Key words: Single dyadic d-dimensional orthonormal wavelets, frames, wavelet set constructions, locally compact abelian group (LCAG), wavelets on LCAGs, p-adic field.

Abstract Sets Ω in d-dimensional Euclidean space are constructed with the property that the inverse Fourier transform of the characteristic function $\mathbb{1}_{\Omega}$ of the set Ω is a single dyadic orthonormal wavelet. The iterative construction is characterized by its generality, its computational implementation, and its simplicity. The construction is transported to the case of locally compact abelian groups G with compact open subgroups H. The best known example of such a group is $G = \mathbb{Q}_p$, the field of p-adic rational numbers (as a group under addition), which has the compact open subgroup $H = \mathbb{Z}_p$, the ring of p-adic integers. Fascinating intricacies arise. Classical wavelet theories, which require a non-trivial discrete subgroup for translations, do not apply to G, which may not have such a subgroup. However, our wavelet theory is formulated on G with new group theoretic operators, which can be thought of as analogues of Euclidean translations. As such, our theory for G is structurally cohesive and of significant generality. For perspective, the Haar and Shannon wavelets are naturally antipodal in the Euclidean setting, whereas their analogues for G are equivalent.

The construction of wavelet sets

John J. Benedetto and Robert L. Benedetto

1 Introduction

1.1 Background

We shall give a general method for constructing single dyadic orthonormal wavelets, which generate wavelet orthonormal bases (ONBs) for the space L^2 of square-integrable functions in two important antipodal cases. The cases are $L^2(\mathbb{R}^d)$, where $\mathbb{R}^d$ is d-dimensional Euclidean space, and $L^2(G)$, where G belongs to the class of locally compact abelian groups (LCAGs) which contain a compact open subgroup, and which are often used in number theoretic applications. The method and associ-

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ated theory for $L^2(\mathbb{R}^d)$ were introduced by Manuel León, Songkiat Sumetkijakan, and the first named author in [20] (1999), [21] (2001), [22] (2002), [23] (2003). The theory for $L^2(G)$ was established by the authors in [17] (2004), [24] (2004).

This *constructive* method, which we refer to as the *neighborhood mapping construction* (NMC) was inspired by groundbreaking operator theoretic work due to Dai and Larson [34] (1998) and Dai, Larson, and Speegle [35] (1997). There was comparably compelling work, contemporaneous to [20] (1999), [21] (2001), in abstract harmonic analysis by Baggett, Medina, and Merrill [11] (1999).

The catalyst for our original research was a preprint of the Soardi-Weiland paper [91] (1998). The aforementioned, as well as less known but equally formidable results by Zakharov [98] (1996), were aimed at establishing the existence of *single dyadic orthonormal wavelets* ψ for $L^2(\mathbb{R}^d),\ d>1,$ i.e., $\{\psi_{m,n} : m \in \mathbb{Z}, n \in \mathbb{Z}^d\}$ is an orthonormal basis (ONB) for $L^2(\mathbb{R}^d)$, where

$$\psi_{m,n}(x) = 2^{md/2}\psi(2^m x - n)\,. \tag{1}$$

It turns out that the Fourier transform of such a function ψ is the characteristic function $\mathbb{1}_\Omega$ of a set Ω, and such sets and their generalizations are called *wavelet sets*. Besides describing the NMC, we shall give a significant list of references to illustrate a range of settings and problems associated with wavelet sets, and to provide perspective about the role and extent of the NMC in wavelet theory and its applications.

For some time there was doubt about the existence of single dyadic orthonormal wavelets ψ for $\mathbb{R}^d$, $d>1$. In fact, the most common construction of wavelet ONBs was from the theory of multiresolution analysis (MRA) which requires 2^d-1 functions $\psi_j, j = 1,\ldots,2^d-1$, to generate the resulting ONB, $\{(\psi_j)_{m,n}\}$, see [81] (1990), [38] (1992), [31] (1994), [76] (1994), [43] (1997), [78] (1998), [77] (1992), [80] (1986), [48] (1992), [95] (1994), [8] (1995), [47] (1995) for MRA theory on

$\mathbb{R}^d$ or, more generally on LCAGs containing cocompact discrete subgroups, cf. the work on minimally supported wavelets [42] (1996), [53] (1996), [54] (1997). Thus, the wavelets we construct, and those in [34] (1998), [35] (1997), [11] (1999) are not derived from any MRA. On the other hand, there are unifying general approaches, e.g., [85] (1998), [11] (1999), [12] (1999), [30] (1999), [84] (1993). There are also results on wavelet theory in a variety of natural settings such as Lie groups and manifolds, sometimes coupled with structural constraints such as MRA, e.g., [72] (1989), [32] (1995), [58] (1995), [97] (1996), [69] (1996), [53] (1996) [4] (1997), [60] (1997), [70] (1998), [71] (1998), [89] (1999), [7] (1999), [52] (1999), [3] (2000), [6] (2000), [65] (2002), [83] (2002), [5] (2004), [59] (2004), [66] (2004), [61] (2005), [45] (2005), [44] (2005), [90] (2009), [62] (2009), and classical work in harmonic analysis on local fields, e.g., [96] (1975) and [37] (1983). This list contains several papers dealing with the p-adics or other local fields on which we have focused in our wavelet analysis of $L^2(G)$, so we note the explicitness, generality, computability, and algebraic cohesiveness of our approach. This will be spelled out in Sections 6–10.

Remark 1.1. a. Wavelet ONBs go far beyond the dyadic case. For example, the $d \times d$, dyadic diagonal matrix A (with 2s along the diagonal), corresponding to (1), can be replaced by real expansive $d \times d$ matrices for which $A(\mathbb{Z}^d) \subseteq \mathbb{Z}^d$. As such, (1) can be replaced by functions of the form

$$\left(\psi_j^A\right)_{m,n}(x) = |\det(A)|^{m/2}\psi_j(A^m x - n), \tag{2}$$

where $j = 1, \ldots J$, $m \in \mathbb{Z}$, $n \in \mathbb{Z}^d$, e.g., see [80] (1986), [64] (1992), [25] (1999). We mention this, since we can define non-separable filters with corresponding matrix A and $J = 1$ to obtain a single MRA wavelet ψ^A for which $\{(\psi^A)_{m,n}\}$ is an ONB for $L^2(\mathbb{R}^d)$.

b. The reason we have not chosen this path to obtain single orthonormal wavelets, in spite of the elegance of MRA, is to make use of the "zooming" property of the dyadic case. In fact, by zooming-in and -out, because of powers of 2 (or of any $n \geq 2$), we can fathom multi-scale phenomena in a function and/or control computational costs vis-à-vis signal resolution in reconstruction.

Remark 1.2. One aspect of the applicability alluded to in Remark 1.1.b is to provide another mathematical tool, along with dimension reduction techniques, for example, with which to manage massive data sets generated by data-creating devices such as supercomputers, internet traffic, CAT scanners, and digital cameras. IDC estimates that the world generated 487 billion gigabytes of information in 2008. This creates formidable problems for obtaining digital representations suitable for storage, transmission, and/or recovery, as well as for handling information accurately, efficiently, and robustly. In the Epilogue we comment on the process of useful implementation of single dyadic orthonormal wavelets for $L^2(\mathbb{R}^d)$, $d >> 0$.

1.2 Notation and outline

We shall employ the usual notation in harmonic analysis and wavelet theory as found in [15] (1997), [38] (1992), [82] (1992), and [94] (1971). The Fourier transform of the function $f : \mathbb{R}^d \longrightarrow \mathbb{C}$ is formally defined by

$$\hat{f}(\gamma) = \int f(x) e^{-2\pi i x \cdot \gamma} \, dx,$$

where $\int$ denotes integration over $\mathbb{R}^d$; and the inverse Fourier transform $F^\vee$ of $F : \widehat{\mathbb{R}}^d \longrightarrow \mathbb{C}$ is formally defined by

$$F^\vee(x) = \int F(\gamma) e^{2\pi i x \cdot \gamma} \, d\gamma, \qquad x \in \mathbb{R}^d,$$

where $\widehat{\mathbb{R}}^d$ is $\mathbb{R}^d$ considered as the spectral domain. $\mathbb{Z}$ is the ring of integers and $\mathbb{T}$ designates the quotient group $\widehat{\mathbb{R}}/\mathbb{Z}$. If F is a $1-$periodic function on $\widehat{\mathbb{R}}$ with Fourier series $S(F)(\gamma) = \sum c_n e^{-2\pi i n\gamma}$, then the Fourier coefficients c_n are designated by $F^\vee[n]$. Further, translation of a function f by x is designated by $\tau_x f$, i.e., $\tau_x f(y) = f(x-y)$. Finally, if $\Omega \subseteq \widehat{\mathbb{R}}^d$, then its Lebesgue measure is denoted by $|\Omega|$. The term *measurable* will mean *Lebesgue measurable*.

The paper is structured as follows. Sections 2–5 deal with the Euclidean theory of wavelet sets and Sections 6–10 deal with the non-Euclidean theory. Section 11, the Epilogue, briefly broadens some of the conventional perspective about wavelet sets and their genuine applicability. Generally, we refer to our original papers for the proofs of theorems. However, there are a few salient exceptions related to our opinion of what constitutes general interest, or where we deem the details or structure of the proof to be particularly informative or surprising. In addition, we present many examples.

Section 2 is devoted to the geometry of Euclidean wavelet sets, as well as to fundamental roots based in Lusin's conjecture (and thus Carleson's theorem) and the Littlewood-Paley theory. Section 3 provides the details of our neighborhood mapping construction (NMC) of wavelet sets. It is highly motivated geometrically, but ultimately rather intricate. In Section 4 we prove a basic theorem about frame wavelet sets which we view as a major means of applying wavelet sets in a host of signal processing applications dealing with large data sets. Finally, in Section 5, for the Euclidean theory, we give geometrical examples with suggestive topological implications, as well as structural implications of the NMC and a hint of the breadth and beauty of NMC constructible wavelet sets.

Early-on we were intrigued by the possibility and utility of wavelet sets in number theory, based on one of the author's ideas about idelic pseudo-measures [14] (1979), [13] (1973). Sections 6–10 are our foray into this area. The background dealing with LCAGs, the p-adic field, and generic wavelet theory in this setting is the subject of section 6. Our fundamental idea to ensure the mathematical cohesiveness and resulting group theoretic canonicity and mathematical beauty of our approach is the subject of Section 7. With this background, Section 8 gives a basic geometrical result for the number theoretic setting analogous to the point of view of Section 2. This substantive theory is the background for the number theoretic construction and algorithm of Section 9, which itself is driven by the ideas of Section 3. Finally, in Section 10, we give examples indicating the incredible breadth of the number theoretic NMC.

2 Geometry of Euclidean wavelet sets

2.1 Wavelet sets, tilings, and congruences

A set Ω, together with the property that $\psi = \mathbb{1}_{\Omega}^{\vee}$ is a single dyadic orthonormal wavelet, is a *wavelet set*. Our construction of such sets Ω is the subject of Section 3, and our basic geometrical approach is not unrelated to constructions of Leonardo da Vinci and Maurits C. Escher.

Remark 2.1. Consider $\Psi = \{\psi^1, \psi^2, \dots \psi^M\} \subseteq L^2(\mathbb{R}^d)$. We say Ψ is a *set of wavelet generators* for $L^2(\mathbb{R}^d)$ if

$$\left\{\psi^i_{m,n}(\cdot) = 2^{md/2}\psi^i(2^m \cdot - n) : m \in \mathbb{Z}, n \in \mathbb{Z}^d, i = 1, \dots, M\right\}$$

is an ONB for $L^2(\mathbb{R}^d)$. Auscher [8] (1995) proved that every set of wavelet generators for $L^2(\mathbb{R})$, whose members satisfy a weak smoothness and decay condition on the Fourier transform side, must come from an MRA. Further, it is known, e.g., see [8] (1995), [82] (1992), that for a given dyadic MRA there is a wavelet collection consisting of $2^d - 1$ elements. There is an analogous assertion for the expansive matrix case.

Because of this remark, and notwithstanding Journé's celebrated example of a non-MRA wavelet basis for $L^2(\mathbb{R})$, e.g., [38] (1992), there was some question during the mid-1990s about the existence of multidimensional single dyadic orthonormal wavelets. Dai, Larson, and Speegle [35] (1997), referenced earlier, proved the existence of such wavelets in $L^2(\mathbb{R}^d)$, $d > 1$. Their proof depended on wavelet sets and used operator algebra methods. Some of the initial reaction was a combination of disbelief and disinterest, the latter response due to the prevailing intuition that such wavelets would be difficult to implement in an effective way.

Definition 2.1. *a.* Let $\Omega \subseteq \widehat{\mathbb{R}}^d$ be measurable. A *tiling* of Ω is a collection $\{\Omega_l : l \in \mathbb{Z}\}$ of measurable subsets of $\widehat{\mathbb{R}}^d$ such that $\bigcup_l \Omega_l$ and Ω differ by a set of measure 0, and, for all $l \neq j$, $\left|\Omega_l \cap \Omega_j\right| = 0$.

b. Let $\Omega, \Theta \subseteq \widehat{\mathbb{R}}^d$ be measurable. If there exist a tiling $\{\Omega_l : l \in \mathbb{Z}\}$ of Ω and a sequence $\{k_l : l \in \mathbb{Z}\} \subseteq \mathbb{Z}^d$ such that $\{\Omega_l + k_l : l \in \mathbb{Z}\}$ is a tiling of Θ, then Ω and Θ are $\mathbb{Z}^d$*-translation congruent* or τ*-congruent*. This is equivalent to the existence of tilings $\{\Omega_l : l \in \mathbb{Z}\}$ and $\{\Theta_l : l \in \mathbb{Z}\}$ of Ω and Θ, respectively, and a sequence $\{n_l : l \in \mathbb{Z}\} \subseteq \mathbb{Z}^d$ such that $\Omega_l = \Theta_l + n_l$, for all $l \in \mathbb{Z}$.

c. Let $\Omega, \Theta \subseteq \widehat{\mathbb{R}}^d$ be measurable. If there exist a tiling $\{\Omega_l : l \in \mathbb{Z}\}$ of Ω and a sequence $\{m_l : l \in \mathbb{Z}\} \subseteq \mathbb{Z}$, where $\{2^{m_l}\Omega_l : l \in \mathbb{Z}\}$ is a tiling of Θ, then Ω and Θ are *dyadic-dilation congruent* or δ*-congruent*. This is equivalent to the existence of tilings $\{\Omega_l : l \in \mathbb{Z}\}$ and $\{\Theta_l : l \in \mathbb{Z}\}$ of Ω and Θ, respectively, and a sequence $\{m_l : l \in \mathbb{Z}\} \subseteq \mathbb{Z}$ such that $\Omega_l = 2^{m_l}\Theta_l$, for all $l \in \mathbb{Z}$.

d. We shall deal with tilings of $\widehat{\mathbb{R}}^d$ by translation or dilation of a measurable set $\Omega \subseteq \widehat{\mathbb{R}}^d$. Thus, $\{\Omega + n : n \in \mathbb{Z}^d\}$ is a *tiling* of $\widehat{\mathbb{R}}^d$ means that $|\widehat{\mathbb{R}}^d \setminus \bigcup_{n \in \mathbb{Z}^d} (\Omega + n)| = 0$ and $|(\Omega + m) \cap (\Omega + n)| = 0$ when $m \neq n$. Similarly, $\{2^m\Omega : m \in \mathbb{Z}\}$ is a *tiling* of $\widehat{\mathbb{R}}^d$ means that $|\widehat{\mathbb{R}}^d \setminus \bigcup_{m \in \mathbb{Z}} (2^m\Omega)| = 0$ and $\left|(2^j\Omega) \cap (2^m\Omega)\right| = 0$ when $j \neq m$.

e. It is not difficult to see that the concept of Ω being $\mathbb{Z}^d$-translation congruent to $\left[-\frac{1}{2},\frac{1}{2}\right]^d$ is equivalent to $\{\Omega+n : n\in\mathbb{Z}^d\}$ being a tiling of $\widehat{\mathbb{R}}^d$.

Remark 2.2. a. The notion of congruence plays a role in several facets of wavelet theory besides the results in this paper. Congruence criteria were used by Albert Cohen in 1990 to characterize the orthonormality of scaling functions defined by infinite products of dilations of a quadrature mirror filter, e.g., [38, pp.182–186] (1992). The same notion of congruence also plays a fundamental role in work on self-similar tilings of $\mathbb{R}^d$ by Gröchenig, Haas, Lagarias, Madych, Yang Wang, et al., e.g., [67] (1997), [68] (2000).

b. The notion of $\mathbb{Z}^d$-translation congruence is intrinsically related to bijective restrictions of the canonical surjection $h : G \longrightarrow G/H$, where G is a locally compact group and H is a closed subgroup. An analysis of this relation is found in [16, Section 3] (1998) in the context of Kluvánek's sampling theorem for locally compact Abelian groups. Kluvánek's sampling formula for a signal f quantitatively relates the sampling rate with the measure of the subsets of a given bandwidth corresponding to the frequency content of f.

Wavelet sets and tilings are related by the following theorem. For an elementary proof, as well as a more complicated one, see [20] (1999). The existence of wavelet sets is not obvious, and this is the point of Section 3.

Theorem 2.1. *Let $\Omega\subseteq\widehat{\mathbb{R}}^d$ be a measurable set. Ω is a wavelet set if and only if*

i. $\{\Omega+n : n\in\mathbb{Z}^d\}$ *is a tiling of* $\widehat{\mathbb{R}}^d$*, and*
ii. $\{2^m\Omega : m\in\mathbb{Z}\}$ *is a tiling of* $\widehat{\mathbb{R}}^d$.

Corollary 2.1. *Let $\Omega\subseteq\widehat{\mathbb{R}}^d$. Ω is a wavelet set if and only if Ω is $\mathbb{Z}^d$-translation congruent to $[0,1)^d$ and Ω is dyadic-dilation congruent to $[-1,1)^d\setminus\left[-\frac{1}{2},\frac{1}{2}\right)^d$.*

Definition 2.2. A collection $\Omega^1,\ldots,\Omega^L$ of measurable subsets of $\widehat{\mathbb{R}}^d$ is a *wavelet collection of sets* if $\{\mathbb{1}^{\vee}_{\Omega^1},\ldots,\mathbb{1}^{\vee}_{\Omega^L}\}$ is a set of wavelet generators for $L^2(\mathbb{R}^d)$.

We have the following generalization of Theorem 2.1. It should be compared with Theorem 8.1, whose more complicated proof is included.

Theorem 2.2. *Let $\Omega^1, \dots, \Omega^L$ be pairwise disjoint measurable subsets of $\widehat{\mathbb{R}}^d$. The family $\{\Omega^l : l = 1, \dots, L\}$ is a wavelet collection of sets if and only if each $|\Omega^l| = 1$ and the following conditions are satisfied:*

i. For each fixed $l = 1, \dots, L$, $\{\Omega^l + k : k \in \mathbb{Z}^d\}$ is a tiling of $\widehat{\mathbb{R}}^d$;
ii. If $\Omega = \bigcup_{l=1}^{L} \Omega^l$, then $\{2^j \Omega : j \in \mathbb{Z}\}$ is a tiling of $\widehat{\mathbb{R}}^d$.

Remark 2.3. In light of our dyadic results in this paper involving functions of the the form $\mathbb{1}_{\Omega}^{\vee}$, we point out that Gu and Han [49] (2000) proved that, in the setting of Equation (2), if $|\det A| = 2$, then there is a measurable set $\Omega \subseteq \widehat{\mathbb{R}}^d$ such that $\{2^{md/2} \mathbb{1}_{\Omega}^{\vee}(A^m x - n) : m \in \mathbb{Z}, n \in \mathbb{Z}^d\}$ is an ONB for $L^2(\mathbb{R}^d)$. This result can be viewed as a converse of the following theorem: *if $\psi \in L^2(\mathbb{R}^d)$ is a single wavelet constructed from an MRA associated with $(\mathbb{Z}^d, A)$, then* $|\det A| = 2$, see [9] (1995) and [50] (1997), cf. [64] (1992) and [28] (1993).

2.2 Kolmogorov theorem and Littlewood-Paley wavelet ONB

In 1922, Kolmogorov [63] (1924) proved that if $F \in L^2(\mathbb{T})$ and $S_N(F)$ is the Nth partial sum of the Fourier series $S(F)$ of F, then

$$\lim_{n \to \infty} S_{2^n}(F)(\gamma) = F(\gamma) \text{ a.e.} \tag{3}$$

His proof is elementary, short, and clever; and the result is still valid when $\{2^n\}$ is replaced by more general lacunary sequences. Writing

$$\Delta_j F(\gamma) = \sum_{2^j \le |n| < 2^{j+1}} F^{\vee}[n] e^{-2\pi i n \gamma}, \qquad j = 0, 1, \dots,$$

Equation (3) can be restated as

$$F(\gamma) = F^{\vee}[0] + \sum_{j=0}^{\infty} \Delta_j F(\gamma) \ \text{ a.e.,} \tag{4}$$

which can be interpreted as a frequency decomposition of F based on dyadic intervals. Equation (4) plays a fundamental role in Littlewood-Paley theory, and was stated and proved as a consequence of deep results in their theory in the setting of $L^p(\mathbb{T})$, $p > 1$, see [74, Theorem 5] (1931) and [75, Theorem 8] (1937). The Littlewood-Paley theory is an important part of 20th century harmonic analysis, e.g., see [29] (1978), [41] (1977), [46] (1991), [92, Chapter 14] (1970), and [93] (1970).

From our point of view, Equation (4) can be adjusted to incorporate time-frequency localization, at least within the constraints of the classical uncertainty principle; and it can be thought of as a primordial wavelet decomposition, e.g., [82, pp.19-20] (1992). In fact, in the setting of $\mathbb{R}$, the decomposition (4), properly localized in time and reformulated in terms of multiresolution analysis, becomes the *Littlewood-Paley* or *Shannon* wavelet orthonormal basis decomposition

$$f = \sum_{m,n} \langle f, \psi_{m,n} \rangle \psi_{m,n}, \qquad \text{for all } f \in L^2(\mathbb{R}), \tag{5}$$

where

$$\widehat{\psi} = \mathbb{1}_{\Omega},\ \Omega = [-1, -\frac{1}{2}) \bigcup [\frac{1}{2}, 1)$$

is the Fourier transform of the Littlewood-Paley or Shannon wavelet ψ.

The decomposition (5) can be proved in several standard ways, but the most convenient is to combine the orthonormality of $\{\psi_{m,n}\}$ with the fact that

$$\sum_{m,n} |\langle f, \psi_{m,n} \rangle| = ||f||^2_{L^2(\mathbb{R})}, \qquad \text{for all } f \in L^2(\mathbb{R}), \tag{6}$$

e.g., see [38, pp. 115-16] (1992) and [46] (1991) for further details. The proof of Equation (6) is the calculation

$$\sum |\langle f, \psi_{m,n}\rangle|^2 = \sum_{m,n} 2^m \left| \int_\Omega \hat{f}(2^m\lambda) e^{2\pi i n\lambda}\, d\lambda \right|^2$$

$$= \sum_{m,n} \left| \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\hat{f}\,(2^m(\gamma-1))\, \mathbb{1}_{[-1,-\frac{1}{2})}(\gamma-1) + \hat{f}\,(2^m(\gamma+1))\, \mathbb{1}_{[\frac{1}{2},1)}(\gamma+1) \right) e^{2\pi i n\gamma} d\gamma \right|^2$$

$$= \sum_m \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \hat{f}\,(2^m(\gamma-1))\, \mathbb{1}_{[-1,-\frac{1}{2})}(\gamma-1) + \hat{f}\,(2^m(\gamma+1))\, \mathbb{1}_{[\frac{1}{2},1)}(\gamma+1) \right|^2 d\gamma \qquad (7)$$

$$= \sum_m 2^m \int_\Omega |\hat{f}(2^m\lambda)|^2\, d\lambda = \sum_m \int_{2^m\Omega} |\hat{f}(\gamma)|^2\, d\gamma = ||f||^2_{L^2(\mathbb{R})}.$$

The points to be made are that (7) is essentially a geometrical argument, and also that it can be generalized. The fact that (7) is a geometrical argument is immediate from the second equality, which depends on the $\mathbb{Z}$-translation congruence of Ω and $\left[-\frac{1}{2}, \frac{1}{2}\right]$, and the last equality, which is due to the fact that $\{2^m\Omega\}$ is a tiling of $\widehat{\mathbb{R}}$.

Thus, the Shannon wavelet ψ does in fact give rise to a dyadic wavelet ONB for $L^2(\mathbb{R})$. Moreover, $\widehat{\psi} = \mathbb{1}_\Omega$ so we are dealing with the wavelet set $\Omega = \left[-1, -\frac{1}{2}\right) \cup \left[\frac{1}{2}, 1\right)$; and, most important, the proof that $\{\psi_{m,n}\}$ is an ONB for $L^2(\mathbb{R})$ depends entirely on the tiling criteria of Theorem 2.1.

In the next section we shall give a general construction of wavelet sets motivated by the tiling criteria of Theorem 2.1. Intuitively, these criteria assert that Ω must have fundamental characteristics of both squares *and* annuli.

3 The construction of Euclidean wavelet sets

3.1 The basic construction

Let $\Omega_0 \subseteq [-N,N]^d \subseteq \widehat{\mathbb{R}}^d$ be a neighborhood of the origin with Lebesgue measure $|\Omega_0| = 1$, and further assume that Ω_0 is $\mathbb{Z}^d$-translation congruent to $\left[-\frac{1}{2},\frac{1}{2}\right]^d$. Ω_0 will be iteratively transformed by the action of a mapping

$$T : \Omega_0 \longrightarrow \overline{[-2N,2N]^d \setminus [-N,N]^d}$$

for some fixed N, where T is defined by the property that, for each fixed $\gamma \in \Omega_0$, $T(\gamma) = \gamma + k_\gamma$ for some $k_\gamma \in \mathbb{Z}^d$.

Because of the requirements of our forthcoming construction, we shall assume that the mapping T, defined in terms of the translation property $T\gamma = \gamma + k_\gamma$, also has the properties that it is a measurable, injective mapping on Ω_0, see [21, Proposition 3.1] (2001).

Algorithm 3.1. We now describe our original NMC construction of wavelet sets Ω depending on Ω_0, N, and T. Let

$$\Lambda_0 = \Omega_0 \cap \left(\bigcup_{j \geq 1} 2^{-j}\Omega_0 \right) \text{ and } \Omega_1 = (\Omega_0 \setminus \Lambda_0) \cup T\Lambda_0.$$

Then,

$$\Omega_0 \setminus \Lambda_0 \subseteq \Omega_0 \text{ and } T\Lambda_0 \subseteq \overline{[-2N,2N]^d \setminus [-N,N]^d}.$$

Next, let

$$\Lambda_1 = \Omega_1 \cap \left(\bigcup_{j \geq 1} 2^{-j}\Omega_1 \right),$$

and let $\Omega_2 = ((\Omega_0 \setminus \Lambda_0) \setminus \Lambda_1) \cup T\Lambda_0 \cup T\Lambda_1$. Then,

$$(\Omega_0 \setminus \Lambda_0) \setminus \Lambda_1 \subseteq \Omega_0$$

and

$$T\Lambda_0 \cup T\Lambda_1 \subseteq \overline{[-2N,2N]^d \setminus [-N,N]^d}.$$

Notationally, we set $((\Omega_0 \setminus \Lambda_0) \setminus \Lambda_1) = \Omega_0 \setminus \Lambda_0 \setminus \Lambda_1$. Generally, for a given Ω_n, let

$$\Lambda_n = \Omega_n \cap \left(\bigcup_{j \geq 1} 2^{-j} \Omega_n \right),$$

and set

$$\Omega_{n+1} = (\Omega_0 \setminus \Lambda_0 \setminus \Lambda_1 \setminus \cdots \setminus \Lambda_n) \cup (T\Lambda_0 \cup T\Lambda_1 \cup \cdots \cup T\Lambda_n). \tag{8}$$

Then,

$$\Omega_0 \setminus \Lambda_0 \setminus \Lambda_1 \setminus \cdots \setminus \Lambda_n \subseteq \Omega_0$$

and

$$T\Lambda_0 \cup T\Lambda_1 \cup \cdots \cup T\Lambda_n \subseteq \overline{[-2N,2N]^d \setminus [-N,N]^d}.$$

We define Ω as

$$\Omega = \left(\Omega_0 \setminus \bigcup_{k=0}^{\infty} \Lambda_k \right) \cup \left(\bigcup_{n=0}^{\infty} T\Lambda_n \right). \tag{9}$$

Denoting

$$\Omega_n^- = \Omega_0 \setminus \Lambda_0 \setminus \Lambda_1 \setminus \cdots \setminus \Lambda_{n-1}$$

and

$$\Omega_n^+ = T\Lambda_0 \cup T\Lambda_1 \cup \cdots \cup T\Lambda_{n-1},$$

we have

$$\Omega_n = \Omega N^- \cup \Omega_n^+, \;\; \Omega = \left(\bigcap_{n=0}^{\infty} \Omega_n^- \right) \cup \left(\bigcup_{n=0}^{\infty} \Omega_n^+ \right), \text{ and } |\Omega_n| = |\Omega_n^-| + |\Omega_n^+| = 1.$$

Thus, the set Ω is obtained by removing from Ω_0 all the Λ_is and sending these disjoint subsets into $\overline{[-2N,2N]^d \setminus [-N,N]^d}$ by means of the mapping T. It should be noted that Ω is $\mathbb{Z}^d$-translation congruent to Ω_0.

Theorem 3.1. *Ω defined by* (9) *is a wavelet set, see [20] (1999), [21] (2001).*

The following is the generalization of Theorem 3.1 corresponding to the geometrical characterization of Theorem 2.2.

Theorem 3.2. *Let $\{\Omega_0^1,\ldots,\Omega_0^L\} \subseteq \widehat{\mathbb{R}}^d$, and assume T and each Ω_0^l satisfy the hypotheses of Algorithm 3.1. Let $\{\Omega^1,\ldots,\Omega^L\}$ be the sequence of sets constructed in Algorithm 3.1. Then, $\{\Omega^1,\ldots,\Omega^L\}$ is a wavelet collection of sets, i.e.,*

$$\{\psi^l : \mathbb{1}^{\vee}_{\Omega^l} : l = 1,\ldots,L\}$$

is a set of wavelet generators for $L^2(\mathbb{R}^d)$.

3.2 A generalization of the neighborhood-mapping construction

It is assumed in the original NMC of Section 3.1 that Ω_0 is contained in $[-N,N]^d$ and that the range of the mapping T is contained in $\overline{[-2N,2N]^d \setminus [-N,N]^d}$. As it turns out, this assumption on the range of T is not necessary. The purpose of the mapping T should only be to move the sets Λ_n, defined below, out of Ω_0. In this section we prove that the procedure produces wavelet sets for a more general class of mappings T, thereby obtaining wavelet sets that we had not been able to obtain by the original construction.

Let Ω_0 be a bounded neighborhood of the origin that is $\mathbb{Z}^d$-translation congruent to the unit cube $\mathrm{Q} = [-\frac{1}{2},\frac{1}{2}]^d$. We shall consider measurable mappings $T : \widehat{\mathbb{R}}^d \to \widehat{\mathbb{R}}^d$ satisfying the following properties.

i. T is a $\mathbb{Z}^d$-translated mapping, i.e.,

$$\forall\gamma \in \widehat{\mathbb{R}}^d,\ \exists n_\gamma \in \mathbb{Z}^d \text{ such that } T(\gamma) = \gamma + n_\gamma.$$

ii. T is injective.

iii. The range of $T - \mathrm{I}$ is bounded, where I is the identity mapping on $\widehat{\mathbb{R}}^d$.

iv. $\left[\bigcup_{i=1}^{\infty} T^i\Omega_0\right] \cap \left[\bigcup_{j=0}^{\infty} 2^{-j}\Omega_0\right] = \varnothing$, where $T^0 = \mathrm{I}$ and $T^i = \underbrace{T \circ \cdots \circ T}_{i\text{-fold}}$.

Compared to the original NMC, the first two conditions on T are unchanged, while the last two relax the earlier assumption on the range of T. Condition *iii* says that $T(\gamma) = \gamma + n_\gamma$ cannot be arbitrarily far from γ. There must be a uniform bound on how far γ moves to $T(\gamma)$ but the range of T does not necessarily lie inside some square box. What condition *iv* says is that for any $\gamma \in \Omega_0$ the sequence $\{T(\gamma), T^2(\gamma), \ldots, T^n(\gamma), \ldots\}$ never returns to Ω_0 or any $2^{-j}\Omega_0, j > 1$. This weakens the earlier artificial assumption that T has to move points in Ω_0 out of a square containing Ω_0.

Algorithm 3.2. Let T satisfy conditions *i*–*iv*. According to [21] (2001), [22] (2002), we iteratively construct a sequence of sets Ω_n each of which is $\mathbb{Z}^d$-translation congruent to Q, and hence tiles $\widehat{\mathbb{R}}^d$ by $\mathbb{Z}^d$-translates, as follows. For each $n = 0, 1, \ldots,$ we define

$$\Lambda_n = \Omega_n \cap [\bigcup_{j=1}^{\infty} 2^{-j}\Omega_n]$$

and

$$\Omega_{n+1} = (\Omega_n \setminus \Lambda_n) \cup T\Lambda_n. \tag{10}$$

This set Ω_{n+1}, defined by (10), is the same as the set Ω_{n+1}, defined by (8). However, by property *iv* and some set theoretic implications of it, we calculate that

$$\Omega_{n+1} = \left[\Omega_0 \setminus \bigcup_{i=0}^{n} \Lambda_i\right] \cup \left[\bigcup_{k=0}^{n} \left(T\Lambda_k \setminus \bigcup_{i=k+1}^{n} \Lambda_i\right)\right]. \tag{11}$$

Because of (11), we define the set

$$\Omega = \left[\Omega_0 \setminus \bigcup_{i=0}^{\infty} \Lambda_i\right] \cup \left[\bigcup_{k=0}^{\infty} \left(T\Lambda_k \setminus \bigcup_{i=k+1}^{n} \Lambda_i\right)\right]. \tag{12}$$

Theorem 3.3. Ω *defined by* (12) *is a wavelet set.*

The proof is found in [23] (2006). It depends on several useful implications of properties *i–iv* including the following result.

Proposition 3.1. *For each* $n = 0, 1, \ldots,$ *we have the following.*

a. Ω_n *tiles* $\widehat{\mathbb{R}}^d$ *by* $\mathbb{Z}^d$*-dyadic translation, and*
b. $\Omega_n \setminus \Lambda_n$ *tiles* $\widehat{\mathbb{R}}^d$ *by dyadic-dilation.*

4 Frame wavelets

The concept of frame introduced by Duffin and Schaeffer [40] (1952) is a natural generalization of an orthonormal basis.

Definition 4.1. *a.* A countable family $\{\phi_i\}_{i\in\mathbb{Z}}$ of functions in $L^2(\mathbb{R}^d)$ is a *frame* for $L^2(\mathbb{R}^d)$ if there exist constants $0 < A \leq B < \infty$ called *frame bounds* for which

$$A||f||^2 \leq \sum_{i\in\mathbb{Z}} |\langle f, \phi_i\rangle|^2 \leq B||f||^2, \qquad \text{for all } f \in L^2(\mathbb{R}^d).$$

When the frame bounds A and B coincide, $\{\phi_i\}$ is called a *tight frame*. If $A = B = 1$, the inequality becomes the Parseval identity and $\{\phi_i\}$ is aptly called a *Parseval frame*, i.e.,

$$\sum_{i\in\mathbb{Z}} |\langle f, \phi_i\rangle|^2 = ||f||^2, \qquad \text{for all } f \in L^2(\mathbb{R}^d).$$

b. An $L^2(\mathbb{R}^d)$ function ψ is a *frame wavelet*, respectively, *tight frame wavelet* and *Parseval frame wavelet*, if the generated family $\{\psi_{m,n} : m \in \mathbb{Z}, n \in \mathbb{Z}^d\}$ is a frame, respectively, tight frame and Parseval frame.

c. $\Omega \subseteq \widehat{\mathbb{R}}^d$ is a *(Parseval) frame wavelet set* if $\psi = \mathbb{1}_{\Omega}^{\vee}$ is a (Parseval) frame wavelet.

The following is the analogue of Theorem 2.1 for the case of Parseval frame wavelets.

Theorem 4.1. *Let $\Omega \subseteq \widehat{\mathbb{R}}^d$ be measurable. The following are equivalent.*

i. *Ω is a Parseval frame wavelet set.*

ii. *Ω is $\mathbb{Z}^d$-translation congruent to a subset of $[0,1)^d$ and Ω is dyadic-dilation congruent to $[-1,1)^d \setminus \left[-\frac{1}{2},\frac{1}{2}\right)^d$.*

iii. *$\{\Omega + k : k \in \mathbb{Z}^d\}$ is a tiling of a subset of $\widehat{\mathbb{R}}^d$ and $\{2^n\Omega : n \in \mathbb{Z}\}$ is a tiling of $\widehat{\mathbb{R}}^d$.*

In recent years, frame wavelets, in particular tight and Parseval frame wavelets, have been studied extensively, e.g., see [50] (1997) by Bin Han, as well as a related paper by Dai, Diao, Gu, and Deguang Han, [33] (2002).

It is a natural question to ask whether the sets Ω_n constructed from finite iterations in the NMC of Section 3 give rise to frame wavelets, respectively, tight frame wavelets and Parseval frame wavelets, i.e., whether the functions $\mathbb{1}_{\Omega_n}^{\vee}$ are frame wavelets, respectively, tight frame wavelets and Parseval frame wavelets. It turns out that $\mathbb{1}_{\Omega_n}^{\vee}$ is a frame wavelet with frame bounds 1 and 2, while we obtain Parseval frame wavelets from the auxiliary sets $\Omega_n \setminus \Lambda_n$ (Theorem 4.4 below). We shall prove this result not only because it is a bit surprising but also because it facilitates the genuine implementation of wavelet set theory.

To this end, we need the characterization of tight frame wavelets due to Bin Han [50] (1997) as well as to Ron and Shen (1997), Bownik (2000), Chui and Shi (2000), and Chui et al. [27] (2002) (Theorem 4.2 below). We also require Daubechies' sufficient condition for a function to be a frame wavelet [38] (1992), combined with refinements by Kugarajah and Zhang (1995) and Hernández and Weiss [55] (1996) (Theorem 4.3 below). The proof for $\mathbb{R}$ of Theorem 5.1 in [26] (2001) can be easily generalized to $\mathbb{R}^d$, and this generalization is Theorem 4.3.

Theorem 4.2. *Let $\psi \in L^2(\mathbb{R}^d)$. The family*

$$\{\psi_{m,n} = 2^{md/2}\psi(2^m \cdot - n) : m \in \mathbb{Z}, n \in \mathbb{Z}^d\}$$

is a Parseval frame if and only if

$$\sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi)|^2 = 1 \quad \textit{and} \quad t_q(\xi) = \sum_{j=0}^{\infty} \widehat{\psi}(2^j\xi)\overline{\widehat{\psi}(2^j(\xi+q))} = 0$$

for almost every $\xi \in \widehat{\mathbb{R}}^d$ and for all $q \in \mathbb{Z}^d \setminus 2\mathbb{Z}^d$.

Theorem 4.3. *Let $a > 1$, $b > 0$, and $\psi \in L^2(\mathbb{R}^d)$ be given. Suppose that*

$$A = \inf_{||\xi|| \in [1,a]} [\sum_{n\in\mathbb{Z}} |\widehat{\psi}(a^n\xi)|^2 - \sum_{k\neq\mathbf{0}}\sum_{n\in\mathbb{Z}} |\widehat{\psi}(a^n\xi)\widehat{\psi}(a^n\xi + k/b)|] > 0,$$

$$B = \sup_{||\xi|| \in [1,a]} [\sum_{n\in\mathbb{Z}} |\widehat{\psi}(a^n\xi)|^2 + \sum_{k\neq\mathbf{0}}\sum_{n\in\mathbb{Z}} |\widehat{\psi}(a^n\xi)\widehat{\psi}(a^n\xi + k/b)|] < \infty.$$

Then $\{a^{jd/2}\psi(a^j \cdot - kb)\}_{j\in\mathbb{Z}, k\in\mathbb{Z}^d}$ is a frame for $L^2(\mathbb{R}^d)$ with frame bounds A/b^d, B/b^d.

We begin the proof of Theorem 4.4 with the following lemma which uses Theorem 4.2. The 1-dimensional version of Lemma 4.1 first appeared in Theorem 4.1 of [50] (1997).

Lemma 4.1. *If a measurable set Θ tiles $\widehat{\mathbb{R}}^d$ by dyadic-dilation and $\Theta \subseteq \Omega$ for some measurable set Ω that tiles $\widehat{\mathbb{R}}^d$ by $\mathbb{Z}^d$-translation, then the function $\psi \in L^2(\mathbb{R}^d)$ defined by $\widehat{\psi} = \mathbb{1}_\Theta$ is a Parseval frame wavelet.*

Proof. Since Θ tiles $\widehat{\mathbb{R}}^d$ by dyadic dilation a.e., we have

$$\sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi)|^2 = \sum_{j\in\mathbb{Z}} \mathbb{1}_\Theta(2^j\xi) = \mathbb{1}_{\bigcup_{j\in\mathbb{Z}} 2^{-j}\Theta}(\xi) = 1 \quad \text{a.e.}$$

We then compute

$$\begin{aligned} t_q(\xi) &= \sum_{j=0}^{\infty} \widehat{\psi}(2^j\xi)\overline{\widehat{\psi}(2^j(\xi+q))} = \sum_{j=0}^{\infty} \mathbb{1}_{2^{-j}\Theta}(\xi)\mathbb{1}_{2^{-j}\Theta-q}(\xi) \\ &= \sum_{j=0}^{\infty} \mathbb{1}_{2^{-j}[\Theta\cap(\Theta-2^jq)]}(\xi). \end{aligned}$$

From the second assumption, $\Theta \cap (\Theta - 2^jq) \subseteq \Omega \cap (\Omega - 2^jq) = \varnothing$. Therefore, $t_q = 0$. Hence, by Theorem 4.2, ψ is a Parseval frame wavelet. $\square$

Theorem 4.4. *For each $n \geq 0$, $\Omega_n \smallsetminus \Lambda_n$ is a Parseval frame wavelet set, and Ω_n is a frame wavelet set with frame bounds 1 and 2, cf. Proposition 2.2 of [39] (2002).*

Proof. By Proposition 3.1, Lemma 4.1, and the inclusion $\Omega_n \smallsetminus \Lambda_n \subseteq \Omega_n$, it is clear that $\Omega_n \smallsetminus \Lambda_n$ is a Parseval frame wavelet set.

Let $\widehat{\psi} = \mathbb{1}_{\Omega_n}$. Then,

$$\sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi)|^2 = \sum_{j\in\mathbb{Z}} \mathbb{1}_{\Omega_n\smallsetminus\Lambda_n}(2^j\xi) + \sum_{j\in\mathbb{Z}} \mathbb{1}_{\Lambda_n}(2^j\xi) = 1 + \mathbb{1}_{\bigcup_{j\in\mathbb{Z}} 2^{-j}\Lambda_n}(\xi).$$

It is straightforward from the definition that the sets $2^{-j}\Lambda_n$, $j \in \mathbb{Z}$, are mutually disjoint. This justifies the second equation. Therefore $\sup_{\xi\in\widehat{\mathbb{R}}^d} \sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi)|^2 = 2$ and $\inf_{\xi\in\widehat{\mathbb{R}}^d} \sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi)|^2 = 1$. Since Ω_n tiles $\widehat{\mathbb{R}}^d$ by $\mathbb{Z}^d$-translation, we have $\widehat{\psi}(2^j\xi)\widehat{\psi}(2^j\xi + k) = \mathbb{1}_{\Omega_n}(2^j\xi)\mathbb{1}_{\Omega_n-k}(2^j\xi) = 0$ for all $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^d \smallsetminus \{0\}$.

Hence, we can invoke Theorem 4.3 to assert that Ω_n is a frame wavelet set with frame bounds 1 and 2. $\square$

5 Examples of Euclidean wavelet sets

Theorem 5.1.

a. A convex set C cannot partition $\widehat{\mathbb{R}}^d$ by dilations; that is, the set $\left\{2^j C : j \in \mathbb{Z}\right\}$ cannot be a tiling of $\widehat{\mathbb{R}}^d$.

b. A convex set is not a wavelet set.

Proof. *a. i*. The statement is clear in $\widehat{\mathbb{R}}$, and, for simplicity, we shall only prove the result in the case $d = 2$. The proof can be easily generalized to $\widehat{\mathbb{R}}^d$ by replacing lines in $\widehat{\mathbb{R}}^2$ with hyperplanes in $\widehat{\mathbb{R}}^d$.

ii. Suppose that C is a convex set and that $\left\{2^j C : j \in \mathbb{Z}\right\}$ is a partition of $\widehat{\mathbb{R}}^2$. Define $S = \left\{\gamma \in \widehat{\mathbb{R}}^2 : \gamma \in C \text{ and } -\gamma \in C\right\} = C \cap (-C) \subseteq C$. It is clear from the definition that the set S is symmetric, i.e., $\gamma \in S$ if and only if $-\gamma \in S$. Clearly, S is also convex. Most important, $|S| = 0$. This is proved by assuming $|S| > 0$ allowing us to verify that C must contain a neighborhood of 0, which, in turn, contradicts the disjointness of $2^j C$, $j \in \mathbb{Z}$, thereby giving $|S| = 0$.

iii. Next, we note that

$$\left(2^j C\right) \cap \left(-2^i C\right) \subseteq \begin{cases} 2^j S & \text{if } j \geq i \\ 2^i S & \text{if } j \leq i. \end{cases} \tag{13}$$

Thus, with our assumption $\bigcup_{j \in \mathbb{Z}} 2^j C = \widehat{\mathbb{R}}^2$, we compute

$$\begin{aligned}\left|\widehat{\mathbb{R}}^2\right| &= \left|\left(\bigcup_{j\in\mathbb{Z}} 2^j C\right) \cap \left(\bigcup_{i\in\mathbb{Z}} -2^i C\right)\right| \\ &= \left|\bigcup_{i,j\in\mathbb{Z}} 2^j C \cap \left(-2^i C\right)\right| \\ &\leq \sum_{i,j\in\mathbb{Z}} \left|2^{\max(i,j)} S\right| = 0\end{aligned} \tag{14}$$

by (13) and since $|S| = 0$. (14) is obviously false, and so the proof is complete.

b. This is immediate from part *a* and Theorem 2.1. □

Theorem 5.2. *Let $C_1, C_2, \ldots, C_n$ be convex sets in $\widehat{\mathbb{R}}^d$. If*

$$\dot{\bigcup}_{i=1}^{n} \dot{\bigcup}_{j\in\mathbb{Z}} 2^j C_i = \widehat{\mathbb{R}}^d,$$

where the left side is a disjoint union, then $n \geq d+1$. In particular, if $C_1, C_2, \ldots, C_n$ are convex sets in $\widehat{\mathbb{R}}^d$ such that $\bigcup_{i=1}^{n} C_i$ is a wavelet set, then $n \geq d+1$, see [23] (2006).

By definition, $W \subseteq \widehat{\mathbb{R}}^d$ is a *subspace wavelet set* if the family

$$\left\{ \left(\mathbb{1}_W^\vee\right)_{m,n} : m \in \mathbb{Z}, n \in \mathbb{Z}^d \right\}$$

is an ONB for a subspace of $L^2(\mathbb{R}^d)$.

Theorem 5.3. *The set $W \subseteq \widehat{\mathbb{R}}^d$ is a subspace wavelet set if and only if the characteristic function $\mathbb{1}_E$ of the set $E = \bigcup_{j<0} 2^j W$ satisfies the following consistency equation:*

$$1 + \sum_{k\in\mathbb{Z}^d} \mathbb{1}_E(\gamma + k) = \sum_{k\in\mathbb{Z}^d} \mathbb{1}_E\left(\frac{1}{2}(\gamma + k)\right) \quad a.e.$$

In particular, W is a wavelet set for all of $\widehat{\mathbb{R}}^d$ if and only if, in addition, $\bigcup_{j\in\mathbb{Z}} 2^j E$ contains, up to a set of measure zero, a neighborhood of the origin, see [23] (2006).

Theorem 5.4. *Every wavelet set in* $[-1,1]^d \smallsetminus [-\frac{1}{4},\frac{1}{4}]^d$ *can be constructed by means of the NMC.*

The proof has some intricacies.

Corollary 5.1. *For every wavelet set* Ω *in* $[-1,1]^d$,

$$\left|\Omega \cap [-\alpha,\alpha]^d\right| > 0 \quad \textit{for all } \alpha > \frac{1}{4}.$$

Proof. If $\left|\Omega \cap \left[-\frac{1}{4},\frac{1}{4}\right]^d\right| > 0$, then we are done. Otherwise, by Theorem 5.4, the wavelet set Ω can be constructed by the NMC. By definition of the mapping T, it can be shown that Λ_2 is the only one of the sets Λ_n that intersects the "square annulus" $\overline{\left[-\frac{5}{16},\frac{5}{16}\right]^d \smallsetminus \left[-\frac{1}{4},\frac{1}{4}\right]^d}$. Again, since T is a $\mathbb{Z}^d$-translated mapping, it is not possible for Λ_2 to cover $[-\alpha,\alpha]^d \smallsetminus \left[-\frac{1}{4},\frac{1}{4}\right]^d$ for any $\alpha > \frac{1}{4}$. This completes the proof. $\square$

Theorem 5.5. *For any* $\alpha < 1$, *a wavelet set* Ω *can not be contained in* $[-\alpha,\alpha]^d$.

Proof. Suppose that $\Omega \subseteq [-\alpha,\alpha]^d$ is a wavelet set. Then the integral translates of Ω will tile $\widehat{\mathbb{R}}^d$, i.e.,

$$\dot{\bigcup}_{k\in\mathbb{Z}^d}(\Omega + k) = \widehat{\mathbb{R}}^d,$$

where $\dot{\bigcup}$ designates disjoint union.

Observe that for each fixed $i = 1,\ldots,d$, the union of all translates $\Omega + (k_1,\ldots,k_d)$ with $k_i \neq 0$ leaves out the band $B_i = \{(x_1,x_2,\ldots,x_d) \in \mathbb{R}^d : \alpha - 1 < x_i < 1-\alpha\}$, i.e.,

$$\dot{\bigcup}_{k_i\neq 0}(\Omega + k) \cap B_i = \varnothing, \ \text{ for each } \ i = 1,2,\ldots,d.$$

Therefore,

$$\dot{\bigcup}_{k\neq(0,\ldots,0)}(\Omega + k) \cap B = \varnothing,$$

where $B = \bigcap_{i=1}^{d} B_i = (\alpha - 1, 1 - \alpha)^d$. This clearly implies that $B \subseteq \Omega$ and that $\varnothing \neq B \subseteq \Omega \cap 2\Omega$, a contradiction to the dyadic-dilation congruence property of wavelet sets. $\square$

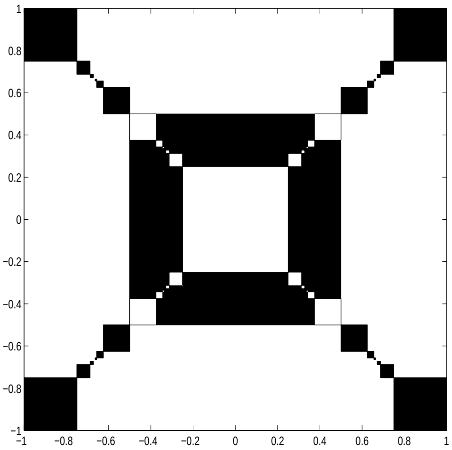

Fig. 1 The set Ω_7 for the 2-dimensional Shannon wavelet set of Example 5.1.

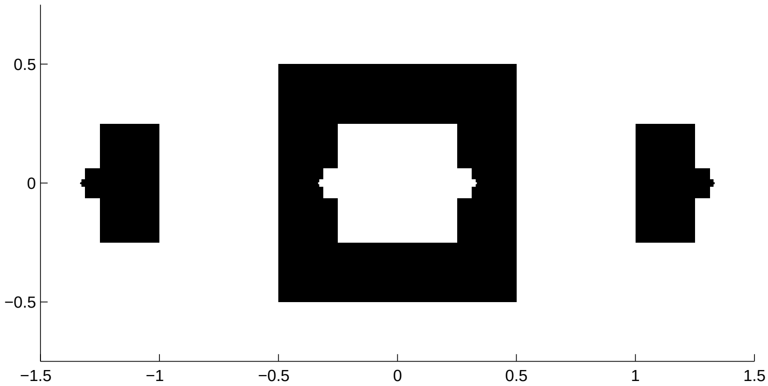

Fig. 2 The wedding cake set of Example 5.2.

Example 5.1. Figure 1 is the $\Omega_7 \subseteq \widehat{\mathbb{R}}^2$ approximant of the 2-dimensional generalization of the Shannon wavelet set described in Section 2.2 for the "variables" $\Omega_0 = Q$ and $T(\gamma_1, \gamma_2) = (\gamma_1, \gamma_2) - (\mathrm{sign}(\gamma_1), \mathrm{sign}(\gamma_2))$.

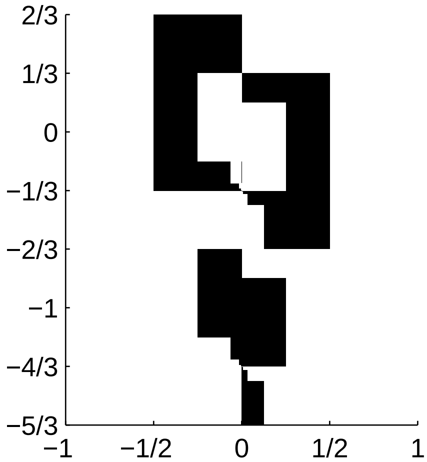

Fig. 3 The wedding night set of Example 5.2.

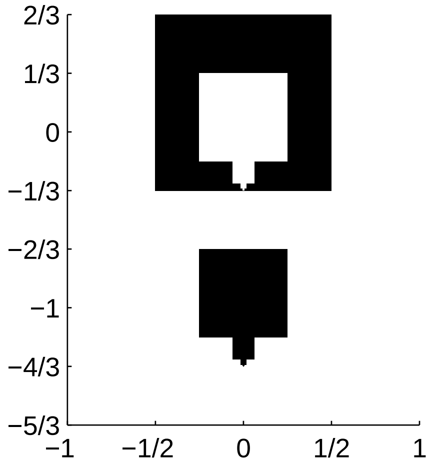

Fig. 4 The wedding cake set of Example 5.2 consisting of 2 connected sets.

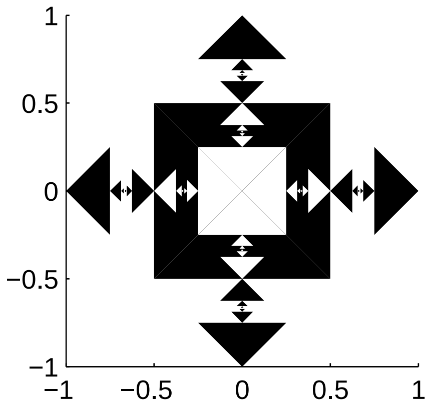

Fig. 5 The connected wavelet set of Example 5.3.

Example 5.2. Wedding sets

The wedding cake wavelet set was defined in [34] (1998), [36] (1998), and it can be constructed by our NMC method with $\Omega_0 = \mathrm{Q}$ and $T(\gamma_1, \gamma_2) = (\gamma_1 + \operatorname{sign}\gamma_1, \gamma_2)$ for $(\gamma_1, \gamma_2) \in \Omega_0$. It was introduced as a simple wavelet set consisting of 3 connected sets, see Figure 2. The interior of each component is connected. We constructed the wedding night set in Figure 3, and it consists of two connected sets. The NMC also

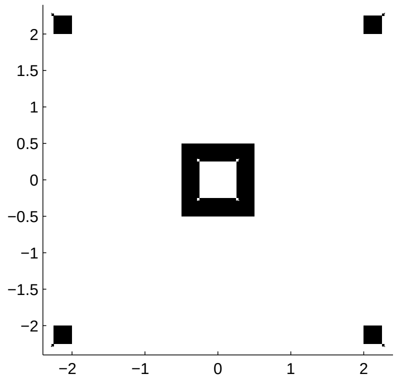

Fig. 6 The 2-dimensional Journé wavelet set of Example 5.4.

allowed us to construct an alternative wedding cake set consisting of two connected sets with connected interiors (Figure 4).

Example 5.3. An example of a connected wavelet set, in which the interior consists of infinitely many components, was given in [21], see Figure 5. It is believed that there is no connected wavelet set with connected interior.

Example 5.4. A generalization of the Journé wavelet set.
Since the Journé wavelet set can be constructed by the NMC with $\Omega_0 = \left[-\frac{1}{2}, \frac{1}{2}\right)$ and $T(\gamma) = \gamma + 2\,\mathrm{sign}(\gamma)$, one of its d-dimensional versions can be produced by setting $\Omega_0 = Q$ and

$$T(\gamma_1, \ldots, \gamma_d) = (\gamma_1 + 2\,\mathrm{sign}(\gamma_1), \ldots, \gamma_d + 2\,\mathrm{sign}(\gamma_d)),$$

see Figure 6 for the corresponding wavelet set in $\widehat{\mathbb{R}}^2$.

It should be noted that the NMC only produces wavelet sets that are bounded away from the origin and infinity, i.e., they have holes at the origin and are bounded sets. Related work can be found in [73] (2002).

6 Locally compact Abelian groups, wavelets, and the p-adic field

6.1 The p-adic field $\mathbb{Q}_p$

Theorem 6.1. *Every locally compact abelian group (LCAG) G is topologically and algebraically isomorphic to $\mathbb{R}^d \times G_o$, where $\mathbb{R}^d$ is Euclidean space and G_o is a LCAG containing a compact open subgroup H_o.*

This fact follows from a result in [1] (1965) combined with [56, Section 9.8] (1963). For the remainder, we shall deal with wavelet theory for functions defined on groups G_o.

A common example of a group G_0 is the field $\mathbb{Q}_p$ of p-adic rationals. Given any prime number p, the field $\mathbb{Q}_p$ is the completion of the field $\mathbb{Q}$ of rationals with respect to the p-adic absolute value $|p^r m/n|_p = p^{-r}$ for all $r, m, n \in \mathbb{Z}$ with m and n not divisible by p. Equivalently, $\mathbb{Q}_p$ may be thought of as the set of Laurent series in the "variable" p, with coefficients $0, 1, \ldots, p$. This means that

$$\mathbb{Q}_p = \left\{ \sum_{n \geq n_0} a_n p^n : n_0 \in \mathbb{Z} \text{ and } a_n \in \{0, 1, \ldots, p-1\} \right\},$$

with addition and multiplication as usual for Laurent series, except with carrying of digits, so that, for example, in $\mathbb{Q}_7$, we have

$$(4 + 1 \cdot 7) + (6 + 5 \cdot 7) = 3 + 0 \cdot 7 + 1 \cdot 7^2.$$

The p-adic absolute value extends naturally to $\mathbb{Q}_p$, and under the operation of addition, $\mathbb{Q}_p$ forms a LCAG, with topology induced by $|\cdot|_p$, and with compact open subgroup $\mathbb{Z}_p$, the ring of p-adic integers, consisting of Taylor series in p. Equivalently, $\mathbb{Z}_p$ is the closure of $\mathbb{Z} \subseteq \mathbb{Q}_p$ with respect to $|\cdot|_p$. Further, $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$.

A few other details about $\mathbb{Q}_p$ appear in Section 6.3. In addition, Section 7.2 contains an informal discussion of the geometry of $\mathbb{Q}_p$ and related groups, including a rough sketch of $\mathbb{Q}_3$ in Figure 7.

6.2 Wavelet theories on $\mathbb{Q}_p$ and related groups

The main obstacle to producing a theory of wavelets on groups like $\mathbb{Q}_p$ is that $\mathbb{Q}_p$ has no nontrivial discrete subgroups, and thus there is no lattice to use for translations. Some other LCAGs have both a compact open subgroup and a discrete cocompact lattice. For example, Lang [69] (1996), [70] (1998), [71] (1998) constructed wavelets for the Cantor dyadic group (known to number theorists as the field $\mathbb{F}_2((t))$ of formal Laurent series over the field of two elements) using the lattice consisting of polynomials in t^{-1} with trivial constant term. Farkov [45] (2005), [44] (2005) later generalized Lang's construction to other LCAGs with a compact open subgroup and a discrete cocompact lattice. However, for $\mathbb{Q}_p$ and other LCAGs with compact open subgroups, the lack of a lattice requires a different strategy.

Several authors [65] (2002), [59] (2004), [62] (2009), [90] (2009) have constructed wavelets on such groups by the following strategy. Given a LCAG G with compact open subgroup H, choose a set $\mathscr{C}$ of coset representatives for G/H, and translate only by elements of $\mathscr{C}$. For example, if $G = \mathbb{Q}_p$ and $H = \mathbb{Z}_p$, we may choose $\mathscr{C}$ to consist of all elements of $\mathbb{Q}_p$ of the form a/p^n, where $n \geq 1$ and $0 \leq a \leq p^n - 1$, so that every element of $\mathbb{Q}_p$ may be represented uniquely as $x+s$, for $x \in \mathbb{Z}_p$ and $s \in \mathscr{C}$. Then, given an appropriate dilation operator $A : G \to G$ (such as multiplication-by-$1/p$, in the case of $\mathbb{Q}_p$), it is possible to develop a corresponding wavelet theory. For example, the p-adic wavelets of [65], as well as of those on certain non-group ultrametric spaces in [61] (2005), are simply inverse transforms of characteristic functions of disks, and they were shown to be wavelets by direct computation. Meanwhile, the wavelets of [59], [62], and [90] all arise from a p-adic version of MRA. Common to all of the above constructions, however, is the use of translations by coset representatives.

Unfortunately, the chosen set of coset representatives is usually not a group; for example, the set $\mathscr{C} \subseteq \mathbb{Q}_p$ of the previous paragraph is closed neither under addition nor under additive inverses. As a consequence, the resulting theory seems limited. In particular, all such wavelets known to date are step functions with a finite number of steps.

Instead, we present a different wavelet theory for such groups, using a different set of operators in place of translation by elements of $\mathscr{C}$. Although our operators are not actual translations, they have the crucial advantage of forming a group. We call them *pseudo-translations*. The resulting theory allows a much wider variety of wavelets, including most of the wavelets produced by other authors, as well as many others. After presenting the general theory and definition of our wavelets, we shall show that it is possible to construct many such wavelets using a theory of wavelet sets, and we shall give an algorithm for constructing a wide variety of wavelet sets. We expect that it should also be possible to develop multiresolution analysis for our wavelet theory, but this has not yet been done.

6.3 Prerequisites about LCAGs

In this section, we set some notation and recall a few standard facts about abstract LCAGs; see [56] (1963), [57] (1970), [86] (1966), [87] (1968), and [88] (1962) for details.

Let G be a LCAG with compact open subgroup H. Denote by $\widehat{G}$ the dual group of G, with action denoted $(x,\gamma) \in \mathbb{C}^\times$, for $x \in G$ and $\gamma \in \widehat{G}$. The *annihilator subgroup* of H in $\widehat{G}$ is

$$H^{\perp} = \{\gamma \in \widehat{G} : \forall x \in H, \quad (x,\gamma) = 1\} \subseteq \widehat{G},$$

which is, in turn, a compact open subgroup of $\widehat{G}$.

The quotient group G/H of course consists of cosets $x+H$, also denoted $[x]$, for $x \in G$. This quotient is discrete, because H is open in G. Moreover, G/H is isomorphic as a LCAG to the dual of $H^{\perp}$. The isomorphism is easy to write down; the element $x+H \in G/H$ acts on $H^{\perp}$ by $(x+H,\gamma) = (x,\gamma)$, for any $\gamma \in H^{\perp}$. Similarly, $\widehat{H}$ and $\widehat{G}/H^{\perp}$ are isomorphic discrete groups.

Set $\mu = \mu_G$ and $\nu = \nu_{\widehat{G}}$ to be Haar measures on G and $\widehat{G}$, respectively, normalized so that $\mu(H) = \nu(H^{\perp}) = 1$. These normalizations induce counting measures on the discrete groups G/H and $\widehat{G}/H^{\perp}$, and they make the Fourier transform, given by

$$\hat{f}(\gamma) = \int_G f(x)\overline{(x,\gamma)}\,d\mu(x), \quad \text{for all } f \in L^2(G),$$

an isometry between $L^2(G)$ and $L^2(\widehat{G})$. See, for instance, [57, Section 31.1] (1970), [87] (1968), and [17, Section 1.3] (2004).

By way of example, consider again the case $G = \mathbb{Q}_p$ and $H = \mathbb{Z}_p$. The quotient $\mathbb{Q}_p/\mathbb{Z}_p$ is isomorphic to μ_{p^∞}, the subgroup of $\mathbb{C}^{\times}$ consisting of all roots of unity ζ for which $\zeta^{p^n} = 1$ for some $n \geq 0$. Meanwhile, $\mathbb{Q}_p$ is self-dual, with duality action given by $(x,\gamma) = \chi(x\gamma)$, where $\chi : \mathbb{Q}_p \to \mathbb{C}$ is the character given by

$$\chi\Big(\sum_{n\geq n_0} a_n p^n\Big) = \exp\Big(2\pi i \sum_{n=n_0}^{-1} a_n p^n\Big).$$

The annihilator $\mathbb{Z}_p^{\perp}$ is just $\mathbb{Z}_p$ under this self-duality.

Our wavelet theory will of course require a dilation operator. Given an automorphism $A : G \to G$, there is a unique positive number $|A|$, the *modulus* of A, with the property that for any measurable set $U \subseteq G$, we have $\mu(AU) = |A|\mu(U)$. Therefore, for any $f \in C_c(G)$, $\int_G f \circ A(x)\,d\mu(x) = |A|^{-1} \int_G f(x)\,d\mu(x)$. See, for example, [56, Section 15.26] (1963). In addition, A has an adjoint element $A^* : \widehat{G} \to \widehat{G}$, defined by $(Ax,\gamma) = (x, A^*\gamma)$ for all $x \in G$ and $\gamma \in \widehat{G}$. We have $(A^*)^{-1} = (A^{-1})^*$, $|A|^{-1} = |A^{-1}|$, and $|A^*| = |A|$.

7 Wavelets for groups with compact open subgroups

7.1 Pseudo-translations

In this section, we present the pseudo-translation operators to be used in our wavelet theory. Rather than translating by one fixed element of each coset $[s] \in G/H$, we shall construct an operator $\tau_{[s]} : L^2(G) \to L^2(G)$ for each $[s] \in G/H$ determined only by the coset $[s] = s + H$, and not by a choice of a particular coset representative $s_0 \in [s]$. In addition, our operators will form a group, in that $\tau_{[s]+[t]} = \tau_{[s]}\tau_{[t]}$. The resulting operators are usually not true translations, but $\tau_{[s]}$ will still be similar in certain ways to the translation-by-s operator.

To construct our operators, however, we *shall* have to make a choice of coset representative; but we choose a set $\mathscr{D}$ of coset representatives in $\widehat{G}$ for $\widehat{G}/H^\perp$, rather than representatives in G for G/H. That is, $\mathscr{D} \subseteq \widehat{G}$ is a discrete subset (probably not forming a subgroup) consisting of exactly one element of every coset $\sigma + H^\perp$. We then define $\tau_{[s]}$ by its induced dual map $\widehat{\tau}_{[s]} : L^2(\widehat{G}) \to L^2(\widehat{G})$, as follows.

Definition 7.1. Let G be a LCAG with compact open subgroup $H \subseteq G$. Let $\mathscr{D} \subseteq \widehat{G}$ be a set of coset representatives in $\widehat{G}$ for the quotient $\widehat{H} = \widehat{G}/H^\perp$.

Define the map $\eta = \eta_{\mathscr{D}} : \widehat{G} \to H^\perp \subseteq \widehat{G}$ by

$$\eta(\gamma) = \text{ the unique } \beta \in H^\perp \text{ such that } \gamma - \beta \in \mathscr{D}.$$

For each $[s] \in G/H$, the pseudo-translation-by-$[s]$ operator $\tau_{[s]} = \tau_{[s],\mathscr{D}} : L^2(G) \to L^2(G)$ is given by

$$\widehat{\tau_{[s]} f}(\gamma) = \overline{\big(s, \eta_{\mathscr{D}}(\gamma)\big)}\,\widehat{f}(\gamma).$$

Note that the true translation-by-s operator $T_s : L^2(G) \to L^2(G)$ acts on the transform side by

$$\widehat{T_s f}(\gamma) = \overline{(s,\gamma)}\,\widehat{f}(\gamma).$$

Thus, $\tau_{[s]}$ resembles a translation operator except for the correction by $\eta_{\mathscr{D}}$. The function $\eta_{\mathscr{D}}(\gamma)$, in turn, should be viewed as giving the difference between γ and the nearest "lattice" point, where we consider $\mathscr{D}$ to be an analog of the dual lattice. In the Euclidean setting, where $\mathscr{D}$ really is a dual lattice and the translating element s really is in a lattice, the corresponding quantity $\big(s, \eta_{\mathscr{D}}(\gamma)\big)$ would exactly equal (s,γ). Thus, $\eta_{\mathscr{D}}$ should be thought of as correcting for the fact that $\mathscr{D}$ is not actually a lattice.

The following proposition shows that the other promised properties of $\tau_{[s]}$ also hold.

Proposition 7.1. *Let G, H, and $\mathscr{D}$ be as in Definition 7.1. We have the following.*

a. $\tau_{[s],\mathscr{D}}$ *is well defined, i.e., if* $s+H = t+H$, *then* $\tau_{[s],\mathscr{D}} = \tau_{[t],\mathscr{D}}$.
b. $\tau_{[0],\mathscr{D}} f = f$ *for all* $f \in L^2(G)$.
c. $\tau_{[s],\mathscr{D}} \circ \tau_{[t],\mathscr{D}} = \tau_{[s+t],\mathscr{D}}$ *for all* $s,t \in G$.

Proof. Given any $s,t \in G$ lying in the same coset $s+H = t+H$ and any $\gamma \in \widehat{G}$, we have

$$\big(t, \eta_{\mathscr{D}}(\gamma)\big) = \big(s, \eta_{\mathscr{D}}(\gamma)\big)\big(t-s, \eta_{\mathscr{D}}(\gamma)\big) = \big(s, \eta_{\mathscr{D}}(\gamma)\big),$$

because $t-s \in H$ and $\eta_{\mathscr{D}}(\gamma) \in H^{\perp}$. Part a follows. Similarly, parts b and c are immediate from the observations that $\big(0, \eta_{\mathscr{D}}(\gamma)\big) = 1$ and $\big(s+t, \eta_{\mathscr{D}}(\gamma)\big) = \big(s, \eta_{\mathscr{D}}(\gamma)\big)\big(t, \eta_{\mathscr{D}}(\gamma)\big)$. □

Besides the elegant properties listed in Proposition 7.1, the reason for the particular forms of $\tau_{[s],\mathscr{D}}$ and $\eta_{\mathscr{D}}$ will become clear in Equation (20), during the proof of Theorem 8.1.

7.2 Expansive automorphisms and dilations

When constructing wavelets in $L^2(\mathbb{R}^d)$, one cannot use just any automorphism $A : G \to G$ as a dilation operator, but rather one with particular properties with respect to the lattice. We now present the corresponding property needed for dilations in our setting.

Definition 7.2. Let G be a LCAG with compact open subgroup $H \subseteq G$, and let $A : G \to G$ be an automorphism. We say that A is *expansive* with respect to H if both of the following conditions hold:

i. $H \subsetneq AH$, and
ii. $\bigcap_{n\leq 0} A^n H = \{0\}$.

As noted in [17, Section 2.2] (2004), if G has a compact open subgroup H and expansive automorphism A, then $|A|$ is an integer strictly greater than 1, G/H is infinite, and G is not compact. In addition, on the dual side, we have $H^\perp \subsetneq A^* H^\perp$, and $\bigcup_{n\geq 0} A^{*n} H^\perp = \widehat{G}$.

The expansiveness condition, together with the original assumption that G has a compact open subgroup, says that G and $\widehat{G}$ both have a self-similar structure. In particular, if we sketch $H^\perp$ as a disk, then $\widehat{G}$ is a union of larger and large dilates of that disk. Meanwhile, each dilate $A^{*n} H^\perp$ contains finitely many (in fact, exactly $|A|^n$) translates (i.e., cosets $\sigma + H^\perp$) of $H^\perp$. Similarly, applying negative powers of A^*, we can see that $H^\perp$ itself consists of $|A|$ translates of the smaller disk $(A^*)^{-1} H^\perp$, each of which itself consists of $|A|$ translates of the still smaller disk $(A^*)^{-2} H^\perp$, and

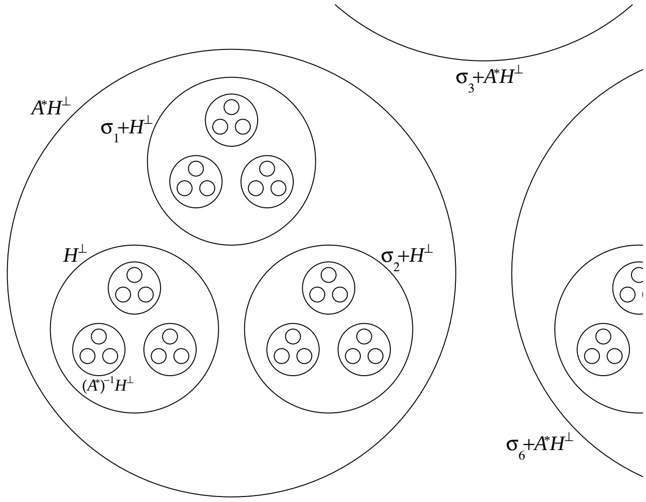

Fig. 7 $\widehat{G}$ for a LCAG G with compact open subgroup H and expansive automorphism A, with $|A| = 3$.

so on. Thus, $H^\perp$ has a fractal structure, much like the Cantor set, while $\widehat{G}$ is an infinite union of translates of $H^\perp$. See Figure 7 for a sketch of such a group $\widehat{G}$ with an expansive automorphism of modulus 3.

For example, if $G = \mathbb{Q}_p$ and $H = \mathbb{Z}_p$, we may choose $A : \mathbb{Q}_p \to \mathbb{Q}_p$ to be $A(x) = x/p$, which maps $\mathbb{Z}_p$ to $(1/p)\mathbb{Z}_p \supsetneq \mathbb{Z}_p$, satisfying condition *i* of Definition 7.2. Condition *ii* also holds, because $\bigcap_{n\leq 0} p^n\mathbb{Z}_p = \{0\}$. The modulus in this case is $|A| = |1/p|_p = p$. Figure 7 may therefore be considered to be a rough sketch of $\mathbb{Q}_3$.

7.3 Wavelets

As in the Euclidean setting, an automorphism $A : G \to G$ induces an operator on $L^2(G)$, sending $f(x)$ to $|A|^{1/2} f(Ax)$; the constant in front, of course, ensures that the resulting operator is unitary. Thus, we may make the following definition.

Definition 7.3. Let G be a LCAG with compact open subgroup $H \subseteq G$, let $\mathscr{D}$ be a choice of coset representatives in $\widehat{G}$ for $\widehat{H} = \widehat{G}/H^{\perp}$, let $A : G \to G$ be an automorphism, and consider $[s] \in G/H$. The *dilated translate* of $f \in L^2(G)$ is defined to be

$$f_{A,[s]}(x) = |A|^{1/2} \cdot (\tau_{[s],\mathscr{D}} f)(Ax). \tag{15}$$

Note that Equation (15) implies that

$$\widehat{f_{A,[s]}}(\gamma) = |A|^{-1/2}\,\widehat{f}\big((A^*)^{-1}\gamma\big)\overline{\big(s, \eta((A^*)^{-1}\gamma)\big)}. \tag{16}$$

Now that we have appropriate dilation and translation operators, we are prepared to define wavelets on our group G.

Definition 7.4. Let G be a LCAG with compact open subgroup $H \subseteq G$, let $\mathscr{D} \subseteq \widehat{G}$ be a choice of coset representatives in $\widehat{G}$ for $\widehat{G}/H^{\perp}$, and let $A : G \to G$ be an automorphism. Consider $\Psi = \{\psi_1, \ldots, \psi_N\} \subseteq L^2(G)$. We say Ψ is a *set of wavelet generators* for $L^2(G)$ with respect to $\mathscr{D}$ and A if

$$\{\psi_{j,m,[s]} : 1 \leq j \leq N, m \in \mathbb{Z}, [s] \in G/H\}$$

forms an ONB for $L^2(G)$, where

$$\psi_{j,m,[s]}(x) = |A|^{m/2} \cdot (\tau_{[s],\mathscr{D}} \psi_j)(A^m x),$$

as in Equation (15). In that case, the resulting basis is called a *wavelet basis* for $L^2(G)$.

If $\Psi = \{\psi\}$, then ψ is a *single wavelet* for $L^2(G)$.

8 Geometry of wavelet sets for G

As we did for $L^2(\mathbb{R}^d)$, we shall use the machinery of wavelet sets, and not MRA, to construct wavelets for $L^2(G)$. Therefore, we state the following definition, cf. [34] (1998), [35] (1997).

Definition 8.1. Let G, H, $\mathscr{D}$, and A be as in Definition 7.4. Let $\Omega_1,\dots,\Omega_N$ be measurable subsets of $\widehat{G}$, and let $\psi_j = \mathbb{1}_{\Omega_j}^{\vee}$ for each $j = 1,\dots,N$. We say that $\{\Omega_1,\dots,\Omega_N\}$ is a *wavelet collection of sets* if $\Psi = \{\psi_1,\dots,\psi_N\}$ is a set of wavelet generators for $L^2(G)$.

If $N = 1$, then $\Omega = \Omega_1$ is a *wavelet set*.

We shall characterize wavelet sets in terms of properties analogous to the Euclidean notions of τ-congruence and δ-congruence, as described in Section 2. See also [17, Section 3.2] (2004) for a broader discussion in our setting.

Definition 8.2. Let G be a LCAG with compact open subgroup $H \subseteq G$, let $\mathscr{D} \subseteq \widehat{G}$ be a choice of coset representatives in $\widehat{G}$ for $\widehat{H} = \widehat{G}/H^{\perp}$, and let $\Omega \subseteq \widehat{G}$ be a subset. We say Ω is $(\tau,\mathscr{D})$-*congruent to* $H^{\perp}$ if there exist measure zero subsets $V_0 \subseteq \Omega$ and $V_0' \subseteq H^{\perp}$, a sequence $\{\sigma_n\}_{n\geq 1} \subseteq \mathscr{D}$, and a countable partition $\{V_n : n \geq 0\}$ of $\Omega \smallsetminus \{V_0\}$ into measurable subsets such that $\{V_n - \sigma_n : n \geq 1\}$ forms a partition of $H^{\perp} \smallsetminus V_0'$.

Definition 8.3. Let $\{W_m : m \in \mathbb{Z}\}$ be a countable set of measurable subsets of $\widehat{G}$. We say that $\{W_m\}$ *tiles* $\widehat{G}$ if

$$\nu\Big(\widehat{G} \smallsetminus \Big[\bigcup_{m\in\mathbb{Z}} W_m\Big]\Big) = 0$$

and

$$\nu(W_m \cap W_n) = 0, \qquad \text{for all } m,n \in \mathbb{Z},\ m \neq n.$$

Our first main result characterizes wavelet collections of sets in terms of the two preceding definitions.

Theorem 8.1. *Let G be a LCAG with compact open subgroup $H \subseteq G$, let $\mathscr{D} \subseteq \widehat{G}$ be a choice of coset representatives in $\widehat{G}$ for $\widehat{G}/H^{\perp}$, and let $A : G \to G$ be an automorphism. A finite set $\{\Omega_1, \dots, \Omega_N\}$ of measurable subsets of $\widehat{G}$ is a wavelet collection of sets if and only if both of the following conditions hold:*

i. $\{A^{*n}\Omega_j : n \in \mathbb{Z}, j = 1, \dots, N\}$ *tiles* $\widehat{G}$, *and*
ii. $\forall j = 1, \dots, N$, Ω_j *is* $(\tau, \mathscr{D})$*-congruent to* $H^{\perp}$.

In that case, $\widehat{G}$ *is* σ*-compact, each* $\nu(\Omega_j) = 1$*, and each* $\mathbb{1}_{\Omega_j} \in L^2(\widehat{G})$.

Proof. See [17, Theorem 3.4]. The centerpiece of the proof is to show that

$$\sum_{j=1}^{N} \sum_{m \in \mathbb{Z}} \sum_{[s] \in G/H} \big|\langle f, \psi_{j,m,[s]} \rangle\big|^2 = \|f\|_2^2, \quad \text{for all } f \in L^2(G), \tag{17}$$

at least under the assumptions that the sum on left side of (17) converges (and in particular, all but countably many terms of the sum are 0), and that properties *i* and *ii* of Theorem 8.1 hold, cf. the calculation (7) in Section 2.2. We now reproduce the argument from [17].

By Plancherel's theorem and (16), we have

$$\begin{aligned}
\sum_{j,m,[s]} \big|\langle f, \psi_{j,m,[s]} \rangle\big|^2 &= \sum_{j,m,[s]} \big|\langle \hat{f}, \widehat{\psi_{j,m,[s]}} \rangle\big|^2 \\
&= \sum_{j,m,[s]} |A|^{-m} \Big| \int_{\widehat{G}} \hat{f}(\gamma) \cdot \overline{\widehat{\psi_j}\big((A^*)^{-m}\gamma\big)} \cdot \big(s, \eta((A^*)^{-m}\gamma)\big)\, d\nu(\gamma) \Big|^2 \\
&= \sum_{j,m,[s]} |A|^{m} \Big| \int_{\Omega_j} \hat{f}(A^{*m}\beta) \cdot \big(s, \eta(\beta)\big)\, d\nu(\beta) \Big|^2,
\end{aligned} \tag{18}$$

where we have substituted $\beta = (A^*)^{-m}\gamma$. By property *ii*, each Ω_j is $(\tau, \mathscr{D})$-congruent to $H^{\perp}$, thereby giving us partitions $\{V_{j,n}\}_{n \geq 0}$ of Ω_j with $\nu(V_{j,0}) = 0$ and sequences $\{\sigma_{j,n}\}_{n \geq 1} \subseteq \mathscr{D}$, as in Definition 8.2. Thus, the right side of (18) becomes

$$\sum_{j,m,[s]} |A|^m \Big| \sum_{n\geq 1} \Big[\int_{V_{j,n}} \hat{f}(A^{*m}\beta) \cdot \big(s, \eta(\beta)\big)\, d\nu(\beta) \Big] \Big|^2 \tag{19}$$
$$= \sum_{j,m,[s]} |A|^m \Big| \sum_{n\geq 1} \Big[\int_{V_{j,n}-\sigma_{j,n}} \hat{f}\big(A^{*m}(\alpha+\sigma_{j,n})\big) \cdot \big(s, \eta(\alpha+\sigma_{j,n})\big)\, d\nu(\alpha) \Big] \Big|^2$$
$$= \sum_{j,m,[s]} |A|^m \Big| \sum_{n\geq 1} \Big[\int_{\widehat{G}} \mathbb{1}_{V_{j,n}-\sigma_{j,n}}(\alpha) \cdot \hat{f}\big(A^{*}(\alpha+\sigma_{j,n})\big) \cdot \big(s, \eta(\alpha+\sigma_{j,n})\big)\, d\nu(\alpha) \Big] \Big|^2,$$

where we have substituted $\alpha = \beta - \sigma_{j,n}$. Since $\alpha \in V_{j,n} - \sigma_{j,n} \subseteq H^{\perp}$, the unique point in $(\alpha + \sigma_{j,n} + H^{\perp}) \cap \mathscr{D}$ is $\sigma_{j,n}$, and therefore

$$\eta(\alpha+\sigma_{j,n}) = (\alpha+\sigma_{j,n}) - \sigma_{j,n} = \alpha. \tag{20}$$

As noted in Section 7.1, the convenient simplification of Equation (20) helps illustrate the reason for the otherwise peculiar-looking description of η and $\tau_{[s],\mathscr{D}}$ in Definition 7.1.

Next, we claim we can exchange the inner summation and integral signs in the last term of (19). After all, we know that $\{V_{j,n} - \sigma_{j,n} : n \geq 1\}$ tiles $H^{\perp}$. Hence, denoting the integrand of (19) by $F_{j,n}$, writing $F_j = \sum_{n\geq 1} F_{j,n}$, and noting that $F_{j,n}$ vanishes off of $V_{j,n} - \sigma_{j,n}$, we see that $F_{j,n}, F_j \in L^2(H^{\perp}) \subseteq L^1(H^{\perp})$, and therefore

$$\sum_{n\geq 1} \int_{\widehat{G}} F_{j,n}(\alpha)\, d\nu(\alpha) = \sum_{n\geq 1} \int_{V_{j,n}-\sigma_{j,n}} F_j(\alpha)\, d\nu(\alpha) = \int_{H^{\perp}} F_j(\alpha)\, d\nu(\alpha).$$

Thus, the right side of (19) becomes

$$\sum_{j,m} |A|^m \sum_{[s]\in G/H} \Big| \int_{H^{\perp}} \Big[\sum_{n\geq 1} \mathbb{1}_{V_{j,n}-\sigma_{j,n}}(\alpha) \cdot \hat{f}\big(A^{*m}(\alpha+\sigma_{j,n})\big) \Big] (s,\alpha)\, d\nu(\alpha) \Big|^2. \tag{21}$$

Because G/H is the (discrete) dual of $H^{\perp}$, Plancherel's theorem tells us that

$$\sum_{[s]\in G/H} \Big| \int_{H^\perp} g(\alpha)(s,\alpha)\, d\nu(\alpha) \Big|^2 = \sum_{[s]\in G/H} \Big| \int_{H^\perp} g(\alpha)([s],\alpha)\, d\nu(\alpha) \Big|^2$$
$$= \int_{H^\perp} |g(\alpha)|^2\, d\nu(\alpha),$$

for any $g \in L^2(H^\perp)$. Thus, (21) becomes

$$\sum_{j,m} |A|^m \int_{H^\perp} \Big| \sum_{n\geq 1} \mathbb{1}_{V_{j,n}-\sigma_{j,n}}(\alpha)\, \hat{f}\big(A^*(\alpha+\sigma_{j,n})\big) \Big|^2 d\nu(\alpha),$$

which, in turn, is

$$\sum_{j,m} |A|^m \int_{H^\perp} \Big[\sum_{n\geq 1} \Big| \hat{f}\big(A^*(\alpha+\sigma_{j,n})\big) \Big|^2 \mathbb{1}_{V_{j,n}-\sigma_{j,n}}(\alpha) \Big]\, d\nu(\alpha), \tag{22}$$

because, for fixed f, the sets $V_{j,n} - \sigma_{j,n}$ are pairwise disjoint. We can now interchange the inner summation and integral as before, and (22) becomes

$$\begin{aligned}
&\sum_{j,m} |A|^m \sum_{n\geq 1} \Big[\int_{V_{j,n}-\sigma_{j,n}} \Big| \hat{f}\big(A^*(\alpha+\sigma_{j,n})\big) \Big|^2 d\nu(\alpha) \Big] \\
&= \sum_{j,m} |A|^m \sum_{n\geq 1} \Big[\int_{V_{j,n}} \Big| \hat{f}\big(A^*(\beta)\big) \Big|^2 d\nu(\beta) \Big] \\
&= \sum_{j,m} |A|^m \int_{\Omega_j} \Big| \hat{f}\big(A^*(\beta)\big) \Big|^2 d\nu(\beta) \\
&= \sum_{j,m} \int_{A^*\Omega_j} \big| \hat{f}(\gamma) \big|^2 d\nu(\gamma).
\end{aligned} \tag{23}$$

However, $\{A^*\Omega_j\}$ tiles $\widehat{G}$. Hence, the right side of (23) is

$$\int_{\widehat{G}} \big| \hat{f}(\gamma) \big|^2 d\nu(\gamma) = \|\hat{f}\|_2^2 = \|f\|_2^2,$$

proving Equation (17). □

9 The construction of wavelet sets for G

9.1 The basic construction

Motivated by the NMC described in Section 3, we now present an algorithm for constructing wavelet collections of sets. As before, G is a LCAG with compact open subgroup H, $\mathscr{D}$ is a choice of coset representatives in $\widehat{G}$ for the quotient $\widehat{G}/H^{\perp}$, and $A : G \to G$ is an automorphism, which we now assume to be expansive with respect to H.

Our algorithm begins with the following data.

i. A nonnegative integer $M \geq 0$. Set $W = (A^*)^M H^{\perp}$.

ii. A positive integer $N \geq 1$.

iii. For each $j = 1, \ldots, N$, a measurable set $\Omega_{j,0} \subseteq W$ that is $(\tau, \mathscr{D})$-congruent to $H^{\perp}$.

iv. For each $j = 1, \ldots, N$, a measurable injective function $T_j : W \to (A^*W) \setminus W$ such that

$$T_j(\gamma) = \gamma - \sigma'_j(\gamma) + \sigma_j(\gamma), \quad \text{for all } \gamma \in W,$$

where $\sigma_j(\gamma) \in \mathscr{D}$, and $\sigma'(\gamma)$ is the unique element of $\mathscr{D} \cap (\gamma + H^{\perp})$.

We also set the following compatibility requirements on the above data.

v. The union $\widetilde{\Omega}_0 = \bigcup_{j=1}^{N} \Omega_{j,0}$ contains the neighborhood $(A^*)^{-\ell} H^{\perp}$ of the origin, for some integer $\ell \geq 0$.

vi. For any distinct $j, k \in \{1, \ldots, N\}$, either

$$T_j W \cap T_k W = \varnothing$$

or

$$T_j = T_k \quad \text{and} \quad \Omega_{j,0} \cap \Omega_{k,0} = \varnothing.$$

Note, however, that we do *not* require the sets $\Omega_{1,0},\dots,\Omega_{N,0}$ to be disjoint. The possibility that two or more of them overlap will be dealt with in the algorithm to be described below. Also, note that, because A is expansive, the set W contains $H^{\perp}$ and $(A^*)^{-1}W$ properly contains W.

Meanwhile, as in Section 3, the mappings T_j should be understood as slicing W into finitely many measurable pieces and then translating each piece, with the injectivity condition requiring that the images of the pieces do not overlap. In Section 3, the translation is by an element of the lattice. In our setting, however, the translation is by an element of the form $\sigma-\sigma'$, where $\sigma,\sigma'\in\mathscr{D}$, and $\sigma'+H^{\perp}$ contains the piece in question, while $\sigma+H^{\perp}$ contains its image. This more complicated description is required for the proof of the algorithm's validity; see Section 9.2.

Algorithm 9.1. Given the initial data described above, our algorithm proceeds inductively, building sets $\Lambda_{j,n+1}$ and $\Omega_{j,n}$ for each $n\geq 0$, as follows. Given the sets $\Omega_{j,n}$ and their union $\widetilde{\Omega}_n=\bigcup_{j=1}^{n}\Omega_{j,n}$ for a particular $n\geq 0$, define $\Lambda_{j,n+1}$ to be the overlap

$$\Lambda_{j,n+1}=\Omega_{j,n}\cap\Big[\bigcup_{m\geq 1}(A^*)^{-m}\widetilde{\Omega}_n\Big]$$

if $n\geq 1$, or

$$\Lambda_{j,1}=\Omega_{j,0}\cap\Big[\Big(\bigcup_{m\geq 1}(A^*)^{-m}\widetilde{\Omega}_n\Big)\cup\Big(\bigcup_{k=1}^{j-1}\Omega_{k,0}\Big)\Big]$$

if $n=0$.

This additional complication at the $n=0$ step could just as well have been used in the Euclidean setting of $\mathbb{R}^d$, but it first appeared in [17] (2004) in the non-Euclidean setting in order to give the resulting algorithm the flexibility required to generate certain wavelets previously constructed by Kozyrev [65] (2002).

Then, for each j, build $\Omega_{j,n+1}$ from $\Omega_{j,n}$ by translating $\Lambda_{j,n+1}\subseteq\Omega_{j,n+1}$ by T_j, i.e.,

$$\Omega_{j,n+1}=\big(\Omega_{j,n}\smallsetminus\Lambda_{j,n+1}\big)\cup T_j\Lambda_{j,n+1}.$$

Finally, for each $j = 1, \ldots, N$, we set

$$\Lambda_j = \bigcup_{m \geq 1} \Lambda_{j,m} \quad \text{and} \quad \Omega_j = (\Omega_{j,0} \smallsetminus \Lambda_j) \cup T_j \Lambda_j. \tag{24}$$

Intuitively, Ω_j is a sort of limit of the sequence of sets $\{\Omega_{j,n}\}_{n \geq 0}$. We refer the reader to [17, Section 4.1] (2004) for a more detailed description of the algorithm, including verification that $\Lambda_{j,n+1}$ does indeed always lie in W, and hence it makes sense to consider $T_j \Lambda_{j,n+1}$.

9.2 *Validity of the algorithm*

The following theorem appeared as [17, Theorem 4.2] (2004).

Theorem 9.1. *Let G be a LCAG with compact open subgroup $H \subseteq G$, let $\mathscr{D} \subseteq \widehat{G}$ be a choice of coset representatives in $\widehat{G}$ for $\widehat{G}/H^{\perp}$, and let $A : G \to G$ be an expansive automorphism. Given the data listed in Section 9.1, the sets $\{\Omega_1, \ldots, \Omega_N\}$ of* (24) *produced by the algorithm of Section 9.1 form a wavelet collection of sets.*

We refer the reader to [17, Section 4.2] (2004) for the proof. The idea of the proof is to verify that $\{\Omega_1, \ldots, \Omega_N\}$ satisfy conditions *i* and *ii* of Theorem 8.1.

To verify condition *i*, we first check that $\bigcup_{m \in \mathbb{Z}} A^{*m} \widetilde{\Omega}$ covers $\widehat{G}$, where $\widetilde{\Omega} = \bigcup_{j=1}^{N} \Omega_j$. This fact follows from the expansiveness of A and the stipulation in the algorithm's initial data that $\widetilde{\Omega}_0 \supseteq (A^*)^{-\ell} H^{\perp}$. To prove that the covering of $\widehat{G}$ is in fact a tiling, we first note that $\Omega_{1,1}, \ldots, \Omega_{N,1}$ are pairwise disjoint, essentially by definition, because $\Lambda_{j,1}$ contains any overlap between $\Omega_{j,0}$ and $\Omega_{k,0}$ for $k < j$. The algorithm maintains this disjointness for $\Omega_{1,n}, \ldots, \Omega_{N,n}$ for each $n \geq 1$, as well as for the limiting sets $\Omega_1, \ldots, \Omega_N$. Meanwhile, the sets $\Lambda_{j,n}$ are the overlaps between $\Omega_{j,n}$ and the union of dilates $\bigcup_{m \geq 1} (A^*)^{-m} \widetilde{\Omega}_n$. By translating them via T_j out to

$(A^*W) \setminus W$, future overlaps should be successively smaller (as they will be compressed by $(A^*)^{-m}$ for $m \geq 1$), so that in the limit, the dilates of $\widetilde{\Omega}$ by different powers of A^{*m} are disjoint. Of course, the details of this verification of condition *i* are much more complicated, but that argument in [17] (2004) is not fundamentally different from the corresponding argument for $\mathbb{R}^d$ in [20] (1999), [21] (2001).

The proof of condition *ii*, on the other hand, requires a slight deviation from the methods of [20] (1999), [21] (2001). In both settings, the proof is relatively straightforward, because each $\Omega_{j,n}$ and Ω_j is of the form $(X \setminus Y) \cup T_j Y$, where $Y \subseteq X \subseteq W$, and X is already known to be $(\tau, \mathscr{D})$-congruent (or, in the $\mathbb{R}^d$ setting, simply τ-congruent) to $H^\perp$. In the $\mathbb{R}^d$ setting, the τ-congruence of the new set is immediate, because the lattice elements used for translations form a group. In our setting, with no lattice, the more complicated definition of T_j is required, with both the subtraction and the addition of an element of $\mathscr{D}$. The resulting $(\tau, \mathscr{D})$-congruence of the new set again follows easily, but the reader should note that the extra step of first subtracting the old element of $\mathscr{D}$ is crucial. Other than that slight complication, however, the proof of condition *ii* is largely similar to those in [20] (1999), [21] (2001).

10 Examples of wavelet sets for G

We now present some examples of wavelet sets. All the examples and figures here are taken from [17] (2004). See also [24] (2003) for more examples.

Example 10.1. Let G be a LCAG with compact open subgroup H, let $\mathscr{D}$ be a choice set of coset representatives in $\widehat{G}$ for $\widehat{G}/H^\perp$, and let A be an expansive automorphism of G.

Take $M = 0$, so that $W = H^\perp$, set $N = |A| - 1 \geq 1$, and let $\sigma_1, \ldots, \sigma_N$ be the N elements of $\mathscr{D} \cap [(A^*H^\perp) \setminus H^\perp]$. For each $j = 1, \ldots, N$, define $T_j(\gamma) = \gamma - \sigma_0' + \sigma_j$, where σ_0' denotes the unique element of $\mathscr{D} \cap H^\perp$, and define $\Omega_{j,0} = H^\perp$. Note that $\{H^\perp, T_1 H^\perp, \dots, T_N H^\perp\}$ is a set of $|A| = N+1$ compact open sets which together tile $A^* H^\perp$. See Figure 8 for a diagram of $\{T_j\}$ and $\Omega_{j,0}$ ($j = 1,2,3$) in the case that $|A| = 4$.

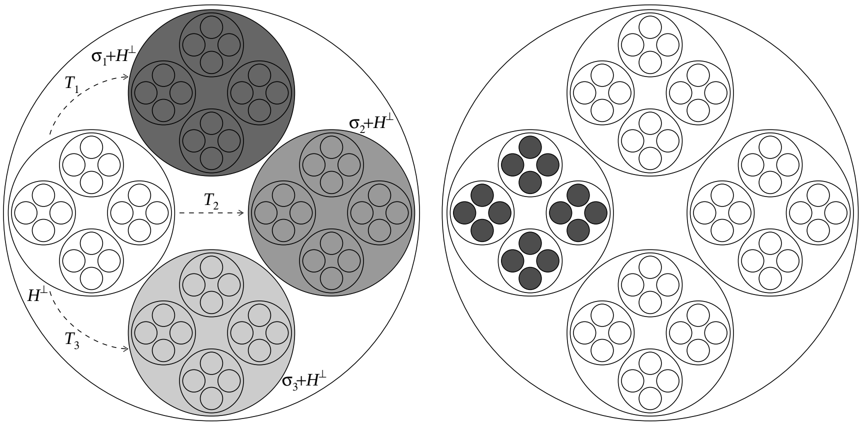

Fig. 8 The maps T_j and the sets $\Omega_{1,0} = \Omega_{2,0} = \Omega_{3,0}$ of Example 10.1, for $|A| = 4$.

As noted in [17, Section 5.1], applying the algorithm of Section 9 to this data gives

$$\Omega_j = \sigma_j + H^\perp \quad \text{for all } j \in \{1, \dots, N\}.$$

Indeed, because the sets $\Omega_{1,0}, \dots, \Omega_{N,0}$ all coincide, the algorithm immediately sets every $\Omega_{j,n}$, for $j \geq 2$ and $n \geq 1$, to be the final set $\Omega_j = \sigma_j + H^\perp$. Meanwhile, the more gradual evolution of $\Omega_{1,n}$ as n increases is illustrated in Figure 9; ultimately, the dark shading will cover precisely the top-most region $\Omega_1 = \sigma_1 + H^\perp$.

In this case, the choice $\mathscr{D}$ of coset representatives is ultimately irrelevant; if σ_j and σ_j' belong to the same coset of $H^\perp$ in $\widehat{G}$, then $\sigma_j + H^\perp = \sigma_j' + H^\perp$. However, as the later examples should illustrate, that happy circumstance is specific to this example, as is the fact that we can actually write down explicit formulas for the resulting wavelets. Indeed, as noted in [17, Proposition 5.1] (2004), the wavelet

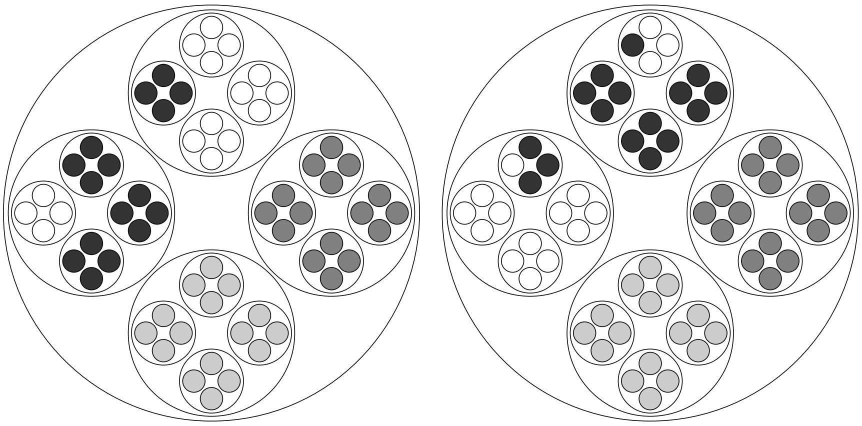

Fig. 9 The sets $\Omega_{j,m}$ ($j = 1,2,3$, $m = 1,2$) of Example 10.1, for $|A_1| = 4$.

generators are

$$\psi_j(x) = (x, \sigma_j)\mathbb{1}_H(x),$$

for $j = 1, \ldots, N$. This simple formula leads to the surprising observation that these wavelets can be considered simultaneously to be analogs of both Haar and Shannon wavelets. See [17, Section 5.1] (2004) and [24, Section 4] (2004). They had been previously discovered in the special cases of the Cantor dyadic group by Lang in [69] (1996), and of $\mathbb{Q}_p$ by Kozyrev in [65, Theorem 2] (2002).

Example 10.2. We can also easily produce single wavelets with the algorithm of Section 9.1. Let G be a LCAG with compact open subgroup H, let $\mathscr{D}$ be a choice of coset representatives in $\widehat{G}$ for $\widehat{G}/H^{\perp}$, and let A be an expansive automorphism of G.

Take $M = 0$, so that $W = H^{\perp}$, set $N = 1$, and let σ_1 be any one of the $|A| - 1$ elements of $\mathscr{D} \cap [(A^* H^{\perp}) \smallsetminus H^{\perp}]$. Define $T_1(\gamma) = \gamma - \sigma_0' + \sigma_1$, where σ_0' denotes the unique element of $\mathscr{D} \cap H^{\perp}$, and define $\Omega_{1,0} = H^{\perp}$. See Figure 10 for a diagram of T_1 and $\Omega_{1,0}$ in the case that $|A| = 4$.

As noted in [17, Section 5.2] (2004), it is easy to check that each $\Lambda_{1,n}$ is a translation of $(A^*)^{-n} H^{\perp}$, which is a dilation of $H^{\perp}$ of measure $\nu(\Lambda_{1,n}) = |A|^{-n}$. Thus, each

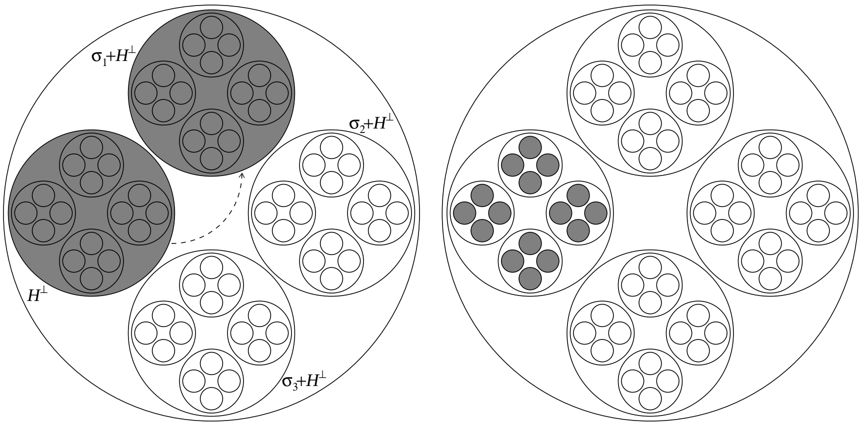

Fig. 10 The map T_1 and set $\Omega_{1,0}$ of Example 10.2, for $|A_1| = 4$.

step of the algorithm translates one more successively smaller translate of $(A^*)^{-n}H^{\perp}$ out of $H^{\perp}$ and into $H^{\perp}+\sigma_1$. See Figure 11 for illustrations of $\Omega_{1,1}$ and $\Omega_{1,2}$ in the case $|A| = 4$; it should be easy to extrapolate what $\Omega_{1,n}$ looks like for any $n \geq 1$, and ultimately, what the wavelet set Ω_1 is.

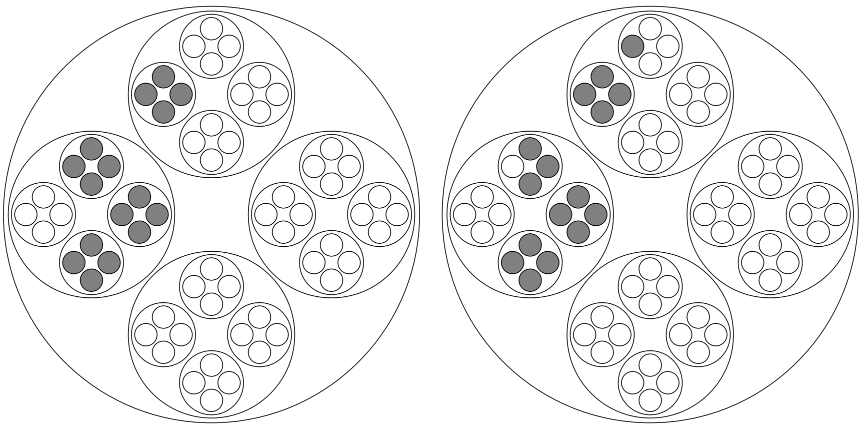

Fig. 11 $\Omega_{1,1}$ and $\Omega_{1,2}$ of Example 10.2.

Example 10.3. We close by giving one more example to illustrate that many other wavelets can be generated by the algorithm of Section 9.1, if one is willing to use more complicated translation functions T_j.

Let $G = \mathbb{Q}_3$, with compact open subgroup $H = \mathbb{Z}_3$, and let A be multiplication-by-$1/3$, so that A is expansive, with $|A| = 3$. As usual, identify $\widehat{G}$ as $\mathbb{Q}_3$ and $H^\perp$ as $\mathbb{Z}_3$. Let $\mathscr{D}$ be a set of coset representatives in $\widehat{G}$ for $\widehat{G}/H^\perp$ including $\sigma_0' = 0$, $\sigma_1 = 1/3$, and $\sigma_2 = 2/3$.

Take $M = 0$, so that $W = H^\perp$, set $N = 1$, and let $\Omega_{1,0} = H^\perp$. For $\gamma \in H^\perp$, define

$$T_1(\gamma) = \begin{cases} \gamma + 2/3 & \text{if } \gamma \in 1 + 3\mathbb{Z}_3, \\ \gamma + 1/3 & \text{if } \gamma \in (3\mathbb{Z}_3) \cup (2 + 3\mathbb{Z}_3), \end{cases}$$

as in Figure 12. Again, our algorithm is guaranteed to produce a single wavelet, but this time, because T_1 breaks $H^\perp$ into two pieces before translating, the wavelet set in question is more intricate than those of Examples 10.1 and 10.2. See Figures 13–14 for some of the resulting sets $\Omega_{1,m}$. Note in particular the very small disk that was moved from $\Omega_{1,2}$ to $\Omega_{1,3}$. The ultimate set Ω_1 will have successively smaller disks moved from $1 + 3\mathbb{Z}_3$ to $5/3 + 3\mathbb{Z}_3$ (i.e., from the left heavily-shaded disk of Figure 12 to the right one) and from $2 + 3\mathbb{Z}_3$ to $7/3 + 3\mathbb{Z}_3$ (i.e., from the lower right lightly-shaded disk of Figure 12 to the upper right one). As noted in [17, Section 5.3] (2004), we can describe this set explicitly as

$$\Omega_1 = \left[\mathbb{Z}_3 \setminus \bigcup_{n=1}^{\infty} \left((-5/8 + 3^{2n-2} + 3^{2n-1}\mathbb{Z}_3) \cup (-7/8 + 3^{2n-1} + 3^{2n}\mathbb{Z}_3) \right) \right]$$
$$\cup \bigcup_{m=1}^{\infty} \left((-7/24 + 3^{2n-2} + 3^{2n-1}\mathbb{Z}_3) \cup (-5/24 + 3^{2n-1} + 3^{2n}\mathbb{Z}_3) \right).$$

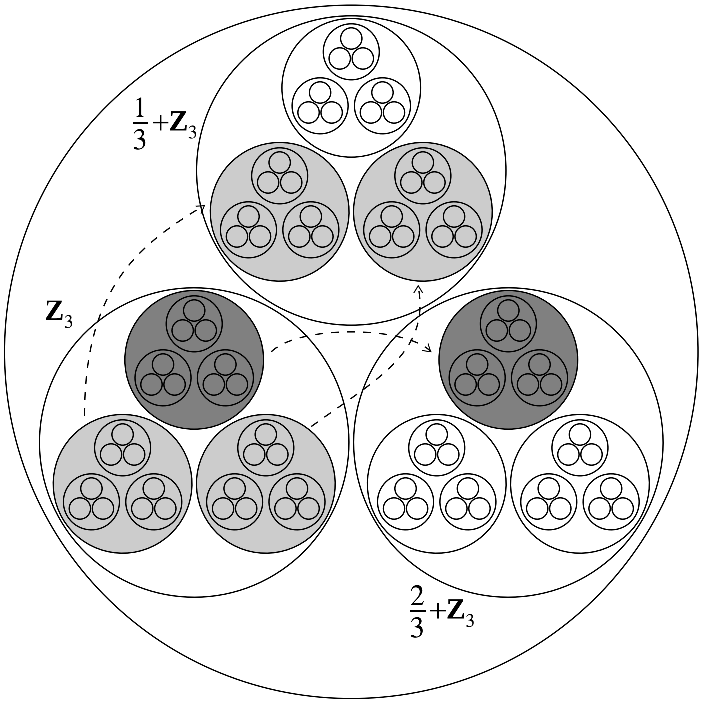

Fig. 12 The map T_1 of Example 10.3.

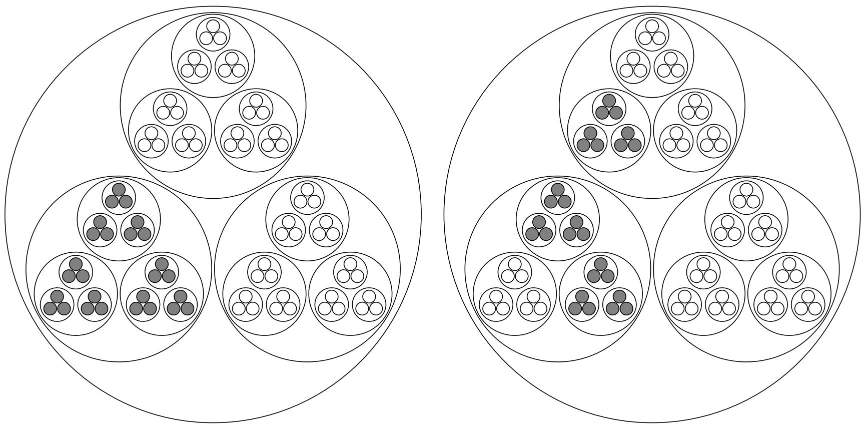

Fig. 13 $\Omega_{1,0}$ and $\Omega_{1,1}$ of Example 10.3.

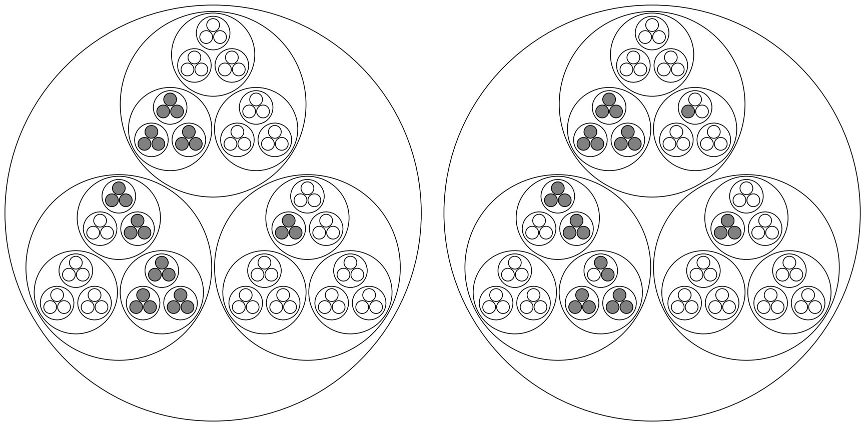

Fig. 14 $\Omega_{1,2}$ and $\Omega_{1,3}$ of Example 10.3.

11 Epilogue

We view the construction of wavelet sets as more than a sidebar in wavelet theory and its applications. There is, of course, the shear beauty and intricacy of many wavelet sets, and the natural questions of generalization, e.g., [20] (1999), as well as what type of theory will be required for such generalization, recalling the theories of [34] (1998) and [11] (1999) in the past.

There is also a host of geometric problems to be resolved. For example, besides the connectivity questions raised by Figures 2, 3, 4, one would like to know if there are connected wavelet sets with connected interior. Further, there are unresolved convexity questions. We know from [23] (2006) that a wavelet set $\Omega \subseteq \widehat{\mathbb{R}}^d$ cannot be decomposed into a union of d or fewer convex sets, and, in particular, wavelet sets cannot be convex, see Theorem 5.1. In recent work, [79] (2008), Merrill has constructed wavelet sets $\Omega \subseteq \widehat{\mathbb{R}}^2$ that are finite unions of 5 or more convex sets. The lower bound "5" for $\widehat{\mathbb{R}}^2$ is not necessarily sharp, and the existence of wavelet sets $\Omega \subseteq \widehat{\mathbb{R}}^d$, $d > 2$, which are finite unions of convex sets is not known.

Another topic of investigation is the tantalizing relation between wavelet sets and fractals, e.g., [22] (2002), see [18] (2009) for background.

Besides the purely mathematical issues of the previous paragraphs, there is the question of applicability of wavelet sets. Naturally, one might be suspicious of ever applying the wavelet sets in Figures 1 and 5 or the even more exotic ones in [20] (1999). However, Theorem 4.4 of Section 4, which we now repeat, provides the basis for implementation.

Theorem 4.4. *For each $n \geq 0$, $\Omega_n \setminus \Lambda_n$ is a Parseval frame wavelet set, and Ω_n is a frame wavelet set with frame bounds 1 and 2.*

In fact, sets such as $\Omega_0 \setminus \Lambda_0$ or Ω_1 can be elementary, computable shapes, and so we can construct a single wavelet frame $\{\psi_{m,n}\}$, where $\widehat{\psi} = \mathbb{1}_{\Omega_1 \setminus \Lambda_1}$, say, for $L^2(\mathbb{R}^d)$, $d >> 0$. Further, if rapid decay of the wavelet is desirable, there are existing frame preserving smoothing results, e.g., [2] (2001), [19] (2009), [10] (2006), [51] (1997), [50] (1997), [54] (1997), and research questions, see [19] (2009). Thus, single wavelet frames can be easily constructed to give computable decompositions of the elements of $L^2(\mathbb{R}^d)$, $d >> 0$, see Remark 1.2 in Section 1 about large data sets.

References

1. Patrick R. Ahern and Robert I. Jewett, *Factorization of locally compact abelian groups*, Illinois J. Math. **9** (1965), 230–235.
2. Hugo A. Aimar, Ana L. Bernardis, and Osvaldo P. Gorosito, *Perturbations of the Haar wavelet by convolution*, Proc. Amer. Math. Soc. **129** (2001), no. 12, 3619–3621 (electronic).
3. Syed Twareque Ali, Jean-Pierre Antoine, and Jean-Pierre Gazeau, *Coherent States, Wavelets and their Generalizations*, Springer-Verlag, New York, 2000.
4. Mikhail V. Altaiski, *p-adic wavelet decomposition vs Fourier analysis on spheres*, Indian J. Pure Appl. Math. **28** (1997), no. 2, 197–205.

5. Mikhail V. Altaisky, *p-adic wavelet transform and quantum physics*, Tr. Mat. Inst. Steklova **245** (2004), no. Izbr. Vopr. p-adich. Mat. Fiz. i Anal., 41–46.
6. Jean-Pierre Antoine, Yebeni B. Kouagou, Dominique Lambert, and Bruno Torrésani, *An algebraic approach to discrete dilations. Application to discrete wavelet transforms*, J. Fourier Anal. Appl. **6** (2000), no. 2, 113–141.
7. Jean-Pierre Antoine and Pierre Vandergheynst, *Wavelets on the* 2*-sphere: a group-theoretical approach*, Appl. Comput. Harmon. Anal. **7** (1999), no. 3, 262–291.
8. Pascal Auscher, *Solution of two problems on wavelets*, J. Geom. Anal. **5** (1995), no. 2, 181–236.
9. Larry Baggett, Alan Carey, William Moran, and Peter Ohring, *General existence theorems for orthonormal wavelets, an abstract approach*, Publ. Res. Inst. Math. Sci. **31** (1995), no. 1, 95–111.
10. Lawrence W. Baggett, Palle E. T. Jorgensen, Kathy D. Merrill, and Judith A. Packer, *Construction of Parseval wavelets from redundant filter systems*, J. Math. Phys. **46** (2005), no. 8, 083502, 28.
11. Lawrence W. Baggett, Herbert A. Medina, and Kathy D. Merrill, *Generalized multi-resolution analyses and a construction procedure for all wavelet sets in* $\mathbb{R}^n$, J. Fourier Anal. Appl. **5** (1999), no. 6, 563–573.
12. Lawrence W. Baggett and Kathy D. Merrill, *Abstract harmonic analysis and wavelets in* $\mathbb{R}^n$, The functional and harmonic analysis of wavelets and frames (San Antonio, TX, 1999), Amer. Math. Soc., Providence, RI, 1999, pp. 17–27.
13. John J. Benedetto, *Idele characters in spectral synthesis on* $\mathbb{R}/2\pi\mathbb{Z}$, Ann. Inst. Fourier **23** (1973), 43–64.
14. ______, *Zeta functions for idelic pseudo-measures*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **4** (1979), no. 6, 367–377.
15. ______, *Harmonic Analysis and Applications*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1997.
16. ______, *Frames, sampling, and seizure prediction*, Advances in Wavelets, edited by K.-S. Lau, Springer-Verlag, New York, 1998.
17. John J. Benedetto and Robert L. Benedetto, *A wavelet theory for local fields and related groups*, J. Geom. Anal. **14** (2004), 423–456.
18. John J. Benedetto and Wojciech Czaja, *Integration and Modern Analysis*, Birkhäuser Advanced texts, Birkhäuser Boston, MA, 2009.

19. John J. Benedetto and Emily J. King, *Smooth functions associated with wavelet sets on* $\mathbb{R}^d$, $d \geq 1$, *and frame bound gaps*, Acta Appl. Math. **107** (2009), no. 1-3, 121–142.

20. John J. Benedetto and Manuel T. Leon, *The construction of multiple dyadic minimally supported frequency wavelets on* $\mathbb{R}^d$, Contemporary Mathematics **247** (1999), 43–74.

21. ______, *The construction of single wavelets in d-dimensions*, J. of Geometric Analysis **11** (2001), 1–15.

22. John J. Benedetto and Songkiat Sumetkijakan, *A fractal set constructed from a class of wavelet sets*, Inverse problems, image analysis, and medical imaging (New Orleans, LA, 2001), Contemp. Math., vol. 313, Amer. Math. Soc., Providence, RI, 2002, pp. 19–35.

23. ______, *Tight frames and geometric properties of wavelet sets*, Adv. Comput. Math. **24** (2006), no. 1-4, 35–56.

24. Robert L. Benedetto, *Examples of wavelets for local fields*, Wavelets, Frames, and Operator Theory (College Park, MD, 2003), Amer. Math. Soc., Providence, RI, 2004, pp. 27–47.

25. Andrea Calogero, *A characterization of wavelets on general lattices*, J. Geom. Anal. **10** (2000), no. 4, 597–622.

26. Ole Christensen, *Frames, Riesz bases, and discrete Gabor/wavelet expansions*, Bull. Amer. Math. Soc. (N.S.) **38** (2001), no. 3, 273–291 (electronic).

27. Charles K. Chui, Wojciech Czaja, Mauro Maggioni, and Guido Weiss, *Characterization of general tight frames with matrix dilations and tightness preserving oversampling*, J. Fourier Anal. Appl. **8** (2002), no. 2, 173–200.

28. Albert Cohen and Ingrid Daubechies, *Nonseparable bidimensional wavelet bases*, Rev. Mat. Iberoamericana **9** (1993), no. 1, 51–137.

29. Ronald R. Coifman and Guido Weiss, *Book Review: Littlewood-Paley and Multiplier Theory*, Bull. Amer. Math. Soc. **84** (1978), no. 2, 242–250.

30. Jennifer Courter, *Construction of dilation-d wavelets*, AMS Contemporary Mathematics **247** (1999), 183–205.

31. Stephan Dahlke, *Multiresolution analysis and wavelets on locally compact abelian groups*, Wavelets, images, and surface fitting (Chamonix-Mont-Blanc, 1993), A K Peters, Wellesley, MA, 1994, pp. 141–156.

32. ______, *The construction of wavelets on groups and manifolds*, General algebra and discrete mathematics (Potsdam, 1993), Heldermann, Lemgo, 1995, pp. 47–58.

33. Xingde Dai, Yuanan Diao, Qing Gu, and Deguang Han, *Frame wavelets in subspaces of* $L^2(\mathbb{R}^d)$, Proc. Amer. Math. Soc. **130** (2002), no. 11, 3259–3267 (electronic).

34. Xingde Dai and David R. Larson, *Wandering vectors for unitary systems and orthogonal wavelets*, Mem. Amer. Math. Soc. **134** (1998), no. 640, viii+68.
35. Xingde Dai, David R. Larson, and Darrin M. Speegle, *Wavelet sets in* $\mathbb{R}^n$, J. Fourier Anal. Appl. **3** (1997), no. 4, 451–456.
36. ______, *Wavelet sets in* $\mathbb{R}^n$*. II*, Wavelets, multiwavelets, and their applications (San Diego, CA, 1997), Contemp. Math., vol. 216, Amer. Math. Soc., Providence, RI, 1998, pp. 15–40.
37. James Daly and Keith Phillips, *On singular integrals, multipliers,* $\mathfrak{H}^1$ *and Fourier series — a local field phenomenon*, Math. Ann. **265** (1983), no. 2, 181–219.
38. Ingrid Daubechies, *Ten Lectures on Wavelets*, CBMS-NSF Series in Applied Math. SIAM, Philadelphia, PA, 1992.
39. Ingrid Daubechies and Bin Han, *The canonical dual frame of a wavelet frame*, Appl. Comput. Harmon. Anal. **12** (2002), no. 3, 269–285.
40. Richard J. Duffin and Albert C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc. **72** (1952), 341–366.
41. Robert E. Edwards and Garth I. Gaudry, *Littlewood-Paley and multiplier theory*, Springer-Verlag, Berlin, 1977, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 90.
42. Xiang Fang and Xihua Wang, *Construction of minimally supported frequency wavelets*, J. Fourier Anal. Appl. **2** (1996), no. 4, 315–327.
43. Yuri A. Farkov, *Orthogonal wavelets on locally compact abelian groups*, Funktsional. Anal. i Prilozhen. **31** (1997), no. 4, 86–88.
44. ______, *Orthogonal p-wavelets on* $\mathbb{R}_+$, Wavelets and Splines, St. Petersburg Univ. Press, St. Petersburg, 2005, pp. 4–26.
45. ______, *Orthogonal wavelets with compact supports on locally compact abelian groups*, Izv. Ross. Akad. Nauk Ser. Mat. **69** (2005), no. 3, 193–220.
46. Michael Frazier, Björn Jawerth, and Guido Weiss, *Littlewood-Paley Theory and the Study of Function Spaces*, CBMS Regional Conference Series in Mathematics, vol. 79, Amer. Math. Soc., Washington, DC, 1991.
47. Gustaf Gripenberg, *A necessary and sufficient condition for the existence of a father wavelet*, Studia Math. **114** (1995), no. 3, 207–226.
48. Karlheinz Gröchenig and Wolodymyr R. Madych, *Multiresolution analysis, Haar bases, and self-similar tilings of* $L^2(\mathbb{R}^n)$, IEEE Trans. Information Theory **38(2)** (1992), 556–568.
49. Qing Gu and Deguang Han, *On multiresolution analysis (MRA) wavelets in* $\mathbb{R}^n$, J. Fourier Anal. Appl. **6** (2000), no. 4, 437–447.

50. Bin Han, *On dual wavelet tight frames*, Appl. Comput. Harmon. Anal. **4** (1997), no. 4, 380–413.

51. Deguang Han, *Unitary systems, wavelets, and operator algebras*, Ph.D. thesis, Texas A&M University, College Station, TX, 1997.

52. Deguang Han, David R. Larson, Manos Papadakis, and Theodoros Stavropoulos, *Multiresolution analyses of abstract Hilbert spaces and wandering subspaces*, The functional and harmonic analysis of wavelets and frames (San Antonio, TX, 1999), Amer. Math. Soc., Providence, RI, 1999, pp. 259–284.

53. Eugenio Hernández, Xihua Wang, and Guido Weiss, *Smoothing minimally supported frequency wavelets. I*, J. Fourier Anal. Appl. **2** (1996), no. 4, 329–340.

54. ______, *Smoothing minimally supported frequency wavelets, part II*, J. Fourier Anal. Appl. **3** (1997), 23–41.

55. Eugenio Hernández and Guido Weiss, *A First Course on Wavelets*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1996, With a foreword by Yves Meyer.

56. Edwin Hewitt and Kenneth A. Ross, *Abstract Harmonic Analysis, Volume I*, Springer-Verlag, New York, 1963.

57. ______, *Abstract Harmonic Analysis, Volume II*, Springer-Verlag, New York, 1970.

58. Matthias Holschneider, *Wavelet analysis over abelian groups*, Appl. Comput. Harmon. Anal. **2** (1995), no. 1, 52–60.

59. Huikun Jiang, Dengfeng Li, and Ning Jin, *Multiresolution analysis on local fields*, J. Math. Anal. Appl. **294** (2004), no. 2, 523–532.

60. Carolyn P. Johnston, *On the pseudodilation representations of Flornes, Grossmann, Holschneider, and Torresani*, J. Fourier Anal. Appl. **3** (1997), no. 4, 377–385.

61. Andrei Yu. Khrennikov and Sergei V. Kozyrev, *Wavelets on ultrametric spaces*, Appl. Comput. Harmon. Anal. **19** (2005), no. 1, 61–76.

62. Andrei Yu. Khrennikov, Vladimir M. Shelkovich, and Maria Skopina, *p-adic refinable functions and MRA-based wavelets*, J. Approx. Theory **161** (2009), no. 1, 226–238.

63. Andrei Kolmogorov, *Une contribution à l'étude de la convergence des séries de Fourier*, Fund. Math. **5** (1924), 96–97.

64. Jelena Kovačević and Martin Vetterli, *Nonseparable multidimensional perfect reconstruction filter banks and wavelet bases for* $\mathscr{R}^n$, IEEE Trans. Inform. Theory **38** (1992), no. 2, part 2, 533–555.

65. Sergei V. Kozyrev, *Wavelet analysis as a p-adic spectral analysis*, Izv. Math **66** (2002), 367–376.

66. ______, *p-adic pseudodifferential operators and p-adic wavelets*, Theoret. and Math. Phys. **138** (2004), no. 3, 322–332.

67. Jeffrey C. Lagarias and Yang Wang, *Integral self-affine tiles in* $\mathbb{R}^n$, *part II lattice tilings*, J. Fourier Anal. Appl. **3** (1997), 83–102.

68. Jeffrey C. Lagarias and Yang Wang, *Orthogonality criteria for compactly supported refinable functions and refinable function vectors*, J. Fourier Anal. Appl. **6** (2000), no. 2, 153–170.

69. W. Christopher Lang, *Orthogonal wavelets on the Cantor dyadic group*, SIAM J. Math. Anal. **27** (1996), 305–312.

70. ______, *Fractal multiwavelets related to the Cantor dyadic group*, Internat. J. Math. Math. Sci. **21** (1998), 307–314.

71. ______, *Wavelet analysis on the Cantor dyadic group*, Houston J. Math. **24** (1998), 533–544 and 757–758.

72. Pierre G. Lemarié, *Base d'ondelettes sur les groupes de Lie stratifiés*, Bull. Soc. Math. France **117** (1989), no. 2, 211–232.

73. Yunzhang Li, *On the holes of a class of bidimensional nonseparable wavelets*, J. Approx. Theory **125** (2003), no. 2, 151–168.

74. John E. Littlewood and Raymond E. A. C. Paley, *Theorems on Fourier series and power series*, J. London Math. Soc. **6** (1931), 230–233.

75. ______, *Theorems on Fourier series and power series III*, Proc. London Math. Soc. **43** (1937), 105–126.

76. Taras P. Lukashenko, *Wavelets on topological groups*, Izv. Ross. Akad. Nauk Ser. Mat. **58** (1994), no. 3, 88–102.

77. Wolodymyr R. Madych, *Some elementary properties of multiresolution analyses of* $L^2(\mathbb{R}^n)$, Wavelets: a Tutorial in Theory and Applications, edited by Charles K. Chui, Academic Press, Inc, San Diego, CA, 1992, pp. 256–294.

78. Steéphane G. Mallat, *A Wavelet Tour of Signal Processing*, Academic Press, Boston, 1998.

79. Kathy D. Merrill, *Simple wavelet sets for integral dilations in* $\mathbb{R}^2$, Representations, Wavelets, and Frames: A Celebration of the Mathematical Work of Lawrence W. Baggett, Birkhäuser, Boston, MA, 2008, pp. 177–192.

80. Yves Meyer, *Ondelettes, fonctions splines et analyses graduées*, Rend. Sem. Mat. Univ. Politec. Torino **45** (1987), no. 1, 1–42 (1988).

81. ______, *Ondelettes et Opérateurs*, Hermann, Paris, 1990.

82. ______, *Wavelets and Operators*, Cambridge Studies in Advanced Mathematics, vol. 37, Cambridge University Press, Cambridge, 1992, Translated from the 1990 French original by D. H. Salinger.

83. Cui Minggen, GuangHong Gao, and Phil Ung Chung, *On the wavelet transform in the field* $\mathbb{Q}_p$ *of p-adic numbers*, Appl. Comput. Harmon. Anal. **13** (2002), no. 2, 162–168.

84. Manos Papadakis, *Generalized frame multiresolution analysis of abstract Hilbert spaces*, Sampling, Wavelets, and Tomography, Birkhäuser, Boston, MA, 2003.

85. Manos Papadakis and Theodoros Stavropoulos, *On the multiresolution analyses of abstract Hilbert spaces*, Bull. Greek Math. Soc. **40** (1998), 79–92.

86. Lev Lemenovich Pontryagin, *Topological Groups, 2nd edition*, Gordon and Breach, Science Publishers, Inc., New York, 1966, Translated from the Russian by Arlen Brown.

87. Hans Reiter, *Classical Harmonic Analysis and Locally Compact Groups*, Oxford University Press, 1968.

88. Walter Rudin, *Fourier Analysis on Groups*, John Wiley and Sons, New York, 1962.

89. Eckart Schulz and Keith F. Taylor, *Extensions of the Heisenberg group and wavelet analysis in the plane*, Spline functions and the theory of wavelets (Montreal, PQ, 1996), Amer. Math. Soc., Providence, RI, 1999, pp. 217–225. MR 99m:42053

90. Vladimir Shelkovich and Maria Skopina, *p-adic Haar multiresolution analysis and pseudo-differential operators*, J. Fourier Anal. Appl. **15** (2009), no. 3, 366–393.

91. Paolo M. Soardi and David Weiland, *Single wavelets in n-dimensions*, J. Fourier Anal. Appl. **4** (1998), 299–315.

92. Elias M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J., 1970.

93. ______, *Topics in Harmonic Analysis Related to the Littlewood-Paley Theory.*, Annals of Mathematics Studies, No. 63, Princeton University Press, Princeton, N.J., 1970.

94. Elias M. Stein and Guido Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, Princeton, N.J., 1971, Princeton Mathematical Series, No. 32.

95. Robert S. Strichartz, *Construction of orthonormal wavelets*, Wavelets: Mathematics and Applications, Stud. Adv. Math., CRC, Boca Raton, FL, 1994, pp. 23–50.

96. Mitchell H. Taibleson, *Fourier Analysis on Local Fields*, Princeton U. Press, Princeton, NJ, 1975.

97. Khalifa Trimèche, *Continuous wavelet transform on semisimple Lie groups and inversion of the Abel transform and its dual*, Collect. Math. **47** (1996), no. 3, 231–268.

98. Victor G. Zakharov, *Nonseparable multidimensional Littlewood-Paley like wavelet bases*, Preprint, 1996.