Fourier Operators in Applied Harmonic Analysis

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Abstract We give a program describing the pervasiveness of the short-time Fourier transform (STFT) in a host of topics including the following: waveform design and optimal ambiguity function behavior for radar and communications applications; vector-valued ambiguity function theory for multi-sensor environments; finite Gabor frames for deterministic compressive sensing and as a background for the HRT conjecture; generalizations of Fourier frames and non-uniform sampling; and pseudo-differential operator frame inequalities.

1 Introduction

1.1 The Short Time Fourier Transform (STFT)

Let $\mathbb{Z}$ denote the ring of integers and let $\mathbb{C}$, respectively $\mathbb{R}$, denote the field of complex, respectively real, numbers. Given an integer N, let $\mathbb{Z}/N\mathbb{Z}$ denote the ring of integers modulo N. Unless otherwise noted, all of the vector spaces herein are complex vector spaces. Let $L^2(\mathbb{R}^d)$ be the space of square-integrable functions defined on the d-dimensional Euclidean space $\mathbb{R}^d$. We let $\widehat{\mathbb{R}}^d$ denote $\mathbb{R}^d$ considered as the Fourier, or spectral, domain. We define the Fourier transform of a Schwartz class function, $f \in \mathscr{S}(\mathbb{R}^d)$, as

$$\forall \gamma \in \widehat{\mathbb{R}}^d, \quad \hat{f}(\gamma) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \gamma}\, dx.$$

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The Fourier transform can be extended to the space $\mathscr{S}'(\mathbb{R}^d)$ of tempered distributions. Some references on harmonic analysis are [12, 99, 100].

Let $f, g \in L^2(\mathbb{R}^d)$. The *short-time Fourier transform* (STFT) of f with respect to g is the function $V_g f$ defined on $\mathbb{R}^{2d}$ as

$$V_g f(x,\omega) = \int_{\mathbb{R}^d} f(t)\overline{g(t-x)}e^{-2\pi i t\cdot\omega}\,dt,$$

see [49, 50]. The STFT is uniformly continuous on $\mathbb{R}^{2d}$. Furthermore, if $f, g \in L^2(\mathbb{R}^d)$, and $F = \hat{f}$ and $G = \hat{g}$, then the *fundamental identity of time-frequency analysis* is

$$V_g f(x,\omega) = e^{-2\pi i x\cdot\omega} V_G F(\omega, -x).$$

If $f, g \in L^2(\mathbb{R}^d)$, then it can be proved that

$$\left\|V_g f\right\|_{L^2(\mathbb{R}^{2d})} = \|f\|_{L^2(\mathbb{R}^d)}\,\|g\|_{L^2(\mathbb{R}^d)}\,. \tag{1}$$

Thus, if $\|g\|_{L^2(\mathbb{R}^d)} = 1$, then (1) allows us to assert that f is completely determined by $V_g f$. Furthermore, for a fixed "window" function $g \in L^2(\mathbb{R}^d)$ with $\|g\|_{L^2(\mathbb{R}^d)} = 1$, we can recover $f \in L^2(\mathbb{R}^d)$ from its STFT, $V_g f$, by means of the vector-valued integral inversion formula,

$$f = \int_{\mathbb{R}^d}\int_{\widehat{\mathbb{R}}^d} V_g f(x,\omega) e_\omega \tau_x g\, d\omega\, dx,$$

where $(e_\omega h)(t) = e^{2\pi i t\cdot\omega}h(t)$ and $\tau_x h(t) = h(t-x)$ represent modulation and translation, respectively.

Remark 1. a. Equation (1) is *Moyal's formula*, that is a special case of a formulation in 1949 due to José Enrique Moyal in the context of quantum mechanics as a statistical theory. When written in terms of the Wigner distribution from quantum mechanics (1932), this formulation is analogous to the orthogonality relations, that give rise to (1) for the STFT. It should also be pointed out that the Ville distribution for signal analysis also appeared in the late 1940s. These ideas are closely related, e.g., see [34, Chapter 8] and [48].

b. Closely related to the STFT and the Wigner and Ville distributions is the *narrow band cross-correlation ambiguity function* of $v, w \in L^2(\mathbb{R})$, defined as

$$\forall (t,\gamma) \in \mathbb{R}\times\widehat{\mathbb{R}}, \quad A(v,w)(t,\gamma) = \int_{\mathbb{R}} v(s+t)\overline{w(s)}e^{-2\pi i s\gamma}\,ds.$$

Note that $A(v,w)(t,\gamma) = e^{2\pi i t\gamma} V_w u(t,\gamma)$. The *narrow band radar ambiguity function*, $A(v)$, of $v \in L^2(\mathbb{R})$ is defined as

$$\begin{aligned}\forall (t,\gamma) \in \mathbb{R}\times\widehat{\mathbb{R}}, \quad A(v)(t,\gamma) &= \int_{\mathbb{R}} v(s+t)\overline{v(s)}e^{-2\pi i s\gamma}\,ds \\ &= e^{\pi i t\gamma}\int_{\mathbb{R}} v(s+\frac{t}{2})\overline{v(s-\frac{t}{2})}e^{-2\pi i s\gamma}\,ds.\end{aligned}$$

P. M. Woodward (1953) introduced the function, $A(v)$, to describe the effect of range and Doppler on matched filter receivers in radar. Underlying the function itself was his idea of using information theory to optimize resolution in terms of radar waveforms. By comparison with Shannon, Woodward dealt with the problem of mapping information into lower dimensions, prescient of current dimension reduction methodologies. This leads to ambiguities whence, the term, *ambiguity function*. Technical examples of such ambiguity abound in the radar literature, e.g., [97, 77]. In Sections 3 and 4, we concentrate on discrete versions of $A(v)$.

Whereas the narrow band ambiguity function is essentially time-frequency analysis, the wide band ambiguity function is essentially a wavelet transform.

c. The STFT can also be formulated in terms of so-called (X, μ) or continuous frames, e.g., see [1, 2, 6, 43, 46].

1.2 Outline and theme

Our theme is to interleave and compare various related decompositions whose coefficients are associated with sampled values of a given function. The tentacles of this process are labyrinthine and diverse.

In Section 2 we give the necessary background from harmonic analysis. We define balayage, sets of spectral synthesis and strict multiplicity, and provide material from the theory of frames.

Motivated by radar and communications applications of waveform design, Section 3 defines and discusses CAZAC sequences and optimal ambiguity function behavior on $\mathbb{Z}/N\mathbb{Z}$, and states a basic result. Because of the importance of dealing effectively with multi-sensor environments, Section 4 is devoted to the development of the vector-valued Discrete Fourier Transform (DFT) and proper definitions of vector-valued ambiguity functions. Perhaps surprisingly, this material requires more than using bold-faced notation.

Section 5 treats two topics dealing with finite Gabor systems: deterministic compressive sensing in terms of Gabor matrices and conditions to assert the linear independence of finite Gabor sums. The former gives elementary results embedded in advanced material developed by others. The latter addresses the HRT (Heil, Ramanathan, Topiwala) conjecture, and solves several special cases.

Sections 6 and 7 use the material on balayage, spectral synthesis, and strict multiplicity to formulate frame inequalities for the STFT and pseudo-differential operators, respectively. It builds on deep work of Beurling and Landau, and it is developed in the spirit of Fourier frames and non-uniform sampling formulas.

We close with a brief Appendix showing how the DFT can be used in practice to *compute* Fourier transforms on $\mathbb{R}$. We omit the required error estimates and generalizations. On the other hand, we include the Appendix since this computation requires the Classical Sampling Theorem (Theorem 17), thereby fitting naturally into our theme.

All of the aforementioned topics are unified by the STFT. Further, most of these topics have a long history with contributions by some of the most profound harmonic analysts. Our presentation has to be viewed in that context. Furthermore, our presentation is meant to integrate [17, 18, 5, 13, 14, 20, 6]. These references do have a common author, who wants to record the relationships between these topics, but who does not want to give the wrong impression about relative importance by having so many of his papers listed in the references.

2 Background from harmonic analysis

2.1 Balayage, spectral synthesis, and multiplicity

Let $M_b(G)$ be the algebra of bounded Radon measures on the locally compact abelian group (LCAG), G, with dual group denoted by $\widehat{G}$. The space $M_b(E)$ designates those elements of $M_b(G)$ for which $\mathrm{supp}(\mu) \subseteq E$, see [15]. We use Beurling's definition of balayage from his 1959-60 lectures.

Definition 1. Let $E \subseteq G$, and $\Lambda \subseteq \widehat{G}$ be closed sets. *Balayage* is possible for $(E,\Lambda) \subseteq G \times \widehat{G}$ if

$$\forall \mu \in M_b(G), \exists \nu \in M_b(E) \text{ such that } \hat{\mu} = \hat{\nu} \text{ on } \Lambda.$$

The notion of balayage originated in potential theory by Christoffel in the early 1870s, see [30], and then by Poincaré in 1890, who used the idea of balayage as a method to solve the Dirichlet problem, see [6] for historical background. The set, Λ, of group characters is the analogue of the original role of Λ in balayage as a collection of potential theoretic kernels. Kahane formulated balayage for the harmonic analysis of restriction algebras, see [63].

We shall also require the definition of spectral synthesis due to Wiener and Beurling.

Definition 2. Let $C_b(G)$ be the set of bounded continuous functions on the LCAG G. A closed set $\Lambda \subseteq \widehat{G}$ is a *set of spectral synthesis*, or *S-set*, if

$$\forall \mu \in M_b(G) \text{ and } \forall f \in C_b(G), \quad \mathrm{supp}(\hat{f}) \subseteq \Lambda \text{ and } \hat{\mu} = 0 \text{ on } \Lambda \implies \int_G f \, d\mu = 0, \tag{2}$$

see [11].

Remark 2. a. Equivalently, a closed set $\Lambda \subseteq \widehat{G}$ is a set of spectral synthesis if for all $T \in A'(\widehat{G})$ and for all $\phi \in A(\widehat{G})$, if $\mathrm{supp}(T) \subseteq \Lambda$ and $\phi = 0$ on Λ, then $T(\phi) = 0$. This equivalence follows from an elementary functional analysis argument. Here, $A(\widehat{G})$ is the Banach algebra of absolutely convergent Fourier transforms on $\widehat{G}$, taken with the transported topology from $L^1(G)$; and $A'(\widehat{G})$ is its dual space.

b. To determine whether or not $\Lambda \subseteq \widehat{G}$ is a set of spectral synthesis is closely related to the problem of determining the ideal structure of the convolution algebra $L^1(G)$, and so a fundamental theorem about sets of spectral synthesis can be thought of in the context of a Nullstellensatz of harmonic analysis. The problem of characterizing *S*-sets emanated from Wiener's Tauberian theorems and was developed by Beurling in the 1940s. It is "synthesis" in that one wishes to approximate $f \in L^\infty(G)$ in the $\sigma(L^\infty(G), L^1(G))$ (weak-$*$) topology by finite sums of characters, $\gamma : L^\infty(G) \to \mathbb{C}$, that is, each γ is a continuous homomorphism $G \to \{z \in \mathbb{C} : |z| = 1\}$ under multiplication. Further, γ can be considered an element of Λ with $\mathrm{supp}(\delta_\gamma) \subseteq \mathrm{supp}(\hat{f})$, where $\mathrm{supp}(\hat{f})$ is the so-called *spectrum* of f. Such an approximation is elementary to achieve with convolutions of the measures δ_γ, but in this case we lose the essential property that the spectra of the approximants be contained in the spectrum of f.

c. The annihilation property of (2) holds when f and μ have balancing smoothness and irregularity. For example, if $\hat{f} \in \mathscr{S}'(\widehat{\mathbb{R}}^d)$, $\hat{\mu} = \phi \in \mathscr{S}(\widehat{\mathbb{R}}^d)$, and $\phi = 0$ on $\mathrm{supp}(\hat{f})$, then $\hat{f}(\phi) = 0$. Similarly, the same annihilation holds for the pairing of $M_b(\widehat{\mathbb{R}}^d)$ and $C_0(\widehat{\mathbb{R}}^d)$.

d. The sphere $S^2 \subseteq \widehat{\mathbb{R}}^3$ is not an S-set (proven by Schwartz in 1947). Also, every non-discrete $\widehat{G}$ has non-S-sets (proven by Malliavin in 1959). Polyhedra are S-sets while the 1/3-Cantor set is an S-set with non-S-subsets, see [11].

Definition 3. A closed set $\Gamma \subseteq \widehat{\mathbb{R}}^d$ is a set of *strict multiplicity* if

$$\exists \mu \in M_b(\Gamma) \setminus \{0\} \text{ such that } \lim_{\|x\| \to \infty} |\check{\mu}(x)| = 0,$$

where $\check{\mu}$ is the inverse Fourier transform of μ and $\|x\|$ denotes the standard Euclidean norm of $x \in \mathbb{R}^d$. This is also well-defined for G and $\widehat{G}$.

The notion of strict multiplicity was motivated by Riemann's study of sets of uniqueness for trigonometric series. In 1916 Menchov showed that there exist a closed $\Gamma \subseteq \widehat{\mathbb{R}}/\mathbb{Z}$ and $\mu \in M(\Gamma) \setminus \{0\}$ such that $|\Gamma| = 0$ and $\check{\mu}(n) = O((\log |n|)^{-1/2})$ as $|n| \to \infty$ ($|\Gamma|$ is the Lebesgue measure of Γ). There have been intricate refinements of Menchov's result by Bary (1927), Littlewood (1936), Salem [94, 95], Ivašev-Mucatov (1957), and Beurling, et al., see [11].

The above concepts are used in the deep proof of the following theorem.

Theorem 1. *Assume that $\Lambda \subseteq \widehat{\mathbb{R}}^d$ is an S-set of strict multiplicity, and that balayage is possible for $(E, \Lambda) \subseteq \mathbb{R}^d \times \widehat{\mathbb{R}}^d$. Let $\Lambda_\varepsilon = \{\gamma \in \widehat{\mathbb{R}}^d : dist(\gamma, \Lambda) \leq \varepsilon\}$. There is $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then balayage is possible for (E, Λ_ε).*

2.2 Frames

Definition 4. Let H be a separable Hilbert space, e.g., $H = L^2(\mathbb{R}^d), \mathbb{R}^d$, or $\mathbb{C}^d$. A sequence $F = \{x_i\}_{i \in I} \subseteq H$ is a *frame* for H if there exist constants $A, B > 0$ such that

$$\forall x \in H, \quad A\,\|x\|^2 \leq \sum_{i\in I} |\langle x, x_i\rangle|^2 \leq B\,\|x\|^2 .$$

The constants A and B are the *lower* and *upper frame bounds*, respectively. If $A = B$, we say that the frame is a *tight frame* for H. We call a finite unit-norm tight frame a *FUNTF*.

Frames are a natural tool for dealing with numerical stability, over-completeness, noise reduction, and robust representation problems. Frames were first defined by Duffin and Schaeffer [37] in 1952 but appeared even earlier in Paley and Wiener's book [83] in 1934. Since then, significant contributions have been made by Beurling [21, 22], Beurling and Malliavin [23, 24], Kahane [62], Landau [76], Jaffard [61], and Seip [96, 82]. Recent expositions on the theory and applications of frames include [32, 72, 73].

Theorem 2. *If $F = \{x_i\}_{i\in I} \subseteq H$ is a frame for H, then*

$$\forall x \in H, \quad x = \sum_{i\in I} \langle x, S^{-1}x_i\rangle x_i = \sum_{i\in I} \langle x, x_i\rangle S^{-1}x_i,$$

where the map, $S : H \to H$, $x \mapsto \sum_{i\in I} \langle x, x_i\rangle x_i$, is a well-defined topological isomorphism.

Theorem 2 illustrates the natural role that frames play in non-uniform sampling formulas.

Let $\Lambda \subseteq \widehat{\mathbb{R}}^d$ be a closed set. The *Paley-Wiener space*, PW_Λ, is defined as

$$PW_\Lambda = \{f \in L^2(\mathbb{R}^d) : \text{supp}(\hat{f}) \subseteq \Lambda\}.$$

Definition 5. Let $\Lambda \subseteq \widehat{\mathbb{R}}^d$ be a compact set and let $E = \{x_i\}_{i\in I} \subseteq \mathbb{R}^d$ be a sequence. For each $x \in E$, define $f_x = (e_{-x}\mathbb{1}_\Lambda)^\vee \in PW_\Lambda$, where $\mathbb{1}_\Lambda$ denotes the characteristic function of the set Λ. The sequence $\{f_x : x \in E\}$ is a *Fourier frame* for PW_Λ if there exist constants $A, B > 0$ such that

$$\forall f \in PW_\Lambda, \quad A\,\|f\|^2_{L^2(\mathbb{R}^d)} \leq \sum_{x\in E} |f(x)|^2 \leq B\,\|f\|^2_{L^2(\mathbb{R}^d)} .$$

Definition 6. A sequence $E \subseteq \mathbb{R}^d$ is *separated* if

$$\exists r > 0 \text{ such that } \inf\{\|x - y\| : x, y \in E \text{ and } x \neq y\} \geq r.$$

The following theorem due to Beurling gives a sufficient condition for the existence of Fourier frames in terms of balayage. The proof uses Theorem 1, and its history and structure are analyzed in [6] as part of a more general program.

Theorem 3 (Beurling). *Assume that $\Lambda \subseteq \widehat{\mathbb{R}}^d$ is an S-set of strict multiplicity and that $E \subseteq \mathbb{R}^d$ is a separated sequence. Further assume that for every $\gamma \in \Lambda$ and for every compact neighborhood $N(\gamma)$, $\Lambda \cap N(\gamma)$ is a set of strict multiplicity. If* *balayage is possible for* (E,Λ)*, then* $\{(e_{-x}\mathbb{1}_\Lambda)^\vee : x \in E\}$ *is a Fourier frame for* PW_Λ.

Example 1. The conclusion of Theorem 3 is the assertion

$$\forall f \in PW_\Lambda, \quad f = \sum_{x\in E} f(x) S^{-1}(f_x) = \sum_{x\in E} \langle f, S^{-1}(f_x)\rangle f_x,$$

where $S(f) = \sum_{x\in E} f(x)\,(e_{-x}\mathbb{1}_\Lambda)^\vee$.

3 Optimal ambiguity function behavior on $\mathbb{Z}/N\mathbb{Z}$

Definition 7. A function, $u : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$, is *Constant Amplitude Zero Autocorrelation* (CAZAC) if

$$\forall m \in \mathbb{Z}/N\mathbb{Z}, \quad |u[m]| = 1, \quad \text{(CA)}$$

and

$$\forall m \in \mathbb{Z}/N\mathbb{Z} \setminus \{0\}, \quad \frac{1}{N}\sum_{k=0}^{N-1} u[m+k]\overline{u[k]} = 0. \quad \text{(ZAC).}$$

Equation (CA) is the condition that u has constant amplitude 1. Equation (ZAC) is the condition that u has zero autocorrelation for $m \in (\mathbb{Z}/N\mathbb{Z}) \setminus \{0\}$, i.e., off the dc-component.

The study of CAZAC sequences and other sequences related to optimal autocorrelation behavior is deeply rooted in several important applications. One of the most prominent applications is the area of waveform design associated with radar and communications. See, e.g., [7, 51, 19, 33, 45, 47, 57, 69, 70, 77, 81, 88, 90, 97, 102, 103]. There has been a striking recent application of low correlation sequences to radar in terms of compressed sensing [58].

There are also purely mathematical roots for the construction of CAZAC sequences. One example, that inspired the role of probability theory in the subject, is due to Wiener, see [16]. Another originated in a question by Per Enflo in 1983 asking about specific Gaussian sequences to deal with the estimation of certain exponential sums, see [93] by Saffari for the role played by Björck, cf. [26, 27].

Do there exist only finitely many non-equivalent CAZAC sequences in $\mathbb{Z}/N\mathbb{Z}$? The answer to this question is "yes" for N prime and "no" for $N = MK^2$, see, e.g., [17, 93]. For the case of non-prime square-free N, special cases are known, and there are published arguments asserting general results.

Definition 8. Let p be a prime number, and so $\mathbb{Z}/p\mathbb{Z}$ is a field. A *Björck CAZAC sequence*, b_p, of length p is defined as

$$\forall k = 0,1,...,p-1, \quad b_p[k] = e^{i\theta_p(k)},$$

where, for $p \equiv 1 \pmod 4$,

$$\theta_p(k) = \arccos\left(\frac{1}{1+\sqrt{p}}\right)\left(\frac{k}{p}\right)$$

and, for $p \equiv 3 \pmod 4$,

$$\theta_p(k) = \frac{1}{2}\arccos\left(\frac{1-p}{1+p}\right)[(1-\delta_k)\left(\frac{k}{p}\right)+\delta_k].$$

Here, δ_k is the Kronecker delta and $\left(\frac{k}{p}\right)$ is the Legendre symbol defined by

$$\left(\frac{k}{p}\right) = \begin{cases} 0, & \text{if } k \equiv 0 \pmod p, \\ 1, & \text{if } k \equiv n^2 \pmod p \text{ for some } n \in \mathbb{Z}, \\ -1, & \text{if } k \not\equiv n^2 \pmod p \text{ for all } n \in \mathbb{Z}. \end{cases}$$

In [25] Björck proved that Björck sequences are CAZAC sequences, and there is a longstanding collaboration of Björck and Saffari in the general area, see [27] for references.

Definition 9. Let $u : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$. The *discrete narrow band ambiguity function*, $A_N(u) : \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$, of u is defined as

$$\forall (m,n) \in \mathbb{Z}/N\mathbb{Z} \times \widehat{\mathbb{Z}/N\mathbb{Z}}, \quad A_N(u)[m,n] = \frac{1}{N}\sum_{k=0}^{N-1} u[m+k]\overline{u[k]}e^{-2\pi ikn/N}. \tag{3}$$

The *discrete autocorrelation* of u is the function, $A_N(u)[\cdot,0] : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$.

The following estimate is proved in [13]. Notwithstanding the difficulty of proof, its formulation was the result of observations by two of the authors of [13] based on extensive computational work by one of them, viz., Woodworth.

Theorem 4. *Let b_p denote the Björck CAZAC sequence of prime length p, and let $A_p(b_p)$ be the discrete narrow band ambiguity function defined on $\mathbb{Z}/p\mathbb{Z} \times \widehat{\mathbb{Z}/p\mathbb{Z}}$. Then,*

$$\forall (m,n) \in (\mathbb{Z}/p\mathbb{Z} \times \widehat{\mathbb{Z}/p\mathbb{Z}}) \setminus (0,0),$$

$$|A_p(b_p)[m,n]| < \tfrac{2}{\sqrt{p}} + \tfrac{4}{p}, \quad \text{if } p \equiv 1 \pmod 4,$$

and

$$|A_p(b_p)[m,n]| < \frac{2}{\sqrt{p}} + \frac{4}{p^{3/2}}, \quad \text{if } p \equiv 3 \pmod 4.$$

The proof of Theorem 4 requires Weil's exponential sum bound [106], which is a consequence of his proof of the Riemann Hypothesis for curves over finite fields [107].

Theorem 4 establishes essentially optimal ambiguity function behavior for b_p, cf. Example 2 and Section 5.1. In this regard, and by comparison, if u is any CAZAC sequence of length p, then

$$\frac{1}{\sqrt{p-1}} \leq \max\{|A_p(u)[m,n]| : (m,n) \in (\mathbb{Z}/p\mathbb{Z} \times \widehat{\mathbb{Z}/p\mathbb{Z}}) \setminus \{(0,0)\}\}.$$

Example 2. a. Let p be a prime number. Alltop [3] defined the sequence, a_p, of length p as

$$\forall k = 0,1,...,p-1, \quad a_p[k] = e^{2\pi i k^3/p}.$$

Clearly, a_p is of constant amplitude (CA). Alltop proved that

$$\forall m \in (\mathbb{Z}/p\mathbb{Z}) \setminus \{0\} \text{ and } \forall n \in \mathbb{Z}/p\mathbb{Z}, \quad |A_p(a_p)[m,n]| = \frac{1}{\sqrt{p}},$$

which is an excellent bound, cf. Theorem 4 and Section 5.1, but also establishes that a_p is *not* a CAZAC sequence in contrast to b_p.

b. The structure of $A_p(b_p)$ is also more complex than that of $A_p(a_p)$ in that $|A_p(b_p)|$ takes values smaller than $1/\sqrt{p}$, a feature that can be used in radar and communications. This goes back to [13] with continuing work by one of those authors and Nava-Tudela.

4 The vector-valued DFT and ambiguity functions

4.1 The vector-valued DFT

Let $N \geq d$. Form an $N \times d$ matrix using any d columns of the $N \times N$ DFT matrix $(e^{2\pi i jk/N})_{j,k=0}^{N-1}$. The rows of this matrix, up to multiplication by $1/\sqrt{d}$, form a FUNTF for $\mathbb{C}^d$.

Definition 10. Let $N \geq d$ and let $s : \mathbb{Z}/d\mathbb{Z} \to \mathbb{Z}/N\mathbb{Z}$ be injective. The rows $\{E_m\}_{m=0}^{N-1}$ of the $N \times d$ matrix,

$$\left(e^{2\pi i m s(n)/N}\right)_{m,n},$$

form an equal-norm tight frame for $\mathbb{C}^d$, that we call a *DFT frame*.

Definition 11. Let $\{E_k\}_{k=0}^{N-1}$ be a DFT frame for $\mathbb{C}^d$. Given $u : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}^d$, we define the *vector-valued discrete Fourier transform* of u by

$$\forall n \in \mathbb{Z}_N, \quad F(u)(n) = \hat{u}(n) = \sum_{m=0}^{N-1} u(m) * E_{-mn},$$

where $*$ is pointwise (coordinatewise) multiplication. We have that

$$F : \ell^2(\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}) \to \ell^2(\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z})$$

is a linear operator.

The following inversion formula for the vector-valued DFT is proved in [5].

Theorem 5. *The vector-valued Fourier transform is invertible if and only if s, the function defining the DFT frame, has the property that*

$$\forall n \in \mathbb{Z}/d\mathbb{Z}, \quad (s(n), N) = 1.$$

The inverse is given by

$$\forall m \in \mathbb{Z}/N\mathbb{Z}, \quad u(m) = F^{-1}\hat{u}(m) = \frac{1}{N}\sum_{n=0}^{N-1} \hat{u}(n) * E_{mn}.$$

In this case we also have that $F^*F = FF^* = NI$, where I is the identity operator.

In particular, the inversion formula is valid for N prime.

We also note here that vector-valued DFT uncertainty principle inequalities are valid, similar to the results [31] in compressive sensing.

4.2 Vector-valued ambiguity functions and frame multiplication

4.2.1 An ambiguity function for vector-valued functions

Given $u : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}^d$. If $d = 1$, then we can write the discrete ambiguity function, $A_N(u)$, as

$$A_N(u)[m,n] = \frac{1}{N}\sum_{k=0}^{N-1} \langle u(m+k), u(k)e_{nk}\rangle, \tag{4}$$

where recall $e_n = e^{2\pi i n/N}$. For $d > 1$, the problem of defining a discrete periodic ambiguity function has two natural settings: either it is $\mathbb{C}$-valued or $\mathbb{C}^d$-valued, i.e., $A_N^1(u)[m,n] \in \mathbb{C}$ or $A_N^d(u)[m,n] \in \mathbb{C}^d$. The problem and its solutions were first outlined in [18] (2008).

Let us consider the case $A_N^1(u)[m,n] \in \mathbb{C}$. Motivated by (4), we must find a sequence $\{E_k\} \subseteq \mathbb{C}^d$ and an operator, $* : \mathbb{C}^d \times \mathbb{C}^d \to \mathbb{C}^d$, so that

$$A_N^1(u)[m,n] = \frac{1}{N}\sum_{k=0}^{N-1} \langle u(m+k), u(k) * E_{nk}\rangle \in \mathbb{C} \tag{5}$$

defines a meaningful ambiguity function.

There is a natural way to proceed motivated by the fact that $e_m e_n = e_{m+n}$. To effect this definition, we shall make the following three *ambiguity function assumptions*. First, we assume that there is a sequence $\{E_k\}_{k=0}^{N-1} \subseteq \mathbb{C}^d$ and an operation,

$*$, with the property that $E_m * E_n = E_{m+n}$ for $m,n \in \mathbb{Z}/N\mathbb{Z}$. Second, to deal with $u(k) * E_{nk}$ in (5), where $u(k) \in \mathbb{C}^d$, we also assume that $\{E_k\}_{k=0}^{N-1} \subseteq \mathbb{C}^d$ is a tight frame for $\mathbb{C}^d$. The multiplication nk is modular multiplication in $\mathbb{Z}/N\mathbb{Z}$. Third, we assume that $* : \mathbb{C}^d \times \mathbb{C}^d \to \mathbb{C}^d$ is bilinear, in particular,

$$\left(\sum_{j=0}^{N-1} c_j E_j\right) * \left(\sum_{k=0}^{N-1} d_k E_k\right) = \sum_{j=0}^{N-1}\sum_{k=0}^{N-1} c_j d_k E_j * E_k.$$

Example 3. Let $\{E_j\}_{j=0}^{N-1} \subseteq \mathbb{C}^d$ satisfy the three ambiguity function assumptions. Then,

$$E_m * E_n = \frac{d^2}{N^2} \sum_{j=0}^{N-1}\sum_{k=0}^{N-1} \langle E_m, E_j\rangle \langle E_n, E_k\rangle E_{j+k}. \tag{6}$$

Further, let $\{E_j\}_{j=0}^{N-1}$ be a DFT frame, and let r designate a fixed column. Assume, without loss of generality, that the $N \times d$ matrix for the frame consists of the first d columns of the $N \times N$ DFT matrix. Then (6) gives

$$(E_m * E_n)(r) = \frac{e^{2\pi i (m+n)r/N}}{\sqrt{d}} = E_{m+n}(r).$$

Consequently, for DFT frames, $*$ is componentwise multiplication in $\mathbb{C}^d$ with a factor of $\sqrt{d}$. In particular, we have shown that if $u : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}^d$, then $A_N^1(u)$ is well-defined and can be written explicitly for the case of DFT frames and componentwise multiplication, $*$, in $\mathbb{C}^d$.

The definition of $*$ is intrinsically related to the “addition” defined on the indices of the frame elements. In fact, it is not pre-ordained that this “addition” must be modular addition on $\mathbb{Z}/N\mathbb{Z}$, as was the case in Example 3. Formally, we could have $E_m * E_n = E_{m \bullet n}$ for some function $\bullet : \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \to \mathbb{Z}/N\mathbb{Z}$. The following example exhibits this phenomenon for the familiar case of cross products from the calculus, see [18].

Example 4 ($A_N^1(u)$ for cross product frames). Define $* : \mathbb{C}^3 \times \mathbb{C}^3 \to \mathbb{C}^3$ to be the cross product on $\mathbb{C}^3$. Let $\{i, j, k\}$ be the standard basis for $\mathbb{C}^3$, e.g., $i = (1,0,0) \in \mathbb{C}^3$. We have that $i * j = k$, $j * i = -k$, $k * i = j$, $i * k = -j$, $j * k = i$, $k * j = -i$, $i * i = j * j = k * k = 0$. The union of tight frames and the zero vector is a tight frame. In fact, $\{0, i, j, k, -i, -j, -k\}$ is a tight from for $\mathbb{C}^3$ with frame constant 2. Let $E_0 = 0$, $E_1 = i$, $E_2 = j$, $E_3 = k$, $E_4 = -i$, $E_5 = -j$, and $E_6 = -k$. The index operation corresponding to the frame multiplication is the non-abelian operation $\bullet : \mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z} \to \mathbb{Z}/7\mathbb{Z}$, where we compute

$$\begin{array}{ccccc}
1 \bullet 2 = 3, & 1 \bullet 3 = 5, & 1 \bullet 4 = 0, & 1 \bullet 5 = 6, & 1 \bullet 6 = 2, \\
2 \bullet 1 = 6, & 2 \bullet 3 = 1, & 2 \bullet 4 = 3, & 2 \bullet 5 = 0, & 2 \bullet 6 = 4, \\
3 \bullet 1 = 2, & 3 \bullet 2 = 4, & 3 \bullet 4 = 5, & 3 \bullet 5 = 1, & 3 \bullet 6 = 0,
\end{array}$$
$$n \bullet n = 0, \qquad n \bullet 0 = 0 \bullet n = 0, \qquad \text{etc.}$$

Thus, the ambiguity function assumptions are valid, with the verification of bilinearity from the definition of the cross product being a tedious calculation. In any case, we can now obtain the following formula:

$$\forall u, v \in \mathbb{C}^3, \quad u * v = \frac{1}{4} \sum_{j=1}^{6} \sum_{k=1}^{6} \langle u, E_j \rangle \langle v, E_k \rangle E_{j \bullet k}.$$

Consequently, $A_N^1(u)$ *is well-defined for the case of this cross product frame and associated bilinear operator*, $*$.

4.2.2 Frame multiplication

The essential idea and requirement to define ambiguity functions for $u : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}^d$ is to formulate an effective notion of *frame multiplication*. This was the purpose of the exposition in Section 4.2.1 and of [18], where we further noted the substantive role of group theory in this process.

In fact, the set $\{0, \pm i, \pm j, \pm k\}$ of Example 4 is a quasi-group, and the quaternion group of order 8, viz., $\{\pm 1, \pm i, \pm j, \pm k\}$, fits into our theory, see Andrews [4] who develops frame multiplication theory for non-abelian finite groups.

We begin this subsection by defining frame multiplication along the lines motivated in Section 4.2.1. Then we shall define frame multiplication associated with a group. Our theory characterizes the groups for which frame multiplication is possible; and, in this case, ambiguity functions can be defined for $\mathbb{C}^d$-valued functions. We shall state some results when the underlying group is abelian, see [5] for the full theory.

Definition 12. *a.* Let $F = \{x_i\}_{i \in I}$ be a frame for a finite dimensional Hilbert space, H, and let $\bullet : I \times I \to I$ be a binary operation. We say $\bullet$ is a *frame multiplication* for F if there is a bilinear map, $* : H \times H \to H$, such that

$$\forall i, j \in I, \quad x_i * x_j = x_{i \bullet j}.$$

Thus, $\bullet$ defines a frame multiplication for F if and only if, for every $x = \sum_{i \in I} a_i x_i$ and $y = \sum_{i \in I} b_i x_i$ in H,

$$x * y = \sum_{i, j \in I} a_i b_j x_{i \bullet j}$$

is well-defined, independent of the frame representations of x and y.

b. Let $(G, \bullet)$ be a finite abelian group, and let $F = \{x_g\}_{g \in G}$ be a frame for a finite dimensional Hilbert space. We say $(G, \bullet)$ defines a *frame multiplication* for F if there is a bilinear map, $* : H \times H \to H$, such that

$$\forall g, h \in G, \quad x_g * x_h = x_{g \bullet h}.$$

Definition 13. Let $(G, \bullet)$ be a finite group. A finite tight frame $F = \{x_g\}_{g \in G}$ for a Hilbert space H is a *G-frame* if there exists $\pi : G \to \mathscr{U}(H)$, a unitary representation of G, such that

$$\forall g, h \in G, \quad \pi(g)x_h = x_{g \bullet h}.$$

Here, $\mathscr{U}(H)$ is the group of unitary operators on H.

Remark 3. The notion of G-frames is a natural one with slightly varying definitions. Definition 13 has been used extensively by Vale and Waldron [105]. Closely related, there are *geometrically uniform frames*, see Bölcskei and Eldar [28], Forney [44], Heath and Strohmer [101], and Slepian [98], as well as a more general formulation due to Han and Larson [52, 53].

We can prove the following theorem, see [5].

Theorem 6. *Let $(G, \bullet)$ be a finite abelian group and let $F = \{x_g\}_{g\in G}$ be a tight frame for a finite dimensional Hilbert space H. Then G defines a frame multiplication for F if and only if F is a G-frame.*

Definition 14. *a*. Let $(G, \bullet)$ be a finite abelian group of order N. Thus, G has exactly N characters, i.e., N group homomorphisms, $\gamma_j : G \to \mathbb{C}^\times$, where $\mathbb{C}^\times$ is the multiplicative group, $\mathbb{C} \setminus \{0\}$. For each i and j, $\gamma_j(x_i)$ is an Nth root of unity; and the set $\{(\gamma_j(x_i))_{i=1}^N : j = 1,...,N\} \subseteq \mathbb{C}^N$ is an orthonormal basis for $\mathbb{C}^N$.

b. Let $I \subseteq \{1,...,N\}$ have cardinality d. Then, for any $U \in \mathscr{U}(\mathbb{C}^d)$, it is straightforward to check that

$$F = \{U(\gamma_j(x_i))_{j\in I} : i = 1,...,N\} \subseteq \mathbb{C}^d$$

is a frame for $\mathbb{C}^d$, and this is the definition of a *harmonic frame*, see [104, 59].

c. If $(G, \bullet)$ is $\mathbb{Z}/N\mathbb{Z}$ with modular addition, and U is the identity, then F is a DFT-FUNTF.

Using Schur's lemma and Maschke's theorem, we see the relationship between frame multiplication and harmonic frames in the following result.

Theorem 7. *Let $(G, \bullet)$ be a finite abelian group and let $F = \{x_g\}_{g\in G}$ be a tight frame for $\mathbb{C}^d$. If $(G, \bullet)$ defines a frame multiplication for F, then F is unitarily equivalent to a harmonic frame, and there exist $U \in \mathscr{U}(\mathbb{C}^d)$ and $c > 0$ such that*

$$\forall g, h \in G, \quad \frac{1}{c}U(x_g * x_h) = \frac{1}{c}U(x_g)\frac{1}{c}U(x_h),$$

where the product on the right side is vector pointwise multiplication.

Corollary 1. *Let $F = \{x_k\}_{k\in\mathbb{Z}/N\mathbb{Z}} \subseteq \mathbb{C}^d$ be a tight frame for $\mathbb{C}^d$. If $\mathbb{Z}/N\mathbb{Z}$ defines a frame multiplication for F, then F is unitarily equivalent to a DFT frame.*

5 Finite Gabor systems

5.1 Gabor matrices

Definition 15. Let $F = \{x_i\}_{i=0}^{N-1} \subseteq \mathbb{C}^d$, $N \geq d$. The *coherence* of F, denoted by $\mu(F)$, is defined as

$$\mu(F) = \max_{j \neq k} \frac{|\langle x_j, x_k \rangle|}{\|x_j\| \, \|x_k\|}.$$

It is well-known that

$$\left(\frac{N-d}{d(N-1)} \right)^{1/2} \leq \mu(F) \leq 1, \tag{7}$$

see [89, 108]. The expression on the left side of (7) is the *Welch bound* for F. If $\mu(F) = 1$, then there are two elements $x_j, x_k \in F$ that are aligned, and we have maximal coherence. If $\mu(F)$ is the Welch bound, then all of the $x_i \in F$ are spread out in $\mathbb{C}^d$, and we say that we have *maximal incoherence* or *minimal coherence*.

Remark 4. In the case that F is a FUNTF, then $\mu(F)$ is the cosine of the smallest angle between the elements of the frame.

A FUNTF, $F = \{x_i\}_{i \in I}$, with $|\langle x_j, x_k \rangle|$ constant for all $j \neq k$ is called an *equiangular frame*. It can be shown that among all FUNTFs of N frame elements in $\mathbb{C}^d$, the equiangular frames are those with minimal coherence. In fact, $\mu(F)$ is the Welch bound if the FUNTF is equiangular. Note that (7) implies that an equiangular frame must satisfy $N \leq d^2$, see [101].

Gabor analysis is centered on the interplay of the Fourier transform, translation operators, and modulation operators. Recall that for a given a function $g : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$, we let $\tau_j g(l) = g(l-j)$ and $e_k(l) = e^{2\pi ikl/N}$, for $l = 0, 1, ..., N-1$, denote translation and modulation on g, respectively. Let $\top$ denote the transpose operator. The $N \times N^2$ *Gabor matrix*, G, generated by g, is defined as

$$G(g) = [G_0 | G_1 | \cdots | G_{N-1}], \tag{8}$$

where each G_j is the $N \times N$ matrix,

$$G_j = [e_0 \tau_{j-N} g | e_1 \tau_{j-N} g | \cdots | e_{N-1} \tau_{j-N} g],$$

and where each $(e_k \tau_{j-N})^\top$ is the $N \times 1$ column vector, $k = 0, 1, ..., N-1$.

Next, we introduce the notation,

$$(g)_k^j = e_k \tau_{j-N} g = (e_k(0) \tau_{j-N} g(0), e_k(1) \tau_{j-N} g(1), ..., e_k(N-1) \tau_{j-N} g(N-1))^\top.$$

We identify the Gabor matrix $G(g)$ with the set of all these vectors, and so we write

$$G_g = \{(g)_k^j\}_{k,j=0}^{N-1}.$$

This set, G_g, of vectors is referred to as the *Gabor system* generated by g, with corresponding Gabor matrix $G(g)$. Clearly, if $g : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$, then G_g consists of N^2 vectors each of length N, corresponding to all N^2 time-frequency shifts in $\mathbb{Z}/N\mathbb{Z} \times \widehat{\mathbb{Z}/N\mathbb{Z}}$.

The following is elementary to prove.

Theorem 8. *Given $g : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$, not identically zero. Then, G_g is a tight frame for $\mathbb{C}^N$.*

In this case of Theorem 8, the Gabor system, G_g, is called a *Gabor frame* for $\mathbb{C}^N$, see [84].

Given $g : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$, not identically zero. Then, for the Gabor frame, G_g, for $\mathbb{C}^N$, (7) becomes

$$\sqrt{\frac{N^2 - N}{N(N^2-1)}} = \frac{1}{\sqrt{N+1}} \leq \mu(G_g).$$

The notion of coherence is useful in obtaining sparse solutions to systems of equations. It is well known that for a full rank matrix $A \in \mathbb{C}^{n\times m}$ with $n < m$, there is an infinite number of solutions, $x \in \mathbb{C}^m$, to the system $Ax = b$. One is interested, especially in the context of signal processing and image compression, in finding the sparsest such solution, x, to the linear system. One measure of sparsity of x is by counting the number of nonzero elements, denoted by the ℓ_0 “norm”,

$$\|x\|_0 = \#\{i : x(i) \neq 0\}.$$

The sparsest solution to the system $Ax = b$ depends on the coherence of the set of column vectors corresponding to A. A basic theorem is the following.

Theorem 9. *If x is a solution to $Ax = b$, and*

$$\|x\|_0 < \frac{1}{2}\left(1 + \frac{1}{\mu(A)}\right), \tag{9}$$

then x is the unique sparsest solution to $Ax = b$, e.g., [29].

Furthermore, the Orthogonal Matching Pursuit (OMP) algorithm constructs x, see [29, 20].

Example 5. We combine Gabor frames, Theorem 9, discrete ambiguity functions, and properties of Alltop and Björck sequences in the following way. The coherence of a Gabor frame, G_g, has an elementary formulaic identity to the discrete ambiguity function of g. Thus, $\mu(G_{a_p})$ and $\mu(G_{b_p})$ are of order $1/\sqrt{p}$ by the comments in Section 3.

Consequently, if we let $n = p$ and $m = p^2$ in the setup of Theorem 9, we see that the right side of (9) is essentially as large as possible. Thus, a large domain is established with regard to unique sparse solutions of $Ax = b$.

5.2 The HRT conjecture

Let $g \in L^2(\mathbb{R})$ and let $\Lambda = \{(\alpha_k, \beta_k)\}_{k=1}^N \subseteq \mathbb{R}^2$ be a collection of N distinct points. The *Gabor system* generated by g and Λ is the set,

$$\mathscr{G}(g,\Lambda) = \{e^{2\pi i \beta_k x} g(x-\alpha_k)\}_{k=1}^N.$$

In [56, 55], the Heil, Ramanathan, and Topiwala (HRT) *conjecture* is stated as follows: *Given $g \in L^2(\mathbb{R}) \setminus \{0\}$ and $\Lambda = \{(\alpha_k, \beta_k)\}_{k=1}^N$ as above; then $\mathscr{G}(g,\Lambda)$ is a linearly independent set of functions in $L^2(\mathbb{R})$.* In this case, we shall say that the HRT conjecture holds for $\mathscr{G}(g,\Lambda)$.

Despite it's simple statement, the HRT conjecture remains an open problem. On the other hand, some special cases for its validity are known, see [56, 79, 75, 8, 35, 36, 14].

Among the results in [14], the authors prove that the HRT conjecture holds in the setting of ultimately positive functions.

Definition 16. We say that a function $f : \mathbb{R} \to \mathbb{C}$ is *ultimately positive* if

$$\exists x_0 > 0 \text{ such that } \forall x > x_0, \quad f(x) > 0.$$

The HRT results for such functions rely on Kronecker's theorem in Diophantine approximations.

Theorem 10 (Kronecker's theorem). *Let $\{\beta_1, ..., \beta_N\} \subseteq \mathbb{R}$ be a linearly independent set over $\mathbb{Q}$, and let $\theta_1, ..., \theta_N \in \mathbb{R}$. If $U, \varepsilon > 0$, then there exist $p_1, ..., p_N \in \mathbb{Z}$ and $u > U$ such that*

$$\forall k = 1, ..., N, \quad |\beta_k - p_k - \theta_k| < \varepsilon,$$

and, therefore,

$$\forall k = 1, ..., N, \quad |e^{2\pi i \beta_k u} - e^{2\pi i \theta_k}| < 4\pi\varepsilon.$$

One proof of Kronecker's theorem relies on the Bohr compactification, [11, Theorem 3.2.7]; see [54, Chapter 23], [71], [68] for different proofs.

We shall use the following lemma in the proof of Theorem 11.

Lemma 1. *Let P be a property that holds for almost every $x \in \mathbb{R}$. For every sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$, there exists a measurable set $E \subseteq \mathbb{R}$ such that $|\mathbb{R} \setminus E| = 0$ and P holds for $x + u_n$ for each $(n,x) \in \mathbb{N} \times E$.*

Proof. If $E = \bigcap_{n\in\mathbb{N}}\{x : P(x+u_n) \text{ holds}\}$, then P holds for $x+u_n$ for each $(n,x) \in \mathbb{N} \times E$. We know that $|\{x : P(x+u_n) \text{ fails}\}| = 0$ for each $n \in \mathbb{N}$, and so $|\bigcup_{n\in\mathbb{N}}\{x : P(x+u_n) \text{ fails}\}| = 0$, i.e., $|\mathbb{R} \setminus E| = 0$. □

Theorem 11 (HRT for ultimately positive functions). *Let $g \in L^2(\mathbb{R})$ be ultimately positive and let $\Lambda = \{(\alpha_k, \beta_k)\}_{k=0}^N \subseteq \mathbb{R}^2$ be a set of distinct points with the property that $\{\beta_0, ..., \beta_N\}$ is linearly independent over $\mathbb{Q}$. Then, the HRT conjecture holds for $\mathscr{G}(g,\Lambda)$.*

Proof. i. We begin by first simplifying the setting. First notice that if $\{\beta_0,...,\beta_N\}$ is linearly independent over $\mathbb{Q}$ then $\{\beta_1-\beta_0,...,\beta_N-\beta_0\}$ is also linearly independent over $\mathbb{Q}$. Furthermore, we can assume that $(\alpha_0,\beta_0)=(0,0)$. In fact, if not, then there exists a metaplectic transform $U : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ with associated linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$ that sends (α_0,β_0) to $(0,0)$ and $\mathscr{G}(g,\Lambda)$ is linearly independent in $L^2(\mathbb{R})$ if and only if $\mathscr{G}(Ug,A(\Lambda))$ is linearly independent in $L^2(\mathbb{R})$). Consequently, without loss of generality, we assume $(\alpha_0,\beta_0)=(0,0)$ and $\{\beta_1,...,\beta_N\}$ is linearly independent over $\mathbb{Q}$.

We suppose that $\mathscr{G}(g,\Lambda)$ is linearly dependent in $L^2(\mathbb{R})$ and obtain a contradiction.

ii. Since $\mathscr{G}(g,\Lambda)$ is linearly dependent, there exist constants $c_1,...,c_N \in \mathbb{C}$ not all zero such that

$$g(x)=\sum_{k=1}^{N} c_k e^{2\pi i\beta_k x} g(x-\alpha_k) \quad \text{a.e.} \tag{10}$$

In fact, we can take each $c_k \in \mathbb{C}\setminus\{0\}$.

By Kronecker's theorem (Theorem 10) and the linear independence of $\{\beta_1,...,\beta_N\}$ over $\mathbb{Q}$, there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ such that $\lim_{n\to\infty} u_n=\infty$, and

$$\forall k=1,...,N, \quad \lim_{n\to\infty} e^{2\pi i\beta_k u_n}=e^{2\pi i\theta_k}, \tag{11}$$

where each

$$\theta_k=\phi_k+1/4 \quad \text{and} \quad \frac{c_k}{|c_k|}=e^{-2\pi i\phi_k},$$

i.e., we have chosen each θ_k in the application of Theorem 10 so that $e^{2\pi i\theta_k}=|c_k|i/c_k$. Therefore, from (11), we compute

$$\forall k=1,...,N, \quad \lim_{n\to\infty} c_k e^{2\pi i\beta_k u_n}=|c_k|i. \tag{12}$$

Then, by Lemma 1, there exists a set $X\subseteq\mathbb{R}$ with $|\mathbb{R}\setminus X|=0$ such that

$$\forall (n,x)\in\mathbb{N}\times X, \quad g(x+u_n)=\sum_{k=1}^{N} c_k e^{2\pi i\beta_k(x+u_n)} g(x+u_n-\alpha_k). \tag{13}$$

iii. Without loss of generality, we may assume that $0\in X$, for if not, then we can replace g with a translated version of g. Since g is ultimately positive and $u_n\to\infty$, then we may also assume without loss of generality that

$$\forall n\in\mathbb{N} \text{ and } \forall k=0,1,...,N, \quad g(u_n-\alpha_k)>0$$

by simply replacing $\{u_n\}_{n\in\mathbb{N}}$ with a subsequence for which this property *does* hold. Then, by the positivity of g, we divide both sides of (13) by $g(x+u_n)$ and evaluate at $x=0$ to obtain

$$1 = \sum_{k=1}^{N} c_k e^{2\pi i \beta_k u_n} \frac{g(u_n - \alpha_k)}{g(u_n)} = \sum_{k=1}^{N} \left(|c_k| i + c_k e^{2\pi i \beta_k u_n} - |c_k| i \right) \frac{g(u_n - \alpha_k)}{g(u_n)} \quad (14)$$
$$\geq \left| \sum_{k=1}^{N} |c_k| i \frac{g(u_n - \alpha_k)}{g(u_n)} \right| - \left| \sum_{k=1}^{N} (c_k e^{2\pi i \beta_k u_n} - |c_k| i) \frac{g(u_n - \alpha_k)}{g(u_n)} \right|$$
$$\geq \sum_{k=1}^{N} |c_k| \frac{g(u_n - \alpha_k)}{g(u_n)} - \sum_{k=1}^{N} |c_k| \left| e^{2\pi i \beta_k u_n} - \frac{|c_k|}{c_k} i \right| \frac{g(u_n - \alpha_k)}{g(u_n)},$$

since $|cd - |c|i| = |c|\,|d - |c|i/c|$ for $c \in \mathbb{C} \setminus \{0\}$ and $d \in \mathbb{C}$.

Now set $\varepsilon = 1/(8\pi)$ and apply Theorem 10 to assert that

$$\exists U > 0 \text{ such that } \forall u_n > U \text{ and } \forall k = 1,..,N, \quad \left| e^{2\pi i \beta_k u_n} - \frac{|c_k|}{c_k} i \right| < \frac{1}{2}.$$

This, combined with (14), gives

$$\forall u_n > U, \quad 2 \geq \sum_{k=1}^{N} |c_k| \frac{g(u_n - \alpha_k)}{g(u_n)}.$$

Hence, $\{g(u_n - \alpha_k)/g(u_n)\}_{n \in \mathbb{N}}$ is a bounded sequence for each $k = 1,...,N$. Therefore there exists a subsequence $\{v_n\}_{n\in\mathbb{N}}$ of $\{u_n\}_{n\in\mathbb{N}}$ and $r_k \in \mathbb{R}$, $k = 1,...,N$, such that

$$\forall k = 1,...,N, \quad \lim_{n\to\infty} \frac{g(v_n - \alpha_k)}{g(v_n)} = r_k.$$

Then, by the equality of (14) and by (12), we have

$$1 = \lim_{n\to\infty} \sum_{k=1}^{N} c_k e^{2\pi i \beta_k v_n} \frac{g(v_n - \alpha_k)}{g(v_n)} = \sum_{k=1}^{N} |c_k| r_k i.$$

The left side is real and the right side is imaginary, giving the desired contradiction. □

Definition 17. We say that a function $f : \mathbb{R} \to \mathbb{R}$ is *ultimately decreasing* if

$$\exists x_0 > 0 \text{ such that } \forall y > x > x_0, \quad f(y) \leq f(x).$$

Kronecker's theorem can also be used to prove that the HRT conjecture holds for a four-element Gabor system generated by an ultimately positive function if $g(x)$ and $g(-x)$ are also ultimately decreasing.

Theorem 12. *Let $g \in L^2(\mathbb{R})$ have the properties that $g(x)$ and $g(-x)$ are ultimately positive and ultimately decreasing, and let $\Lambda = \{(\alpha_k, \beta_k)\}_{k=0}^{3} \subseteq \mathbb{R}^2$ be a set of distinct points. Then, the HRT conjecture holds for $\mathcal{G}(g, \Lambda)$.*

The proof in [14] is omitted here.

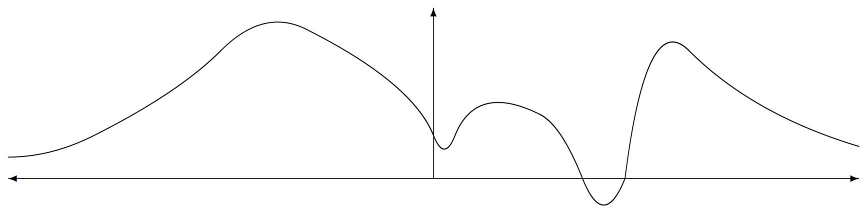

Fig. 1: An illustration of an ultimately positive and ultimately decreasing function.

Because of the importance of various independent sets in harmonic analysis, Kronecker's theorem motivates the following definition.

Definition 18. Let $E \subseteq \widehat{R}$ be compact, and let $C(E)$ be the space of complex-valued continuous functions on E. The set E is a *Kronecker set* if

$$\forall \varepsilon > 0 \text{ and } \forall \varphi \in C(E) \text{ for which } \forall x \in E, |\varphi(x)| = 1, \exists x \in \mathbb{R} \text{ such that } \forall \gamma \in E,$$
$$|\varphi(\gamma) - e^{2\pi i x \gamma}| < \varepsilon.$$

The resemblance to Kronecker's theorem is apparent. Further, it is clear from Definition18 how to define Kronecker sets for $E \subseteq \Gamma$, a LCAG. In this setting and going back to the definition of strict multiplicity in Definition 3, we say that a closed set $E \subseteq \Gamma$ is a *Riemann set of uniqueness*, or *U-set*, if it is not a set of multiplicity, i.e., it is not a closed set, F, for which $A'_0(\Gamma) \cap A'(F) \neq \{0\}$. Here, $A'_0(\Gamma) = \{T \in A'(\Gamma) : \widehat{T} \in L^\infty(G)$ vanishes at infinity$\}$, and $A'(F) = \{T \in A'(\Gamma) : \mathrm{supp}(T) \subseteq F\}$. This definition of a U-set is correct but not highly motivated; however, see [11] for history, motivation, open problems, and important references.

From the point of view of Kronecker's theorem, it is interesting to note that Kronecker sets are sets of strong spectral resolution, i.e., $A'(E) = M_b(E)$, and these, in turn, are U-sets (Malliavin, 1962). There are many other intricacies and open problems in this area combining harmonic analysis, in particular, spectral synthesis, with number theory, see [92, 64, 68, 62, 10, 78, 80, 10, 11].

6 Short-time Fourier transform frame inequalities on $\mathbb{R}^d$

Let $g_0(x) = 2^{d/4} e^{-\pi \|x\|^2}$. Then $G_0(\gamma) = \hat{g}_0(\gamma) = 2^{d/4} e^{-\pi \|\gamma\|^2}$ and $\|g_0\|_{L^2(\mathbb{R}^d)} = 1$, see [15] for properties of g_0.

Definition 19. The *Feichtinger algebra*, $\mathscr{S}_0(\mathbb{R}^d)$, is defined as

$$\mathscr{S}_0(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) : \|f\|_{\mathscr{S}_0(\mathbb{R}^d)} = \left\|V_{g_0} f\right\|_{L^1(\mathbb{R}^{2d})} < \infty\}.$$

The Fourier transform of $\mathscr{S}_0(\mathbb{R}^d)$ is an isometric isomorphism onto itself, and, in particular, $f \in \mathscr{S}_0(\mathbb{R}^d)$ if and only if $F \in \mathscr{S}_0(\widehat{\mathbb{R}}^d)$, see, e.g., [38, 40, 39, 42, 49].

The Feichtinger algebra, as well as its generalization to modulation spaces, provides a natural setting for proving non-uniform sampling theorems for the STFT analogous to Beurling's non-uniform sampling theorem, Theorem 3, for Fourier frames. The theory for the STFT is given in [6].

The following is Gröchenig's non-uniform Gabor frame theorem, and it was also influenced by earlier work with Feichtinger, see [48, Theorem S] and [49], cf. [41, 42] for a precursor of this result.

Theorem 13. *Given any $g \in \mathscr{S}_0(\mathbb{R}^d)$. There is $r = r(g) > 0$ such that if $E = \{(s_n, \sigma_n)\}_{n=1}^{\infty} \subseteq \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ is a separated sequence with the property that*

$$\bigcup_{n=1}^{\infty} \overline{B((s_n, \sigma_n), r(g))} = \mathbb{R}^d \times \widehat{\mathbb{R}}^d,$$

then the frame operator $S = S_{g,E}$ defined by

$$S_{g,E} f = \sum_{n=1}^{\infty} \langle f, \tau_{s_n} e_{\sigma_n} g \rangle \tau_{s_n} e_{\sigma_n} g,$$

is invertible on $\mathscr{S}_0(\mathbb{R}^d)$. Further, every $f \in \mathscr{S}_0(\mathbb{R}^d)$ has a non-uniform Gabor expansion,

$$f = \sum_{n=1}^{\infty} \langle f, \tau_{s_n} e_{\sigma_n} g \rangle S_{g,E}^{-1}(\tau_{s_n} e_{\sigma_n} g),$$

where the series converges unconditionally in $\mathscr{S}_0(\mathbb{R}^d)$.

It should be noted that the set E depends on g.

The following is proved in [6] and can be compared with Theorem 13.

Theorem 14. *Let $E = \{(s_n, \sigma_n)\}_{n=1}^{\infty} \subseteq \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ be a separated sequence; and let $\Lambda \subseteq \widehat{\mathbb{R}}^d \times \mathbb{R}^d$ be an S-set of strict multiplicity that is compact, convex, and symmetric about $0 \in \widehat{\mathbb{R}}^d \times \mathbb{R}^d$. Assume balayage is possible for (E, Λ). Further, let $g \in L^2(\mathbb{R}^d)$, $\hat{g} = G$, have the property that $\|g\|_{L^2(\mathbb{R}^d)} = 1$. We have that*

$$\exists A, B > 0, \text{ such that } \forall f \in \mathscr{S}_0(\mathbb{R}^d), \text{ for which } \operatorname{supp}(\widehat{V_g f}) \subseteq \Lambda,$$
$$A \|f\|_{L^2(\mathbb{R}^d)}^2 \leq \sum_{n=1}^{\infty} |V_g f(s_n, \sigma_n)|^2 \leq B \|f\|_{L^2(\mathbb{R}^d)}^2 .$$

Consequently, the frame operator $S = S_{g,E}$ is invertible in $L^2(\mathbb{R}^d)$-norm on the subspace of $\mathscr{S}_0(\mathbb{R}^d)$, whose elements f have the property that $\operatorname{supp}(\widehat{V_g f}) \subseteq \Lambda$.

Moreover, every $f \in \mathscr{S}_0(\mathbb{R}^d)$ satisfying the support condition, $\operatorname{supp}(\widehat{V_g f}) \subseteq \Lambda$, has a non-uniform Gabor expansion,

$$f = \sum_{n=1}^{\infty} \langle f, \tau_{s_n} e_{\sigma_n} g \rangle S_{g,E}^{-1}(\tau_{s_n} e_{\sigma_n} g),$$

where the series converges unconditionally in $L^2(\mathbb{R}^d)$.

It should be noted that here the set E does not depend on g.

Example 6. In comparing Theorem 14 with Theorem 13, a possible weakness of the former is the dependence of the set E on g, whereas a possible weakness of the latter is the hypothesis that $\mathrm{supp}(\widehat{V_g f}) \subseteq \Lambda$. We now show that this latter constraint is of no major consequence.

Let $f, g \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. We know that $V_g f \in L^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$, and

$$\widehat{V_g f}(\zeta, z) = \int_{\widehat{\mathbb{R}}^d} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} f(t) g(t-x) e^{-2\pi i t \cdot \omega} \, dt \right) e^{-2\pi i (x \cdot \zeta + z \cdot \omega)} \, dx \, d\omega.$$

The right side is

$$\int_{\widehat{\mathbb{R}}^d} \int_{\mathbb{R}^d} f(t) \left(\int_{\mathbb{R}^d} g(t-x) e^{-2\pi i x \cdot \zeta} \, dx \right) e^{-2\pi i t \cdot \omega} e^{-2\pi i z \cdot \omega} \, dt \, d\omega,$$

where the interchange in integration follows from the Fubini-Tonelli theorem and the hypothesis that $f, g \in L^1(\mathbb{R}^d)$. This, in turn, is

$$\begin{aligned} &\hat{g}(-\zeta) \int_{\widehat{\mathbb{R}}^d} \left(\int_{\mathbb{R}^d} f(t) e^{-2\pi i t \cdot \zeta} e^{-2\pi i t \cdot \omega} \, dt \right) e^{-2\pi i z \cdot \omega} \, d\omega \\ &= \hat{g}(-\zeta) \int_{\widehat{\mathbb{R}}^d} \hat{f}(\zeta + \omega) e^{-2\pi i z \cdot \omega} \, d\omega = e^{-2\pi i z \cdot \zeta} f(-z) \hat{g}(-\zeta). \end{aligned}$$

Consequently, we have shown that

$$\forall f, g \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d), \quad \widehat{V_g f}(\zeta, z) = e^{-2\pi i z \cdot \zeta} f(-z) \hat{g}(-\zeta). \tag{15}$$

Let $d = 1$ and let $\Lambda = [-\Omega, \Omega] \times [-T, T] \subseteq \widehat{\mathbb{R}}^d \times \mathbb{R}^d$. We can choose $g \in PW_{[-\Omega,\Omega]}$, where $\hat{g}$ is even and smooth enough so that $g \in L^1(\mathbb{R})$. For this window, g, we can take any even $f \in L^2(\mathbb{R})$ which is supported in $[-T, T]$. Equation (15) applies.

7 Pseudo-differential operator frame inequalities on $\mathbb{R}^d$

Let $\sigma \in \mathscr{S}'(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$. The operator, K_σ, formally defined as

$$(K_\sigma f)(x) = \int_{\widehat{\mathbb{R}}^d} \sigma(x, \gamma) \hat{f}(\gamma) e^{2\pi i x \cdot \gamma} \, d\gamma,$$

is the *pseudo-differential operator* with Kohn-Nirenberg symbol, σ, see [49] Chapter 14, [50] Chapter 8, [60], and [99]. For consistency with the notation and setting of the previous sections, we shall define pseudo-differential operators, K_s, with tempered distributional Kohn-Nirenberg symbols, $s \in \mathscr{S}'(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$, as

$$(K_s \hat{f})(\gamma) = \int_{\mathbb{R}} s(y,\gamma) f(y) e^{-2\pi i y\cdot\gamma}\, dy.$$

Furthermore, we shall deal with Hilbert-Schmidt operators, $K : L^2(\widehat{\mathbb{R}}^d) \to L^2(\widehat{\mathbb{R}}^d)$; and these, in turn, can be represented as $K = K_s$, where $s \in L^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$. Recall that $K : L^2(\widehat{\mathbb{R}}^d) \to L^2(\widehat{\mathbb{R}}^d)$ is a *Hilbert-Schmidt operator* if

$$\sum_{n=1}^{\infty} \|Ke_n\|^2_{L^2(\widehat{\mathbb{R}}^d)} < \infty$$

for some orthonormal basis, $\{e_n\}_{n=1}^{\infty}$, for $L^2(\widehat{\mathbb{R}}^d)$, in which case the *Hilbert-Schmidt norm* of K is defined as

$$\|K\|_{HS} = \left(\sum_{n=1}^{\infty} \|Ke_n\|^2_{L^2(\widehat{\mathbb{R}}^d)} \right)^{1/2},$$

and $\|K\|_{HS}$ is independent of the choice of orthonormal basis.

The following theorem on Hilbert-Schmidt operators can be found in [91].

Theorem 15. *If $K : L^2(\widehat{\mathbb{R}}^d) \to L^2(\widehat{\mathbb{R}}^d)$ is a bounded linear mapping and $(K\hat{f})(\gamma) = \int_{\widehat{\mathbb{R}}^d} m(\gamma,\lambda) \hat{f}(\lambda)\, d\lambda$, for some measurable function m, then K is a Hilbert-Schmidt operator if and only if $m \in L^2(\widehat{\mathbb{R}}^{2d})$ and, in this case, $\|K\|_{HS} = \|m\|_{L^2(\mathbb{R}^{2d})}$.*

The following theorem about pseudo-differential operator frame inequalities is proved in [6].

Theorem 16. *Let $E = \{x_n\} \subseteq \mathbb{R}^d$ be a separated sequence, that is symmetric about $0 \in \mathbb{R}^d$; and let $\Lambda \subseteq \widehat{\mathbb{R}}^d$ be an S-set of strict multiplicity, that is compact, convex, and symmetric about $0 \in \widehat{\mathbb{R}}^d$. Assume balayage is possible for (E,Λ). Furthermore, let K be a Hilbert-Schmidt operator on $L^2(\widehat{\mathbb{R}}^d)$ with pseudo-differential operator representation,*

$$(K\hat{f})(\gamma) = (K_s \hat{f})(\gamma) = \int_{\mathbb{R}^d} s(y,\gamma) f(y) e^{-2\pi i y\cdot\gamma}\, dy,$$

where $s_\gamma(y) = s(y,\gamma) \in L^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ is the Kohn-Nirenberg symbol and where we furthermore assume that

$$\forall \gamma \in \widehat{\mathbb{R}}^d, \quad s_\gamma \in C_b(\mathbb{R}^d) \text{ and } \operatorname{supp}(s_\gamma e_{-\gamma})\check{} \subseteq \Lambda.$$

Then,

$$\exists A, B > 0 \text{ such that } \forall f \in L^2(\mathbb{R}^d) \setminus \{0\},$$

$$A\frac{\left\|K_s\hat{f}\right\|^4_{L^2(\widehat{\mathbb{R}}^d)}}{\|f\|^2_{L^2(\mathbb{R}^d)}} \le \sum_{x\in E} |\langle (K_s\hat{f})(\cdot), \overline{s(x,\cdot)}e_x(\cdot)\rangle|^2 \le B\,\|s\|^2_{L^2(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)} \left\|K_s\hat{f}\right\|^2_{L^2(\widehat{\mathbb{R}}^d)}.$$

Example 7. We shall define a Kohn-Nirenberg symbol class whose elements, s, satisfy the hypotheses of Theorem 16.

Choose $\{\lambda_j\} \subseteq \mathrm{int}(\Lambda)$, where Λ is as described in Theorem 16. Choose $\{a_j\} \subseteq C_b(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$ and $\{b_j\} \subseteq C_b(\widehat{\mathbb{R}}^d)\cap L^2(\widehat{\mathbb{R}}^d)$ with the following properties:

i. $\sum_{j=1}^\infty |a_j(y)b_j(y)|$ is uniformly bounded and converges uniformly on $\mathbb{R}^d\times\widehat{\mathbb{R}}^d$;
ii. $\sum_{j=1}^\infty \left\|a_j\right\|_{L^2(\mathbb{R}^d)} \left\|b_j\right\|_{L^2(\widehat{\mathbb{R}}^d)} < \infty$;
iii. $\forall j\in\mathbb{N}$, $\exists\varepsilon_j>0$ such that $\overline{B(\lambda_j,\varepsilon_j)} \subseteq \Lambda$ and $\mathrm{supp}(\hat{a}_j) \subseteq \overline{B(0,\varepsilon_j)}$.

These conditions are satisfied for a large class of functions a_j and b_j.

The Kohn-Nirenberg symbol class consisting of functions, s, defined by

$$s(y,\gamma) = \sum_{j=1}^\infty a_j(y)b_j(\gamma)e^{-2\pi i y\cdot\lambda_j}$$

satisfy the hypotheses of Theorem 16. To see this, first note that condition *i* guarantees that if we set $s_\gamma(y) = s(y,\gamma)$, then

$$\forall\gamma\in\widehat{\mathbb{R}}^d, \quad s_\gamma \in C_b(\mathbb{R}^d).$$

Condition *ii* allows us to assert that $s\in L^2(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)$ since Minkowski's inequality can be used to make the following estimate:

$$\begin{aligned}\|s\|_{L^2(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)} &\le \sum_{j=1}^\infty \left(\int_{\widehat{\mathbb{R}}^d}\int_{\mathbb{R}^d}\left|b_j(\gamma)a_j(y)e^{-2\pi i y\cdot(\lambda_j-\gamma)}\right|^2 dy\,d\gamma\right)^{1/2}\\ &= \sum_{j=1}^\infty \left\|a_j\right\|_{L^2(\mathbb{R}^d)}\left\|b_j\right\|_{L^2(\widehat{\mathbb{R}}^d)}.\end{aligned}$$

Finally, using condition *iii*, we obtain the support hypothesis, $\mathrm{supp}(s_\gamma e_{-\gamma})\hat{} \subseteq \Lambda$, of Theorem 16 for each $\gamma\in\widehat{\mathbb{R}}^d$, because of the following calculations:

$$\mathrm{supp}(s_\gamma e_{-\gamma})\hat{}(\omega) = \sum_{j=1}^\infty b_j(\gamma)(\hat{a}_j * \delta_{-\lambda_j})(\omega)$$

and, for each j,

$$\mathrm{supp}(\hat{a}_j*\delta_{-\lambda_j}) \subseteq \overline{B(0,\varepsilon_j)} + \{\lambda_j\} \subseteq \overline{B(\lambda_j,\varepsilon_j)} \subseteq \Lambda.$$

Remark 5. Pfander and collaborators combine the theory of Gabor frames and Hilbert-Schmidt operators to obtain results in *operator sampling*. The goal of operator sampling is to determine an operator completely from its action on a single input function or distribution. The question of determining which operators can be identified in this way was addressed in basic work of Kailath [65, 66, 67] and Bello [9], who found that the identifiability of a communication channel is characterized by the area of the support of its so-called *spreading function*. The spreading function, $\eta_H(t,\nu)$, of the Hilbert-Schmidt operator, H, on $L^2(\mathbb{R})$ is the *symplectic Fourier transform* of its Kohn-Nirenberg symbol, σ, viz,

$$\eta_H(t,\nu) = \int_{\widehat{\mathbb{R}}}\int_{\mathbb{R}} \sigma(x,\gamma)\hat{f}(\gamma)e^{-2\pi i(\nu x-\gamma t)}\,dx\,d\gamma;$$

and we have the representation,

$$Hf(x) = \int_{\mathbb{R}}\int_{\widehat{\mathbb{R}}} \eta_H(t,\nu)\tau_t e_\nu f(x)\,d\nu\,dt.$$

In this sense, an operator H whose spreading function has compact support can be said to have a *bandlimited symbol*. This motivates the definition of an *operator Paley-Wiener space* and an associated sampling theorem [86]. The aforementioned communications application was put on a firm mathematical footing, first proving Kailath's conjectures in [74] (Kozek and Pfander) and then proving Bello's assertions in [86] (Pfander and Walnut). Results and an overview of the subject are given in [87, 85].

Appendix

The Classical Sampling Theorem goes back to papers by Cauchy (1840s), see [12, Theorem 3.10.10] for proofs of Theorem 17. It has had a significant impact on various topics in mathematics, including number theory and interpolation theory, long before Shannon's application of it in communications.

Theorem 17 (Classical Sampling Theorem). *Let $T,\Omega > 0$ satisfy the condition that $0 < 2T\Omega \leq 1$, and let s be an element of the Paley-Wiener space $PW_{1/(2T)}$ satisfying the condition that $\hat{s} = S = 1$ on $[-\Omega,\Omega]$ and $S \in L^\infty(\widehat{\mathbb{R}})$. Then*

$$\forall f \in PW_\Omega, \quad f = T\sum_{n\in\mathbb{Z}} f(nT)\tau_{nT}s, \tag{16}$$

where the convergence of (16) is in the $L^2(\mathbb{R})$ norm and uniformly in $\mathbb{R}$. One possible sampling function s is

$$s(t) = \frac{\sin(2\pi\Omega t)}{\pi t}.$$

We can compute Fourier transforms numerically using the following result, whose proof requires Theorem 17.

Theorem 18. *Let $T,\Omega > 0$ satisfy the property that $2T\Omega = 1$, let $N \geq 2$ be an even integer, and let $f \in PW_\Omega \cap L^1(\mathbb{R})$. Consider the dilation $f_T(t) = Tf(Tt)$ as a con-*

tinuous function on $\mathbb{R}$, as well as a function on $\mathbb{Z}$ defined by $m \mapsto f_T[m]$, where $f_T[m] = f_T(m)$. Assume that $f_T \in \ell^1(\mathbb{Z})$. Then for every integer $n \in [-\frac{N}{2}, \frac{N}{2}]$, we have

$$\hat{f}\left(\frac{2\Omega n}{N}\right) = \hat{f}\left(\frac{n}{NT}\right) = \sum_{m=0}^{N-1} (f_T)_N^0[m] W_N^{mn}, \tag{17}$$

where $W_N = e^{-2\pi i/N}$ and $(f_T)_N^0[m] = \sum_{k\in\mathbb{Z}} f_T[m-kN]$.

In practice, the computation (17) requires natural error estimates and the FFT.

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