Balayage and pseudo-differential operator frame inequalities

Enrico Au-Yeung
Department of Mathematical Sciences
DePaul University, Chicago, IL 60614 USA
Email: eauyeun1@depaul.edu

John J. Benedetto
Norbert Wiener Center, Department of Mathematics
University of Maryland, College Park, MD 20742 USA
Email: jjb@math.umd.edu

***Abstract*—Using his formulation of the potential theoretic notion of balayage and his deep results about this idea, Beurling gave sufficient conditions for Fourier frames in terms of balayage. The analysis makes use of spectral synthesis, due to Wiener and Beurling, as well as properties of strict multiplicity, whose origins go back to Riemann. In this setting and with this technology, with the goal of formulating non-uniform sampling formulas, we show how to construct frames using pseudo-differential operators. This work fits within the context of the short time Fourier transform (STFT) and time-frequency analysis.**

I. Introduction

A. Background and theme

Our main result, Theorem III.3, formulates pseudo-differential operator frame inequalities, giving rise to associated frame expansions. The proof is based on Beurling's theory of balayage. Our immediate aim is to provide a new tool in the theory of operator sampling, see [1], [2], [3], [4], [5], in the context of time-frequency analysis and short time Fourier transforms with regard to the importance of pseudo-differential operators in operator sampling.

Generally, frames provide a natural tool for dealing with signal reconstruction in the presence of noise in the setting of overcomplete sets of atoms, and with the goals of numerical stability and robust signal representation.

Fourier frames (Definition I.3) were the first frames analyzed in harmonic analysis, and they were originally studied for the case of non-harmonic Fourier series by Duffin and Schaeffer [6], with a history going back to Paley and Wiener [7] (1934) and further, and with significant activity in the 1930s and 1940s, e.g., see [8]. Since [6], there have been significant contributions by Beurling (unpublished 1959-1960 lectures), [9], [10], Beurling and Malliavin [11], [12], Kahane [13], Landau [14], Jaffard [15], and Seip [16], [17].

The remainder of Section I is devoted to motivation (Subsection I-B) and definitions (Subsection I-C), respectively. In Section II, we define and expand on the definition of balayage. Section III gives the requisite background on pseudo-differential operators and states our main result, Theorem III.3. It is apparent from the statement of Theorem III.3 that a deeper understanding of its hypotheses is necessary for any useful applicability. It is for this reason that we include Section IV.

B. Motivation

There are analogues and extensions of the classical result that the set $\{e^{-2\pi i n\omega}\colon n \in \mathbb{Z}\}$ of exponentials forms an orthonormal basis for the space $L^2(\Lambda)$ of square-integrable functions on $\Lambda = [0,1]$. As such, we ask if there is a unifying theory that ties together these analogues and extensions? Consider the general sampling formulas,

$$f(x) = \sum f(x_n)s_n, \tag{1}$$

for non-uniformly spaced sequences $\{x_n\}$, for specific sequences of sampling functions s_n depending on x_n, and for classes of functions f for which such formulas are true.

Are there general theoretical justifications for the often intricate relations that occur between the sequences of sampling points and the support sets of the spectra of functions in equations such as (1)? Such questions are the basis for our *motivation* for the setting of Theorem III.3.

To be more precise, and to illustrate a specific case of such a relation before we get to pseudo-differential operators, we give the following example.

Example I.1. This is a result of Olevskii and Ulanovskii [18] (2008) concerning universal sets of stable sampling for band-limited functions.

Consider an analogue of the aforementioned classical result, where the interval $[0,1]$ is now replaced by a possibly unbounded set $\Lambda \subseteq \widehat{\mathbb{R}}$, where $\widehat{\mathbb{R}}$ is defined in Subsection I-C, in which Λ has Lebesgue measure $|\Lambda|$ strictly less than 1 and, speaking intuitively, for which Λ is not too spread-out.

Let $E = \{n + 2^{-|n|}\colon n \in \mathbb{Z}\}$ and let $\mathcal{E}(E) = \{e_{-x}\colon x \in E\}$, where $e_x(\gamma) = e^{2\pi x\gamma}$. Then $\mathcal{E}(E)$ is complete in $L^2(\Lambda)$ for every measurable set $\Lambda \subseteq \widehat{\mathbb{R}}$ satisfying $|\Lambda| < 1$ and for which $|\Lambda \cap \{\gamma\colon k-1 < |\gamma| < k\}| \leq C\,2^{-k}$, where C is independent of k. This means that for any $F \in L^2(\Lambda)$, where F is zero outside the set Λ, if F is orthogonal to each function in $\mathcal{E}(E)$, then we can conclude that $F = 0\,a.e.$ This is equivalent to saying that for any $f \in L^2(\mathbb{R})$, for which $f(x) = \int_\Lambda F(\gamma)e^{2\pi i x\gamma}\, d\gamma$, for some $F \in L^2(\Lambda)$ (and so f is continuous on $\mathbb{R}$ since $|\Lambda| < \infty$), the condition that $f = 0$ on E implies that $f = 0\,a.e.$ The hypothesis, $|\Lambda \cap \{\gamma\colon k-1 < |\gamma| < k\}| \leq C\,2^{-k}$, where C is independent of k, can be weakened but not eliminated. Thus, although Λ

can be an unbounded set, there is a restriction that Λ cannot be too thin or too spread-out over $\widehat{\mathbb{R}}$.

This is the type of intricate relation we are referring to above, in this case between the set E of sampling points and the support set Λ of the Fourier transform $\mathcal{F}(f)$ of a function f, see Subsection I-C for the definition of $\mathcal{F}(f)$.

C. Definitions

Definition I.2. (Frame) Let H be a separable Hilbert space. A sequence $\{x_n\}_{n\in\mathbb{Z}} \subseteq H$ is a *frame* for H if there are positive constants A and B such that

$$\forall\ f \in H, \quad A\|f\|^2 \le \sum_{n\in\mathbb{Z}} |\langle f, x_n\rangle|^2 \le B\|f\|^2.$$

The constants A and B are lower and upper frame bounds, respectively. The smallest possible value of B and the largest possible value of A are the optimal frame bounds.

Our overall goal beyond this paper is to extend the theory of Fourier frames to more general types of frames in time-frequency analysis. To accomplish this, we formulate non-uniform sampling formulas parametrized by the space $M(\mathbb{R}^d)$ of bounded Radon measures, see [19]. This formulation provides a natural way to generalize non-uniform sampling to the setting of short time Fourier transforms (STFTs) [20], Gabor theory [21], [22], [23], and pseudo-differential operators [20], [24], The techniques are based on Beurling's methods from 1959-1960, [10], [9], which incorporate balayage, spectral synthesis, and strict multiplicity. In this paper, we show how to achieve this goal for pseudo-differential operators.

We define the Fourier transform $\mathcal{F}(f)$ of $f \in L^2(\mathbb{R}^d)$ and its inverse Fourier transform $\mathcal{F}^{-1}(f)$ by

$$\mathcal{F}(f)(\gamma) = \widehat{f}(\gamma) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i x\cdot\gamma}\ dx,$$

and

$$\mathcal{F}^{-1}(\widehat{f})(\gamma) = f(x) = \int_{\widehat{\mathbb{R}}^d} \widehat{f}(\gamma)e^{2\pi i x\cdot\gamma}\ d\gamma.$$

$\widehat{\mathbb{R}}^d$ denotes $\mathbb{R}^d$ considered as the spectral domain. We write $F^\vee(x) = \int_{\widehat{\mathbb{R}}^d} F(\gamma)e^{2\pi i x\cdot\gamma}\, d\gamma$. The notation "$\int$" designates integration over $\mathbb{R}^d$ or $\widehat{\mathbb{R}}^d$. When f is a bounded continuous function, its Fourier transform is defined in the sense of distributions. If $X \subseteq \mathbb{R}^d$, where X is closed, then $M_b(X)$ is the space of bounded Radon measures μ with the support of μ contained in X. $C_b(\mathbb{R}^d)$ denotes the space of complex valued bounded continuous functions on $\mathbb{R}^d$.

Definition I.3. (Fourier frame) Let $E \subseteq \mathbb{R}^d$ be a sequence and let $\Lambda \subseteq \widehat{\mathbb{R}}^d$ be a compact set. Notationally, let $e_x(\gamma) = e^{2\pi i x\cdot\gamma}$. The sequence $\mathcal{E}(E) = \{e_{-x} : x \in E\}$ is a *Fourier frame* for $L^2(\Lambda)$ if there are positive constants A and B such that

$$\forall\ F \in L^2(\Lambda), \quad A\|F\|^2_{L^2(\Lambda)} \le \sum_{x\in E} |\langle F, e_{-x}\rangle|^2 \le B\|F\|^2_{L^2(\Lambda)}.$$

Define the *Paley-Wiener space*,

$$PW_\Lambda = \{f \in L^2(\mathbb{R}^d) : \mathrm{supp}(\widehat{f}) \subseteq \Lambda\}.$$

Clearly, $\mathcal{E}(E)$ is a Fourier frame for $L^2(\Lambda)$ if and only if the sequence,

$$\{(e_{-x}\ \mathbb{1}_\Lambda)^\vee : x \in E\} \subseteq PW_\Lambda,$$

is a frame for PW_Λ, in which case it is called a *Fourier frame* for PW_Λ. Note that $\langle F, e_{-x}\rangle = f(x)$ for $f \in PW_\Lambda$, where $\widehat{f} = F \in L^2(\widehat{\mathbb{R}}^d)$ can be considered an element of $L^2(\Lambda)$.

II. Balayage

Beurling introduced the following definition in his 1959-1960 lectures.

Definition II.1. (Balayage) Let $E \subseteq \mathbb{R}^d$ and $\Lambda \subseteq \widehat{\mathbb{R}}^d$ be closed sets. *Balayage* is possible for $(E, \Lambda) \subseteq \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ if

$$\forall \mu \in M_b(\mathbb{R}^d),\ \exists \nu \in M_b(E) \text{ such that } \widehat{\mu} = \widehat{\nu} \text{ on } \Lambda.$$

Balayage originated in potential theory, where it was introduced by Christoffel (early 1870s) and by Poincaré (1890). Kahane formulated balayage for the harmonic analysis of restriction algebras. The set, Λ, of group characters (in this case $\mathbb{R}^d$) is the analogue of the original role of Λ in balayage as a set of potential theoretic kernels.

Let $\mathcal{C}(\Lambda) = \{f \in C_b(\mathbb{R}^d) : \mathrm{supp}(\widehat{f}) \subseteq \Lambda\}$.

Definition II.2. (Spectral synthesis) A closed set $\Lambda \subseteq \widehat{\mathbb{R}}^d$ is a set of *spectral synthesis (S-set)* if

$$\forall f \in \mathcal{C}(\Lambda) \text{ and } \forall \mu \in M_b(\mathbb{R}^d), \quad \widehat{\mu} = 0 \text{ on } \Lambda \Rightarrow \int f\, d\mu = 0, \tag{2}$$

see [25].

Closely related to spectral synthesis is the ideal structure of L^1, which can be thought of as the Nullstellensatz of harmonic analysis. As examples of sets of spectral synthesis, polyhedra are S-sets; and the middle-third Cantor set is an S-set which contains non-S-sets. Laurent Schwartz (1947) showed that $S^2 \subseteq \widehat{\mathbb{R}}^3$ is not an S-set; and, more generally, Malliavin (1959) proved that every non-discrete locally compact abelian group contains non-S sets. See [25] for a unified treatment of this material.

Definition II.3. (Strict multiplicity) A closed set $\Gamma \subseteq \widehat{\mathbb{R}}^d$ is a set of *strict multiplicity* if

$$\exists \mu \in M_b(\Gamma) \setminus \{0\} \text{ such that } \lim_{\|x\|\to\infty} |\mu^\vee(x)| = 0.$$

The notion of strict multiplicity was motivated by Riemann's study of sets of uniqueness for trigonometric series. Menchov (1906) showed that there exists a closed set $\Gamma \subseteq \widehat{\mathbb{R}}/\mathbb{Z}$ and $\mu \in M(\Gamma) \setminus \{0\}$, such that $|\Gamma| = 0$ and $\mu^\vee(n) = O((\log|n|)^{-1/2}), |n| \to \infty$. There have been intricate refinements of Menchov's result by Bary (1927), Littlewood (1936), Beurling, et al., see [25].

The above concepts are used in the deep proof of the following theorem, due to Beurling.

Theorem II.4. *Assume that Λ is an S-set of strict multiplicity, and that balayage is possible for (E, Λ). Let $\Lambda_\epsilon = \{\gamma \in \widehat{\mathbb{R}}^d :$*

dist $(\gamma, \Lambda) \leq \epsilon\}$. *Then, there is* $\epsilon_0 > 0$ *such that if* $0 < \epsilon < \epsilon_0$, *then balayage is possible for* (E, Λ_ϵ).

Definition II.5. A sequence $E \subseteq \mathbb{R}^d$ is *separated* if

$$\exists\, r > 0 \text{ such that } \inf\{\|x - y\| : x, y \in E \text{ and } x \neq y\} \geq r.$$

The following theorem, also due to Beurling, gives a sufficient condition for Fourier frames in terms of balayage. Its history and structure are analyzed in [19] as part of a more general program.

Theorem II.6. *Assume that* $\Lambda \subseteq \widehat{\mathbb{R}}^d$ *is an S-set of strict multiplicity and that* $E \subseteq \mathbb{R}^d$ *is a separated sequence. If balayage is possible for* (E, Λ)*, then* $\mathcal{E}(E)$ *is a Fourier frame for* $L^2(\Lambda)$*, i.e.,* $\{(e_{-x}\, \mathbb{1}_\Lambda)^\vee : x \in E\}$ *is a Fourier frame for* PW_Λ.

III. Pseudo-differential operators and Time-frequency analysis

Definition III.1. Let $f, g \in L^2(\mathbb{R}^d)$. The *short time Fourier transform* (STFT) of f with respect to g is the function $V_g f$ on $\mathbb{R}^{2d}$ defined as

$$V_g f(x, \omega) = \int_{\mathbb{R}^d} f(t)\overline{g(t - x)}\; e^{-2\pi i t\cdot\omega}\; dt,$$

see [20], [24] (Chapter 8).

The STFT is uniformly continuous on $\mathbb{R}^{2d}$. Further, for a fixed "window" $g \in L^2(\mathbb{R}^d)$ with $\|g\|_2 = 1$, we can recover the original function $f \in L^2(\mathbb{R}^d)$ from its STFT $V_g f$ by means of the vector-valued integral inversion formula,

$$f = \int\int_{\mathbb{R}^{2d}} V_g f(x, \omega)\; e_\omega \tau_x\; d\omega\; dx, \tag{3}$$

where $(\tau_x g)(t) = g(t - x)$ and $(e_\omega g)(t) = e^{2\pi i t\cdot\omega} g(t)$.

Let $\sigma \in \mathcal{S}'(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$. The operator, K_σ, formally defined as

$$(K_\sigma f)(x) = \int \sigma(x, \gamma)\widehat{f}(\gamma) e^{2\pi i x\cdot\gamma}\; d\gamma,$$

is the *pseudo-differential operator* with Kohn-Nirenberg symbol, σ, see [24] Chapter 8, [26], and [27], Chapter VI. We can manipulate the above formula to illustrate its connection to time frequency analysis. In fact, we compute as follows:

$$\begin{aligned}
(K_\sigma f)(x) &= \int \sigma(x, \gamma)\widehat{f}(\gamma) e^{2\pi i x\cdot\gamma}\; d\gamma \\
&= \int\int \sigma(x, \gamma) e^{2\pi i (x - y)\cdot\gamma} f(y)\; d\gamma\; dy \\
&= \int\int \widehat{\sigma}(\omega, y - x) e^{2\pi i \omega\cdot x} f(y)\; dy\; d\omega \\
&= \int\int \widehat{\sigma}(\omega, u) e^{2\pi i \omega\cdot x} f(u + x)\; du\; d\omega.
\end{aligned}$$

This last expression allows us to view K_σ formally as a superposition of time-frequency shifts, that is,

$$K_\sigma = \int\int \widehat{\sigma}(\omega, u) e_\omega \tau_{-u}\; du\; d\omega,$$

where e_ω and τ_u are, respectively, the modulation and translation operators, defined in (3). These two operators are the fundamental operators in time-frequency analysis, and we see that a pseudo-differential operator with the right symbol class is a superposition of modulation and translations. The function $\widehat{\sigma}$ is the spreading function associated to the operator K_σ.

For consistency with the notation of the previous section, we shall define pseudo-differential operators, K_s, with tempered distributional Kohn-Nirenberg symbols, $s \in \mathcal{S}'(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$, as

$$(K_s \widehat{f})(\gamma) = \int s(y, \gamma) f(y) e^{-2\pi i y\cdot\gamma}\; dy.$$

Further, we shall deal with Hilbert-Schmidt operators, $K\colon L^2(\widehat{\mathbb{R}}^d) \to L^2(\widehat{\mathbb{R}}^d)$; and these, in turn, can be represented as $K = K_s$, where $s \in L^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$. Recall that $K\colon L^2(\widehat{\mathbb{R}}^d) \to L^2(\widehat{\mathbb{R}}^d)$ is a *Hilbert-Schmidt operator* if

$$\sum\nolimits_{n=1}^{\infty} \|K e_n\|_2^2 < \infty$$

for some orthonormal basis, $\{e_n\}_{n=1}^\infty$, for $L^2(\widehat{\mathbb{R}}^d)$, in which case the *Hilbert-Schmidt norm* of K is defined as

$$\|K\|_{HS} = \left(\sum_{n=1}^{\infty} \|K e_n\|_2^2\right)^{1/2},$$

and $\|K\|_{HS}$ is independent of the choice of orthonormal basis. The first theorem about Hilbert-Schmidt operators is the following [28]:

Theorem III.2. *If* $K\colon L^2(\widehat{\mathbb{R}}^d) \to L^2(\widehat{\mathbb{R}}^d)$ *is a bounded linear mapping and* $(K\widehat{f})(\gamma) = \int m(\gamma, \lambda)\widehat{f}(\lambda)\; d\lambda$, *for some measurable function* m*, then* K *is a Hilbert-Schmidt operator if and only if* $m \in L^2(\widehat{\mathbb{R}}^{2d})$ *and, in this case,* $\|K\|_{HS} = \|m\|_{L^2(\mathbb{R}^{2d})}$.

The following is our main result in this paper. It shows the construction of frames using pseudo-differential operators.

Theorem III.3. *Let* $E = \{x_n\} \subseteq \mathbb{R}^d$ *be a separated sequence, that is symmetric about* $0 \in \mathbb{R}^d$*; and let* $\Lambda \subseteq \widehat{\mathbb{R}}^d$ *be an S-set of strict multiplicity, that is compact, convex, and symmetric about* $0 \in \widehat{\mathbb{R}}^d$. *Assume balayage is possible for* (E, Λ). *Further, let* K *be a Hilbert-Schmidt operator on* $L^2(\widehat{\mathbb{R}}^d)$ *with pseudo-differential operator representation,*

$$(K\widehat{f})(\gamma) = (K_s \widehat{f})(\gamma) = \int s(y, \gamma) f(y) e^{-2\pi i y\cdot\gamma}\; dy,$$

where $s_\gamma(y) = s(y, \gamma) \in L^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ *is the Kohn-Nirenberg symbol and where we make the further assumption that*

$$\forall \gamma \in \widehat{\mathbb{R}}^d, \quad s_\gamma \in C_b(\mathbb{R}^d) \quad \text{and} \quad \operatorname{supp}\,(s_\gamma e_{-\gamma})\widehat{\ } \subseteq \Lambda.$$

Then,

$$\exists A,\, B > 0 \quad \text{such that} \quad \forall f \in L^2(\mathbb{R}^d)\backslash\{0\},$$

$$\begin{aligned}
&A \frac{\|K_s \widehat{f}\|_2^4}{\|f\|_2^2} \leq \sum_{x \in E} |\langle (K_s \widehat{f})(\cdot), \overline{s(x, \cdot)}\; e_x(\cdot)\rangle|^2 \\
&\leq B\; \|s\|^2_{L^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)} \|K_s \widehat{f}\|_2^2.
\end{aligned}$$

The proof of Theorem III.3 can be found in [19]. The framework developed in [29] and [30] are presently our guide for formulating the next stage of this theorem in a more useful and elegant form.

IV. The Beurling covering theorem

The material in this section is due to [29] and [30], and it goes back to the late 1990s. The underlying analysis is due to Beurling [10], but Benedetto and Wu emphasized the essential nature of coverings in the process. The setting is for Fourier frames. However, it is this *point of view* that we hope will be useful in constructing implementable frames from Theorem III.3. In particular, the hypothesis of balayage in Theorem III.3 would be replaced by the analogues of the analytic and geometrical hypotheses of an inequality and a covering property, respectively, in Theorem IV.2.

Let $\Lambda \subseteq \widehat{\mathbb{R}}^d$ be a convex, compact set which is symmetric about the origin and has non-empty interior. Then $\|\cdot\|_\Lambda$, defined by

$$\forall \gamma \in \widehat{\mathbb{R}}^d, \quad \|\gamma\|_\Lambda = \inf\{\rho > 0 : \gamma \in \rho\Lambda\},$$

is a norm on $\widehat{\mathbb{R}}^d$ equivalent to the Euclidean norm. The polar set $\Lambda^* \subseteq \mathbb{R}^d$ of Λ is defined as

$$\Lambda^* = \{x \in \mathbb{R}^d : x \cdot \gamma \leq 1, \text{ for all } \gamma \in \Lambda\}.$$

It is elementary to check that Λ^* is a convex, compact set which is symmetric about the origin, and that it has non-empty interior.

Example IV.1. Let $\Lambda = [-1,1]\times[-1,1]$. Then, for $(\gamma_1, \gamma_2) \in \widehat{\mathbb{R}}^2$,

$$\|(\gamma_1,\gamma_2)\|_\Lambda = \inf\{\rho > 0 : |\gamma_1| \leq \rho, |\gamma_2| \leq \rho\} = \|(\gamma_1,\gamma_2)\|_\infty .$$

The polar set of Λ is

$$\Lambda^* = \{(x_1,x_2) : |x_1|+|x_2| \leq 1\} = \{(x_1,x_2) : \|(x_1,x_2)\|_1 \leq 1\}.$$

Theorem IV.2. *(Beurling covering theorem) Let $\Lambda \subseteq \widehat{\mathbb{R}}^d$ be a convex, compact set which is symmetric about the origin and has non-empty interior, and let $E \subseteq \mathbb{R}^d$ be a separated set satisfying the covering property,*

$$\bigcup_{y\in E} \tau_y \Lambda^* = \mathbb{R}^d.$$

If $\rho < 1/4$, then $\{(e_{-x}\, \mathbb{1}_\Lambda)^\vee : x \in E\}$ is a Fourier frame for $PW_{\rho\Lambda}$.

Theorem IV.2 [29], [30] involves the Paley-Wiener theorem and properties of balayage, and, as mentioned, it depends on the theory developed in [10] as well as [9], pages 341-350, and [14]. Unaware of our work from the late 1990s there is the recent development[31].

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