT)},$$

where the convergence is uniform on $\mathbb{R}$ *and in* $L^2(\mathbb{R})$.

4.2. Definition. Let $\{t_n : n \in \mathbf{Z}\} \subseteq \mathbf{R}$ be a strictly increasing sequence for which $\lim_{n\to\pm\infty} t_n = \pm\infty$, and for which

$$\exists d > 0 \quad \text{such that} \quad \forall m \neq n, |t_m - t_n| \geq d. \tag{4.1}$$

Sequences satisfying (4.1) are *uniformly discrete*. A uniformly discrete sequence $\{t_n\}$ is *uniformly dense* with *uniform density* $\Delta > 0$ if

$$\exists L > 0 \quad \text{such that} \quad \forall n \in \mathbf{Z}, |t_n - \frac{n}{\Delta}| \leq L.$$

A central result for the modern theory of irregular sampling is the remarkable Duffin-Schaeffer theorem [DS].

4.3. Theorem. *Let $\{t_n : n \in \mathbf{Z}\} \subseteq \mathbf{R}$ be a uniformly dense sequence with uniform density Δ. If $0 < 2\Omega < \Delta$ then $\{e_{-t_n}(\gamma)\}$ is a frame for $L^2[-\Omega,\Omega]$.*

After 40 years, *Theorem 4.3* is still difficult to prove. Among other notions and estimates, its proof involves fundamental properties of entire functions of exponential type associated with the work of Plancherel-Pólya and Boas.

Using the Duffin-Schaeffer theorem (*Theorem 4.3*) for one direction, Jaffard [J] has provided the following characterization of frames $\{e_{-t_n}\}$ for $L^2[-\Omega,\Omega]$.

4.4. Theorem. *Let $\{t_n\} \subseteq \mathbf{R}$ be a strictly increasing sequence for which $\lim_{n\to\pm\infty} t_n = \pm\infty$, and let $I \subseteq \mathbf{R}$ denote an interval.*

a. *The following two assertions are equivalent:*
   i. *There is $I \subseteq \mathbf{R}$ for which $\{e_{-t_n}\}$ is a frame for $L^2(I)$;*
   ii. *The sequence $\{t_n\}$ is a disjoint union of a uniformly dense sequence with uniform density Δ and a finite number of uniformly discrete sequences.*
b. *In the case assertion ii of part a holds, then $\{e_{-t_n}\}$ is a frame for $L^2(I)$ for each $I \subseteq \mathbf{R}$ for which $|I| < \Delta$.*

The following general irregular sampling formula originated in [BH].

4.5. Theorem. *Suppose $\Omega > 0$ and $\Omega_1 > \Omega$, and let $\{t_n\} \subseteq \mathbf{R}$ be a strictly increasing sequence for which $\lim_{n\to\pm\infty} t_n = \pm\infty$. Assume the sequence $\{t_n\}$ is a disjoint union of a uniformly dense sequence with uniform density $\Delta > 2\Omega_1$ and a finite number of uniformly discrete sequences. Let $s \in L^2(\mathbf{R})$ have the properties that $\hat{s} \in L^\infty(\hat{\mathbf{R}})$, $\operatorname{supp}\hat{s} \subseteq [-\Omega_1,\Omega_1]$, and $\hat{s} = 1$ on $[-\Omega,\Omega]$.*

a. *The sequence $\{e_{-t_n}\}$ is a frame for $L^2[-\Omega_1,\Omega_1]$ with frame operator S.*
b. *Each $f \in PW_\Omega$ has the representation*

$$f = \sum c_f[n]\tau_{t_n}s \quad \text{in } L^2(\mathbf{R}),$$

where

$$c_f[n] \equiv \langle S^{-1}(\hat{f}\,\mathbf{1}_{(\Omega_1)}), e_{-t_n}\rangle_{[-\Omega_1,\Omega_1]}$$

and

$$\{c_f[n]\} \in \ell^2(\mathbf{Z}).$$

Proof. Part a is a restatement of *Theorem 4.4.*

For part b, since $\{e_{-t_n}\}$ is a frame for $L^2[-\Omega_1,\Omega_1]$ and supp $\hat{f} \subseteq [-\Omega,\Omega]$, we have

$$\hat{f} = \hat{f}\,\mathbf{1}_{(\Omega_1)} = \sum \langle S^{-1}(\hat{f}\,\mathbf{1}_{(\Omega_1)}), e_{-t_n}\rangle_{[-\Omega_1,\Omega_1]} e_{-t_n} \quad \text{in } L^2[-\Omega_1,\Omega_1]. \tag{4.2}$$

Equation (4.2) is a direct consequence of the frame decomposition formula, e.g., [B2, Equation (2.3)], and the fact that S^{-1}, being a positive operator, is self-adjoint. Using the hypothesis, $f \in PW_\Omega$, we can rewrite (4.2) as

$$\hat{f} = \sum c_f[n](e_{-t_n}\mathbf{1}_{(\Omega_1)}) \quad \text{in } L^2(\hat{\mathbb{R}}). \tag{4.3}$$

In fact,

$$\begin{aligned}\|\hat{f} - \sum_{-M}^{N} c_f[n](e_{-t_n}\mathbf{1}_{(\Omega_1)})\|_2^2 &\\ &= \int_{\Omega}^{\Omega} |\hat{f}(\gamma) - \sum_{-M}^{N} c_f[n]e_{-t_n}(\gamma)|^2\, d\gamma\\ &= \|\hat{f} - \sum_{-M}^{N} c_f[n]e_{-t_n}\|^2_{L^2[-\Omega_1,\Omega_1]}.\end{aligned}$$

Next, we note that $\hat{f} = \hat{f}\hat{s}$, and, hence,

$$\begin{aligned}\|\hat{f} - \sum_{-M}^{N} c_f[n](e_{-t_n}\hat{s})\|_2^2 &\\ &= \|\hat{f}\hat{s} - \sum_{-M}^{N} c_f[n](e_{-t_n}\mathbf{1}_{(\Omega_1)})\hat{s}\|_2^2\\ &\leq \|\hat{s}\|_\infty^2 \|\hat{f} - \sum_{-M}^{N} c_f[n](e_{-t_n}\mathbf{1}_{(\Omega_1)})\|_2^2.\end{aligned}$$

Using this estimate, equation (4.3), and the hypotheses on s, we obtain

$$\hat{f} = \sum c_f[n](e_{-t_n}\hat{s}) \quad \text{in } L^2(\hat{\mathbb{R}}). \tag{4.4}$$

The proof is completed by taking the Fourier transform in equation (4.4). ∎

Theorem 4.5 is a *sampling theorem* in the conventional sense that the coefficients $c_f[n]$ can be described in terms of values of f on the sampling set $\{t_m\}$. However, because a given sampling set is irregularly spaced in *Theorem 4.5*, we can not, in general, write $c_f[n] = f(t_n)$.

5. Uncertainty

We begin by stating the classical uncertainty principle inequality.

5.1. Theorem. *Let* $(t_0, \gamma_0) \in \mathbf{R} \times \hat{\mathbf{R}}$. *Then*

$$\forall f \in \mathcal{S}(\mathbf{R}), \qquad \|f\|_2^2 \leq 4\pi \|(t - t_0)f(t)\|_2 \|(\gamma - \gamma_0)\hat{f}(\gamma)\|_2, \tag{5.1}$$

and there is equality in (5.1) if and only if

$$f(t) = Ce^{2\pi i t \gamma_0} e^{-s(t-t_0)^2}$$

for $C \in \mathbf{C}$ *and* $s > 0$.

The proof results from an elementary calculation involving integration by parts, Hölder's inequality, and the Plancherel theorem, e.g., [B2]. There are weighted generalizations of (5.1) and the main ingredients for their proofs are the same: integration by parts or conceptually similar ideas such as generalizations of Hardy's inequality, Hölder's inequality, and weighted norm inequalities for the Fourier transform, of which the Plancherel theorem is a special case. We now present one such generalization, cf., [B2] for others.

We shall deal with weighted spaces of the form

$$L_v^p(\mathbf{R}^d) = \{f : \mathbf{R}^d \to \mathbf{C} : \|f\|_{p,v} \equiv \left(\int |f(t)|^p \, v(t) dt \right)^{1/p} < \infty\},$$

where $v > 0$ a.e.

5.2. Definition. The *Hardy operator* is the positive linear operator P_d defined as

$$P_d(f)(x) = \int_0^{x_d} \cdots \int_0^{x_1} f(t_1, \cdots, t_d) dt_1 \cdots dt_d \equiv \int_{\langle 0,x \rangle} f(t)\, dt$$

for Borel measurable functions f on $\mathbf{R}^{+d}$. The *dual Hardy operator* P_d' is defined as

$$P_d'(f)(x) = \int_{x_d}^{\infty} \cdots \int_{x_1}^{\infty} f(t_1, \cdots, t_d) dt_1 \cdots dt_d \equiv \int_{\langle x,\infty \rangle} f(t)\, dt,$$

where $x > 0$, i.e., each $x_j > 0$ for $x = (x_1, \cdots, x_d)$.

Hardy's inequality (1920) is

$$\int_0^{\infty} P_1(f)(t)^p t^{-p}\, dt < \left(\frac{p}{p-1} \right)^p \int_0^{\infty} f(t)^p\, dt, \tag{5.2}$$

where $p > 1$ and $f \geq 0 (f \not\equiv 0)$ is Borel measurable.

5.3. Lemma. *[He, Theorem 3.1]. Given $1 < p \le q < \infty$ and non-negative Borel measurable functions u and v on $X \subseteq \mathbf{R}^d$. Suppose $P : L^p_v(X) \to L^q_u(X)$ is a positive linear operator with canonical dual operator $P' : L^{q'}_{u^{-q'/q}}(X) \to L^{q'}_{v^{-p'/p}}(X)$ defined by the duality $\int_X P(f)(x)g(x)\,dx = \int_X f(x)P'(g)(x)\,dx$. Assume there exist $K_1, K_2 > 0$ such that*

$$\forall g \in L^{(q/p)'}(X), \quad \text{for which} \quad g \ge 0 \quad \text{and} \quad \|g\|_{(q/p)'} \le 1,$$

there are non-negative functions,

$$f_1 \in L^p_v(X),\ h_1 \in L^p_{u^{p/q}g}(X),\ f_2 \in L^{p'}_{u^{-p'/q}g}(X),\ h_2 \in L^{p'}_{v^{-p'/p}}(X),$$

with the properties

$$P(f_1) \le K_1 h_1 \quad \text{and} \quad P'(f_2 g) \le K_2 h_2 \tag{5.3}$$

and

$$v = f_1^{-p/p'} h_2 \quad \text{and} \quad u = h_1^{-q/p'} f_2^{q/p}.$$

Then $P \in \mathcal{L}(L^p_v(X), L^q_u(X))$, $P' \in \mathcal{L}(L^{q'}_{u^{-q'/q}}(X), L^{p'}_{v^{-p'/p}}(X))$, and $\|P\|, \|P'\| \le K_1^{1/p'} K_2^{1/p}$. ∎

Setting

$$\begin{aligned}
f_1 &= v^{-p'/p} P_d(v^{-p'/p})^{-1/p},\\
h_1 &= P_d(v^{-p'/p})^{-1/p'},\\
f_2 &= u^{p/q} P'_d(u)^{-p/(qp')},\\
h_2 &= P_d(v^{-p'/p})^{-1/p'},
\end{aligned}$$

it is easy to verify (5.3) for any non-negative $g \in L^{(q/p)'}(\mathbf{R}^{+d})$, for which $\|g\|_{(q/p)'} \le 1$, as long as (5.4), (5.5), and (5.6) are assumed. As a result Hernandez obtained the following version of Hardy's inequality on $\mathbf{R}^{+d}$.

5.4. Theorem *[He, Section 4.2]. Given $1 < p \le q < \infty$ and non-negative Borel measurable functions u and v on $\mathbf{R}^{+d}$. Assume there exist K, $C_1(p), C_2(p) > 0$ such that*

$$\sup_{s>0} \left(\int_{\langle s,\infty\rangle} u(x)\,dx \right)^{1/q} \left(\int_{\langle 0,s\rangle} v(x)^{-p'/p}\,dx \right)^{1/p'} = K, \tag{5.4}$$

$$\begin{aligned}
\forall x \in \mathbf{R}^{+d}, \quad & P_d(v^{-p'/p} P_d(v^{-p'/p})^{-1/p})(x)\\
& \le C_1(p) P_d(v^{-p'/p})^{-1/p'},
\end{aligned} \tag{5.5}$$

and

(5.6)
$$\forall x \in \mathbb{R}^{+d}, \quad P_d'(u(P_d'u)^{-1/p'})(x), \\ \le C_2(p)^{q/p}(P_d'u)^{1/p}.$$

Then $P_d \in \mathcal{L}(L_v^p(\mathbb{R}^{+d}), L_u^q(\mathbb{R}^{+d}))$, $P_d' \in \mathcal{L}(L^{q'}_{u^{-q'/q}}(\mathbb{R}^{+d}), L^{p'}_{v^{-p'/p}}(\mathbb{R}^{+d}))$, *and* $\|P_d\|, \|P_d'\| \le KC_1(p)^{1/p'}C_2(p)^{1/p}$.

Let Ω be the subgroup of the orthogonal group whose corresponding matrices with respect to the standard basis are diagonal with ± 1 entries. Each element $\omega \in \Omega$ can be identified with an element $(\omega_1, \cdots, \omega_d) \in \{-1, 1\}^d$, and $\omega\gamma = (\omega_1\gamma_1, \cdots, \omega_d\gamma_d)$. Thus,

$$\int F(\gamma)\, d\gamma = \sum_{\omega \in \Omega} \int_{\hat{\mathbb{R}}^{+d}} F(\omega\gamma)\, d\gamma,$$

and since

$$\sum_{\omega\in\Omega} a_\omega^{1/r} b_\omega^{1/r'} \le \left(\sum_{\omega\in\Omega} a_\omega\right)^{1/r} \left(\sum_{\omega\in\Omega} b_\omega\right)^{1/r'},$$

for $1 < r < \infty$ and $a_\omega, b_\omega \ge 0$, we have the following regrouping lemma.

5.5. Lemma. *Given* $1 < r < \infty$ *and suppose* $F \in L^r(\hat{\mathbb{R}}^d), G \in L^{r'}(\hat{\mathbb{R}}^d)$. *Then*

$$\sum_{\omega\in\Omega} \left(\int_{\hat{\mathbb{R}}^{+d}} |F(\omega\gamma)|^r\, d\gamma\right)^{1/r} \left(\int_{\hat{\mathbb{R}}^{+d}} |G(\omega\gamma)|^{r'}\, d\gamma\right)^{1/r'} \le \|F\|_r \|G\|_{r'}.$$

5.6. Definition. $\mathcal{S}_{oa}(\mathbb{R}^d) \equiv \{f \in \mathcal{S}(\mathbb{R}^d) : \hat{f}(\gamma) = 0 \text{ if some } \gamma_j = 0\} \subseteq \mathcal{S}_o(\mathbb{R}^d)$. Thus, $f \in \mathcal{S}(\mathbb{R}^d)$ is an element of $\mathcal{S}_{oa}(\mathbb{R}^d)$ if $\hat{f} = 0$ on the coordinate axes.

Combining Hardy's inequality (*Theorem 5.4*) and the regrouping lemma (*Lemma 5.5*) we obtain the following uncertainty principle inequality.

5.7. Theorem. *Given* $1 < r < \infty$ *and non-negative Borel measurable weights* v *and* w. *Suppose* $u = w^{-r'/r}$, *and assume that, for all* $\omega \in \Omega$, *the weights* $u(\omega\gamma)$ *and* $v(\omega\gamma)$ *satisfy conditions (5.4), (5.5), and (5.6) on* $\hat{\mathbb{R}}^{+d}$ *for* $p = q = r'$ *and constants* $K(\omega), C_1(p, \omega)$, *and* $C_2(p, \omega)$. *If* $C = \sup_{\omega\in\Omega} K(\omega)C_1(p,\omega)^{1/p'}C_2(p,\omega)^{1/p}$ *then*

(5.7)
$$\forall f \in \mathcal{S}_{oa}(\mathbb{R}^d), \quad \|f\|_2^2 \le C\|\hat{f}\|_{r,w}\|\partial_1 \cdots \partial_d \hat{f}\|_{r',v}.$$

The right side of (5.7) will have the form of the right side of (5.1) in the case

(5.8)
$$\|\partial_1 \cdots \partial_d \hat{f}\|_{r',v} \le C\|t_1 \cdots t_d f\|_{s,u}.$$

Thus, *weighted Fourier transform norm inequalities* such as (5.8) are *critical for obtaining generalizations of the classical uncertainty principle*. As such we shall turn our attention to such inequalities in the next section.

Theorem 5.7 seems burdened with laborious hypotheses. There are some attractive corollaries. For example, we can take $v \equiv 1$, $w(\gamma) \equiv |\gamma_1 \cdots \gamma_d|^r$, $1 < r \le 2$, and prove –

5.8. Corollary. *If $1 < r \le 2$ then*

$$\forall f \in \mathcal{S}_{oa}(\mathbf{R}^d), \quad \|f\|_2^2 \le (2\pi r)^d B_d(r) \|t_1 \cdots t_d f(t)\|_r \|\gamma_1 \cdots \gamma_d \hat{f}(\gamma)\|_r,$$

where $B_d(r) \equiv (r^{1/r}(r')^{-1/r'})^{d/2}$ is the Babenko (1961)-Beckner (1975) constant.

6. Weighted Fourier transform norm inequalities

A weighted Fourier transform norm inequality has the form

$$\forall f \in X, \quad \|\hat{f}\|_{q,\mu} \le C\|f\|_{p,v}, \tag{6.1}$$

where $v > 0$ a.e., μ is a positive measure, $X \subseteq L^1(\mathbf{R}^d) \cap L^p_v(\mathbf{R}^d)$, and $\bar{X} = L^p_v(\mathbf{R}^d)$. Conditions for establishing (6.1) are fairly well understood. For example, in 1982 Heinig and I proved that if $\mu = u > 0$ a.e., $1 < p \le q < \infty$, u and v even, and $1/u$ and v increasing on $(0, \infty)$, then (6.1) is valid for $X \equiv \mathcal{S}(\mathbf{R}) \cap L^p_v(\mathbf{R})$ if and only if

$$\sup_{s>0} \left(\int_0^{1/s} u(\gamma)\, d\gamma \right)^{1/q} \left(\int_0^s v(t)^{-p'/p}\, dt \right)^{1/p'} < \infty,$$

cf., [H], [JS], [Mu] for related early results and [B2], [BL] for more recent developments.

In the case an inequality such as (6.1) is valid, there is a unique continuous linear map,

$$\mathcal{F} : L^p_v(\mathbf{R}^d) \longrightarrow L^q_\mu(\mathbf{R}^d),$$

with the property that $\mathcal{F}f = \hat{f}$ for all $f \in X$.

6.1 Question. Under what circumstances can we say that $\mathcal{F}f = \hat{f}$?

This question is quantified and partially answered in [BL]. The remainder of this section is taken from [BL], and we deal with one special case. Although [BL] develops a fairly large theory, there are still many questions to be answered.

6.2 Theorem. *Let $u \equiv \mu$ and v be locally integrable and positive a.e., let $1 < p, q < \infty$, and assume (6.1) and $L^p_v(\mathbf{R}^d) \subseteq \mathcal{S}'(\mathbf{R}^d)$. Suppose the following weak uncertainty inequality:*

$$\forall g \in \mathcal{S}(\mathbf{R}^d), \quad \|\hat{g}\|_{q', u^{1-q'}} \|g\|_{p', v^{1-p'}} < \infty. \tag{6.2}$$

If $f \in L^p_v(\mathbf{R}^d)$ then $\mathcal{F}f$ agrees with the ordinary Fourier transform $\hat{f}$ in the sense of tempered distributions.

Proof. Let $f \in L^p_v(\mathbf{R}^d), \lim \|f_n - f\|_{p,v} = 0$ for $\{f_n\} \subseteq X$, and $g \in \mathcal{S}(\mathbf{R}^d)$. Using (6.1) we compute

$$\begin{aligned} |\langle \mathcal{F}f - \hat{f}, \hat{g}\rangle| &\le |\langle \mathcal{F}f - \hat{f}_n, \hat{g}\rangle| + |\langle \hat{f}_n - \hat{f}, \hat{g}\rangle| \\ &\le C\|f_n - f\|_{p,v}(\|\hat{g}\|_{q',u^{1-q'}} + \|g\|_{p',v^{1-p'}}), \end{aligned}$$

and the proof is complete by (6.2). ∎

In the case of measures μ, if (6.1) is valid and $\mathcal{F}$ is defined by a limiting argument, then an essential part of the *Question* is to ask under what circumstances we can assert that

$$\forall f \in L^1(\mathbf{R}^d) \cap L^p_v(\mathbf{R}^d), \quad \mathcal{F}f(\gamma) = \hat{f}(\gamma) \ \mu \ \text{ a.e.} \tag{6.3}$$

6.3 Definition. a. Let $1 \leq p, q < \infty$. The *Wiener amalgam space* $W(L^p, \ell^q)(\mathbf{R}^d)$ is the Banach space of functions for which

$$\|f\|_{W(L^p,\ell^q)} \equiv \left(\sum_n \left(\int_{Q_n} |f(t)|^p \, dt\right)^{q/p}\right)^{1/q} < \infty,$$

where Q_n is the translate by $n \in \mathbf{Z}^d$ of the unit cube $[0,1)^d$.

b. A thorough treatment of the *Question* requires distinguishing whether or not $v^{1-p'} \in L^1_{loc}(\mathbf{R}^d)$. It turns out that if not only $v^{1-p'} \in L^1_{loc}(\mathbf{R}^d)$ but further $v^{1-p'} \in W(L^1, \ell^\infty)(\mathbf{R}^d)$, then we have the continuous imbedding,

$$\forall p \in [1,2], \quad L^p_v(\mathbf{R}^d) \subseteq W(L^1, \ell^2)(\mathbf{R}^d). \tag{6.4}$$

In fact,

$$\begin{aligned}
\left(\sum \left(\int_{Q_n} |f(t)| \, dt\right)^2\right)^{1/2} &= \left(\sum \left(\int_{Q_n} |f(t)| v(t)^{1/p} \, v(t)^{-1/p} \, dt\right)^2\right)^{1/2} \\
&\leq \left(\sum \left(\int_{Q_n} |f(t)|^p v(t) dt\right)^{2/p} \left(\int_{Q_n} v(t)^{1-p'} \, dt\right)^{2/p'}\right)^{1/2} \\
&\leq C \left(\sum \left(\int_{Q_n} |f(t)|^p v(t) \, dt\right)^{2/p}\right)^{1/2},
\end{aligned}$$

and the last term is finite since $f \in L^p_v(\mathbf{R}^d)$ and $2/p \geq 1$.

Condition (6.4) allows us to deal with (6.3) in the following result.

6.4 Theorem. *Given* $p \in [1,2], 1 \leq q < \infty, v > 0$ *a.e. and locally integrable, and* μ *a positive measure. Assume (6.1) and (6.4). Then (6.3) is valid if and only if there is a constant* C' *such that*

$$\forall f \in X \quad \text{and} \quad \forall \gamma \in \hat{\mathbf{R}}^d, \; \|\tau_\gamma \hat{f}\|_{q,\mu} \leq C' \|f\|_{p,v}. \tag{6.5}$$

The proof that (6.5) is a necessary condition is elementary and does not require (6.4). The sufficiency is more difficult to prove.

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