Irregular Sampling and Frames

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Abstract. *A theory of irregular sampling is developed for the class of real sampling sequences for which there are L^2-convergent sampling formulas. The sampling sequences are effectively characterized, and the formulas are accompanied by methods of computing coefficients. These sampling formulas depend on the theory of coherent state (Gabor) frames and an analysis of the inverse frame operator. The results include regular sampling theory and the irregular sampling theory of Paley-Wiener, Levinson, Beutler, and Yao-Thomas. The chapter also presents a new aliasing technique, perspective on stability and uniqueness, and references to recent contributions by others.*

CONTENTS

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§1. Introduction

The subject of sampling, whether as method, point of view, or theory, weaves its fundamental ideas through a panorama of engineering, mathematical, and scientific disciplines. Results have been discovered in one or another discipline independently of similar results in other disciplines. The spectacular expositions and research-tutorials of Bützer et al. [21] and Higgins [31] not only establish the pervasiveness of sampling, but leave the reader with a sense that the time has arrived for an efficacious synthesis of the subject. As an example from the past, Schwartz's treatise [57] on distribution theory was a compendium of diverse past accomplishments, a unification of technologies, an original formulation of ideas and techniques both new and old, and a research manual leading to new mathematics and applications, cf. a similar phenomenon in wavelet theory at the present time, *e.g.*, Meyer's treatise [47]. The stage is set for a comparable development in sampling theory.

Our more focused and realistic goal in this chapter is to use the theory of frames to formulate applicable sampling formulas in an elementary and unified way for irregularly spaced sampling sequences. The treatment is general with regard to the spacing of the sampling sequences, and the sampling formulas are mathematical theorems when mild hypotheses are made. Our basic technique involves an analysis of the inverse frame operator, and the formulas include a good deal of the existing theory as well as new material. We have also developed an algorithm associated with the sampling formulas, and there are undoubtedly other algorithms for various specific applications. The theory of frames is due to Duffin and Schaeffer [24], and there is an extraordinary presentation by Young [64] on the subject. Our basic formulas are stated in §8, and these results, as well as other observations throughout the chapter, are part of a collaborative venture with Heller, *e.g.*, [9], [10]. Item [9] is referenced in various sections, and item [10] deals with multidimensional sampling and applications of our algorithm.

We do not repeat material covered in [21], [31], except to round out a discussion now and then. In particular, we reference, rather than prove, certain well-known and occasionally difficult theorems.

The titles for §§2-4 are self-explanatory. In §5 we utilize coherent state frames to obtain the results of §2 in another way. This leads to a new point of view on aliasing, and sets the stage for the case of irregular sampling in §6 and §8. §6 presents the sampling theory of Paley-Wiener, Levinson, Beutler, and Yao-Thomas, but it does so in terms of the theory of exact frames and the inverse frame operator. §7 provides results so that the formulas in §8 can be implemented. §9 is perhaps the most idiosyncratic section of the chapter, and it deals with stability and uniqueness. In §9, we have also chosen to point out the deep work for Beurling [12], [13], Beurling and Malliavin [14], [15], and Landau [41], and to hint at exciting questions which remain to be answered.

§10 is divided into two parts. The first part is brief but important. It lists references to new ideas in irregular sampling. It is, of course, dangerous to make such a list, since I am undoubtedly unaware of other excellent work besides that listed. However, irregular sampling, in the context of coherent states and wavelets, is a subject whose time has come; and these new contributions fit into the theme of the chapter. The second part of §10 shows how aliasing problems provide a transition from the coherent state setting of this chapter to the threshold of the wavelet and wavelet packet setting of some of our other work.

Besides the usual notation in analysis as found in the books by Hörmander [32], Katznelson [38], Schwartz [57], and Stein and Weiss [59], we use the conventions and notation described in §11.

§2. The Classical Sampling Theorem

$L^1(\mathbb{R})$ is the space of complex-valued integrable functions f defined on the real-line $\mathbb{R}$. The L^1*-norm* of $f \in L^1(\mathbb{R})$ is

$$\|f\|_1 = \int |f(t)|\, dt < \infty,$$

where "$\int$" designates integration over $\mathbb{R}$. The *Fourier transform* $\hat{f}$ of $f \in L^1(\mathbb{R})$ is defined as

$$\hat{f}(\gamma) = \int f(t) e^{-2\pi i t \gamma}\, dt$$

for each $\gamma \in \hat{\mathbb{R}}(= \mathbb{R})$. $A(\hat{\mathbb{R}})$ is the space of such Fourier transforms. The A*-norm* of $F = \hat{f} \in A(\hat{\mathbb{R}})$ is

$$\|F\|_A \equiv \|f\|_1.$$

$L^2(\mathbb{R})$ is the space of complex-valued square integrable functions, *i.e.*, the space of *finite-energy signals.* The L^2*-norm* of $f \in L^2(\mathbb{R})$ is

$$\|f\|_2 = \left(\int |f(t)|^2\, dt \right)^{\frac{1}{2}} < \infty;$$

and the Fourier transform of $f \in L^2(\mathbb{R})$ is a well-defined element of $L^2(\hat{\mathbb{R}})$. Katznelson's book [38] is a standard and excellent reference for the Fourier analysis we shall use.

The *classical sampling theorem* is —

Theorem 1. *Let $f \in L^1(\mathbb{R}) \cap A(\mathbb{R})$, or let $f \in L^2(\mathbb{R})$. Assume there are constants $T, \Omega > 0$ such that*

$$supp\, \hat{f} \subseteq [-\Omega, \Omega], \tag{1}$$

i.e., $\hat{f} = 0$ off of the interval $[-\Omega, \Omega]$, and

$$0 < 2T\Omega \leq 1. \tag{2}$$

Then

$$f(t) = 2T\Omega \sum f(nT) \frac{\sin[2\pi\Omega(t-nT)]}{2\pi\Omega(t-nT)}, \tag{3}$$

where the convergence is pointwise on $\mathbb{R}$ for $f \in L^1(\mathbb{R}) \cap A(\mathbb{R})$, or the convergence is uniform on $\mathbb{R}$ and in L^2-norm for $f \in L^2(\mathbb{R})$.

Proof: Our proofs shall be honest but brief, to highlight their simplicity.

a. Let $f \in L^1(\mathbb{R}) \cap A(\mathbb{R})$. For each $t \in \mathbb{R}$,

$$\begin{aligned} f(t) &= \int \hat{f}(\gamma) e^{2\pi i t\gamma}\, d\gamma \\ &= \int_{-\Omega}^{\Omega} G(\gamma) e^{2\pi i t\gamma}\, d\gamma \\ &= \sum c_n \int_{-\Omega}^{\Omega} e^{2\pi i(t-nT)\gamma}\, d\gamma \\ &= \sum c_n \frac{sin[2\pi\Omega(t-nT)]}{\pi(t-nT)}, \end{aligned} \tag{4}$$

where G is defined as

$$G(\gamma) = \begin{cases} \hat{f}(\gamma), & \text{if} \quad |\gamma| \leq \Omega, \\ 0, & \text{if} \quad \Omega < |\gamma| \leq \frac{1}{2T}, \end{cases}$$

extended $\frac{1}{T}$-periodically on $\hat{\mathbb{R}}$. The *Fourier series* of G is

$$G(\gamma) \equiv \sum c_n\, e^{-\pi i n\gamma(2T)},$$

where

$$c_n = T \int_{-\frac{1}{2T}}^{\frac{1}{2T}} G(\gamma) e^{\pi i n\gamma(2T)}\, d\gamma.$$

Thus, $c_n = Tf(nT)$.

The calculation (4) is justified as follows. The first equation is a consequence of a classical inversion theorem for $f \in L^1(\mathbb{R}) \cap A(\mathbb{R})$, the second equation follows by the definition of G, the third equation results from the fact that Fourier series of integrable functions can be integrated term by term, and the fourth equation is clear by a simple calculation.

b. Let $f \in L^2(\mathbb{R})$. By the definition of the L^2-Fourier transform,

$$\left\| f(t) - \int_{-\Omega}^{\Omega} G(\gamma) e^{2\pi i t\gamma}\, d\gamma \right\|_2 = 0. \tag{5}$$

If $S_N G$ is the N-th partial sum of the Fourier series of G then

$$\begin{aligned} &\left\| \int_{-\Omega}^{\Omega} G(\gamma) e^{2\pi i t\gamma}\, d\gamma - \int_{-\Omega}^{\Omega} (S_N G)(\gamma) e^{2\pi i t\gamma}\, d\gamma \right\|_2 \\ &\qquad = \left\| \int \mathbf{1}_{(\Omega)}(\gamma)\big(G(\gamma) - (S_N G)(\gamma)\big) e^{2\pi i t\gamma}\, d\gamma \right\|_2 \\ &\qquad = \left\| \mathbf{1}_{(\Omega)}(G - S_N G) \right\|_2 = \left\| G - S_N G \right\|_{L^2[-\Omega,\Omega]}, \end{aligned} \tag{6}$$

where the second equation is a consequence of the Plancherel theorem. Also, we have

$$\lim_{N\to\infty} \|G - S_N G\|_{L^2[-\Omega,\Omega]} = 0,$$

since

$$\lim_{N\to\infty} \|G - S_N G\|_{L^2[-\frac{1}{2T},\frac{1}{2T}]} = 0$$

because of properties of Fourier series of square integrable functions. Combining this information with (5), (6), Hölder's inequality, and the definition of c_n, we obtain (3) with convergence in the L^2-norm, cf. the proof of Proposition 9.

Assuming the result for L^2-convergence, we prove the uniform convergence on $\mathbb{R}$ by means of Hölder's inequality and the Plancherel theorem. ■

Discussion 2. a. A common and essential feature of both proofs of Theorem 1 is the interplay between Fourier series and Fourier transforms.

b. The space $L^1(\mathbb{R}) \cap A(\mathbb{R})$, with norm $\|f\| = \|f\|_1 + \|\hat{f}\|_1$, is a reflexive Banach space. Similarly, the space $L^1(\mathbb{R}) \cap A(\mathbb{R}) \cap L^2(\mathbb{R})$, with norm $\|f\| = \|f\|_1 + \|\hat{f}\|_1 + \|f\|_2$, is a reflexive Banach space. Along with the Hilbert space $L^2(\mathbb{R})$, both these spaces are useful in sampling theory. In Theorem 1, the hypotheses $f \in L^1(\mathbb{R}) \cap A(\mathbb{R})$ and (1) allow us to conclude that $f \in L^2(\mathbb{R})$, so that the second proof also works for the first case.

c. There is an intriguing and labyrinthine history associated with Equation (3), *e.g.*, [31].

Calculation 3. *Classical sampling and the Poisson summation formula.*

There is another attractive proof of (3) which we shall now outline. The proof uses the Poisson summation formula,

$$\left(\sum \delta_{nT}\right)^{\wedge} = \frac{1}{T} \sum \delta_{n/T}, \tag{7}$$

where δ_{nT} is the Dirac δ-measure supported by the point $nT \in \mathbb{R}$. We are not awarding this (correct) proof "Theorem" status since we do not wish to stress smoothness requirements (in the function version of (7)) or distributions in this section, *e.g.*, [32], [38, pages 130-131, number 15], and [57].

Implementing (7), we see that if f and s are well-behaved functions, then

$$f = \left[\left(\sum \delta_{nT}\right) f\right] * s \tag{8}$$

if and only if

$$\hat{f} = \left(\frac{1}{T} \sum \tau_{n/T} \hat{f}\right) \hat{s}, \tag{9}$$

where $(\tau_{n/T}\hat{f})(\gamma) = \hat{f}(\gamma - \frac{n}{T})$. Assuming (1) and (2), we obtain the validity of (9) in the case $\hat{s} = T$ on $[-\Omega, \Omega]$ and $supp\, \hat{s} \subseteq [-\frac{1}{2T}, \frac{1}{2T}]$. For such an s and using the fact,

$$\delta_{nT} f = f(nT)\, \delta,$$

the equivalence of (8) and (9) allows us to conclude that

$$f = \sum f(nT)\,\tau_{nT}\,s. \tag{10}$$

Equation (10) reduces to (3) for the case,

$$\hat{s} = T\,\mathbf{1}_{(\Omega)},$$

or, equivalently,

$$s(t) = T\,\frac{\sin 2\pi\Omega t}{\pi t}. \tag{11}$$

Example 4. It is natural to investigate an appropriate converse of Theorem 1. *Suppose we assume Equation* (3),*with convergence in the* L^2*-norm, for all* $f \in L^2(\mathbb{R})$ *satisfying* (1). *Can we conclude the validity of* (2), *i.e., is*

$$2T\Omega \leq 1?$$

We shall answer this question in the *positive.* Assume $T, \Omega > 0$ satisfy $2T\Omega > 1$ and define

$$f(t) = \frac{\sin\left(\frac{\pi t}{T}\right)}{\frac{\pi t}{T}} \in L^2(\mathbb{R}).$$

Clearly,

$$\hat{f} = T\,\mathbf{1}_{(\frac{1}{2T})},$$

and, in particular, (1) is satisfied. Since

$$f(nT) = \begin{cases} 0, & \text{if} \quad n \neq 0, \\ 1, & \text{if} \quad n = 0, \end{cases}$$

then the right side of the sampling formula (3) is

$$g(t) = 2T\Omega\frac{\sin 2\pi\Omega t}{2\pi\Omega t}.$$

The functions f and g are not equal since both are continuous on $\mathbb{R}$, and $f(0) = 1$ and $g(0) = 2T\Omega > 1$. (For another proof that $f \neq g$, note that $\hat{g} = T\,\mathbf{1}_{(\Omega)}$.)

Definition 5. a. In the sampling formula (3), the *sampling period* is T, the *sampling sequence* is $\{nT : n \in \mathbb{Z}\}$, and the sequence of *sampled values* is $\{f(nT) : n \in \mathbb{Z}\}$.

b. Let 2Ω be a given frequency bandwidth (of a linear time-invariant system having an even frequency response). For example, consider the ideal lowpass filter with cutoff frequency Ω. Because of Theorem 1 and Example 4, the minimum rate at which each element $f \in L^2(\mathbb{R})$, for which *supp* $\hat{f} \subseteq$

$[-\Omega, \Omega]$, must be sampled for exact reconstruction is 2Ω samples per unit time. This sampling rate, 2Ω, is the *Nyquist rate.*

If the unit time is seconds (for convenience) and we define

$$T \equiv \frac{1}{2\Omega},$$

then the Nyquist rate or *sampling frequency* is $1/T$ samples per second, *i.e.*, 1 sample per T seconds, from which we obtain the maximal sampling period of T.

The effect of undersampling continuous (in fact, analytic) signals $f \in L^2(\mathbb{R})$ satisfying (1), *vis a vis* the goal of exact reconstruction by means of (3), is called *aliasing.*

Definition/Discussion 6. *Aliasing.*

a. Suppose $2T\Omega > 1$. For simplicity, let us further assume that $1 \geq T\Omega$. Consider the Fourier series of G, defined in the proof of Theorem 1. Since $G$is $\frac{1}{T}$-periodic, it is of the form $\sum F(\gamma - \frac{n}{T})$, where, from the analysis of Theorem 1, F must vanish off of $[-\frac{1}{2T}, \frac{1}{2T}]$. By our hypotheses on T and Ω, $[\Omega - \frac{1}{T}, -\Omega + \frac{1}{T}]$ is the only subset of $[-\frac{1}{2T}, \frac{1}{2T}]$ where the Fourier series of G can be expected to converge to G. That this is the case follows since both $\tau_{1/T}F$ and $\tau_{-1/T}F$ are non-zero on subsets of $[-\frac{1}{2T}, \frac{1}{2T}]$. The ensuing phenomenon, caused by this non-convergence of the Fourier series on all of $[-\frac{1}{2T}, \frac{1}{2T}]$, is *aliasing.* The term "aliasing", due to Tukey, catches the flavor of high and low frequencies from the Fourier series "assuming the alias of each other", cf. the more quantitative discussion of aliasing in §§5 and 10 as well as [50].

b. Old motion pictures of fast moving events produce "jumpy" video, and this is a classical example of aliasing. In fact, each frame of film is a sampled value, but the sampling rate is not sufficiently high to produce exact reconstruction of the event.

In Definition 5b, we referred to the notion of a "linear time-invariant system". This is an elementary engineering concept, *e.g.*, [48], [51], which also has a long history in mathematics, *e.g.*, [1, pages 216-217]. To be a little more precise, let X and Y be Banach sub-algebras of the convolution algebra $M_b(\mathbb{R})$ of bounded Radon-measures, let $\delta \in X$, assume $X \subseteq Y$, and let $L : X \to Y$ be a continuous linear map. L is a *linear time-invariant system* if the impulse response $L\delta = h$ has the property that

$$\forall f \in X, \quad Lf = h * f.$$

The Fourier transform $\hat{h}$ is the *frequency response* of the system, and L or h is called a *filter* if the inclusion *supp* $\hat{h} \subseteq \hat{\mathbb{R}}$ is proper.

We omit the proof of the following result. The proof is similar to that of Theorem 1, and it becomes valid when natural hypotheses are added to the statement of the "theorem". The "theorem", itself, provides the classical sampling formula for linear time-invariant systems.

Theorem 7. *Let $L : X \to Y$ be a linear time-invariant system with impulse response $h \in Y$ and frequency response $\hat{h}$. Assume there are constants $T, \Omega > 0$ such that*

$$0 < 2T\Omega \leq 1,$$

and define the function

$$s(t) = T \int_{-\Omega}^{\Omega} \frac{1}{\hat{f}(\gamma)} e^{2\pi i t\gamma}\, d\gamma. \tag{12}$$

Let $f \in X$ satisfy the support condition,

$$supp\, \hat{f} \subseteq [-\Omega, \Omega].$$

Then

$$f = \sum (Lf)(nT)\tau_{nT} s. \tag{13}$$

The function s, in Equations (10), (11), (12) and (13), is a *sampling function*. In the case of (11) we obtain the classical sampling formula (3). Our goal in the sequel is to obtain sampling formulas of the form,

$$f = \sum f(t_n) s_n \tag{14}$$

and

$$f = \sum c_n(f) \tau_{t_n} s, \tag{15}$$

where $\{t_n\}$ is an irregular sampling sequence, *i.e.*, $\{t_n\}$ is not uniformly spaced. Later, we shall comment on the other parameters in (14) and (15). Also, there are versions of (14) and (15) corresponding to (13).

§3. The Paley-Wiener Theorem

In light of the support hypothesis in Theorem 1 and Theorem 7, we make the following definition.

Definition/Notation 8. a. Let $\Omega > 0$, and define the space,

$$PW_\Omega = \{f \in L^2(\mathbb{R}) : supp\, \hat{f} \subseteq [-\Omega, \Omega]\}.$$

PW_Ω is the *Paley-Wiener space*, and it is a Hilbert space, considered as a closed subspace of $L^2(\mathbb{R})$ taken with the L^2-norm. The support condition, $supp\, \hat{f} \subseteq [-\Omega, \Omega]$, is described by saying that f is *Ω-bandlimited*.

b. Let

$$d(t) = \frac{\sin t}{\pi t},$$

and define the L^1-dilation

$$d_\lambda(t) = \lambda d(t\lambda).$$

("d" is for Dirichlet.) Equation (3) can be written as

$$f(t) = T \sum f(nT)(\tau_{nT}\, d_{2\pi\Omega})(t).$$

PW_Ω is a *reproducing kernel Hilbert space* in the sense that

$$PW_\Omega = \{f \in L^2(\mathbb{R}) : f * d_{2\pi\Omega} = f\},$$

cf., the usual development of reproducing kernel Hilbert space in [30], [61], [64].

c. If $f \in PW_\Omega$ then $\hat{f} \in L^2[-\Omega, \Omega] \subseteq L^1[-\Omega, \Omega]$, where the inclusion is a consequence of Hölder's inequality. Further, if we define the function,

$$g(z) \equiv \int_{-\Omega}^{\Omega} \hat{f}(\gamma) e^{2\pi i z \gamma}\, d\gamma, \quad z = t + iy \in \mathbb{C},$$

then g is a continuous function on $\mathbb{C}$ and $g = f$ a.e. on $\mathbb{R}$.

In the other direction, if $F \in L^2[-\Omega, \Omega]$ and we define the continuous function

$$f(t) \equiv \int_{-\Omega}^{\Omega} F(\gamma) e^{2\pi i t \gamma} d\gamma, \quad t \in \mathbb{R},$$

then $f \in PW_\Omega$ and $F = \hat{f}$ a.e. on $\hat{\mathbb{R}}$. These observations are elementary facts from real and harmonic analysis, *e.g.*, [2], [38].

d. A function $f : \mathbb{C} \to \mathbb{C}$ which is analytic at every point of $\mathbb{C}$ is *entire*; and an entire function is of *exponential type* A if for each $B > A$

$$\exists C = C(B) > 0 \quad \text{such that} \quad \forall z \in \mathbb{C}, \qquad |f(z)| \leq C e^{B|z|}. \tag{16}$$

Proposition 9. *If $f \in PW_\Omega$ then f is an entire function of exponential type $2\pi\Omega$, and*

$$\forall t \in \mathbb{R}, \qquad f(t) = \int_{-\Omega}^{\Omega} \hat{f}(\gamma) e^{2\pi i t \gamma}\, d\gamma,$$

i.e., pointwise equality as well as L^2-norm equality.

Proof: If $f \in PW_\Omega$ then

$$g(z) \equiv \int_{-\Omega}^{\Omega} \hat{f}(\gamma) e^{2\pi i z \gamma}\, d\gamma, \qquad z = t + iy \in \mathbb{C},$$

is a well-defined continuous function on $\mathbb{C}$. One proof that g is entire follows from Fubini's and Morera's theorems. Finally,

$$\begin{aligned} |g(z)| &\leq \int_{-\Omega}^{\Omega} |\hat{f}(\gamma)| e^{-2\pi y \gamma}\, d\gamma \\ &\leq e^{2\pi |y| \Omega} \int_{-\Omega}^{\Omega} |\hat{f}(\gamma)|\, d\gamma \leq e^{2\pi |z| \Omega} \int_{-\Omega}^{\Omega} |\hat{f}(\gamma)|\, d\gamma \\ &\equiv C e^{2\pi |z| \Omega} < \infty. \end{aligned}$$

Now, the Fourier inversion theorem for $L^2(\mathbb{R})$ asserts that

$$f(t) = \int \hat{f}(\gamma) e^{2\pi i t\gamma}\, d\gamma$$

in $L^2(\mathbb{R})$. Consequently, $\|f - g\|_2 = 0$ so that $f = g$ a.e., and, hence, f can be identified with g on $\mathbb{R}$. ■

In this case, $\inf\{B > 2\pi\Omega : (16) \text{ is valid}\} = 2\pi\Omega$ also satisfies (16) with the constant $C = \|\hat{f}\|_1$, cf. [19, page 104, Equations (6.8.4)-(6.8.6)].

The converse of Proposition 9 is the Paley-Wiener theorem which plays a role in much of what follows.

Theorem X. (Paley-Wiener, [49, Theorem X]). *Let $f \in L^2(\mathbb{R})$. Then $f \in PW_\Omega$ if and only if f is an entire function of exponential type $2\pi\Omega$.*

Discussion 11. a. There are a number of similar formulations of Theorem 10; and, besides the proof in [49], proofs of Theorem 10 can be found in [19, pages 103-108], [56, pages 370-372]. The sufficient condition in Theorem 10, in order that $f \in L^2(\mathbb{R})$ be an element of PW_Ω, can be replaced by the condition,

$$\exists C > 0 \quad \text{such that} \quad \forall z \in \mathbb{C}, \quad |f(z)| \leq C e^{2\pi|z|\Omega}.$$

b. Because of our interest in d-dimensional sampling, we note that d-dimensional versions of the Paley-Wiener theorem were proved early-on, *e.g.*, [52]. Further, there are distributional versions in $\mathbb{R}^d$, *e.g.*, [32, volume I, pages 181-182 (cf. pages 21-22 of Hörmander's first edition)], [57]. A proof of the Paley-Wiener theorem for $L^2(\mathbb{R}^d)$ is found in [59, pages 112-114].

Example 12. There is an analogue of Theorem 10 for the usual Laplace transform, which was also proved by Paley and Wiener [49, Theorem V]. To state this result, we first define the classical *Hardy space* H^2 to be the set of functions f, analytic in the right half plane, with the property that

$$\sup_{t>0} \left(\int |f(t + iy)|^2\, dy \right)^{\frac{1}{2}} < \infty,$$

cf., Lemma 7.7a. Further, if $F : \hat{\mathbb{R}} \to \mathbb{C}$ is supported by $[0, \infty)$, its Laplace transform is

$$L(F)(z) = \int_0^\infty F(\gamma) e^{-z\gamma}\, d\gamma.$$

The Laplace transform analogue of Theorem 10 is the representation theorem: *if $f \in H^2$ then there is $F \in L^2(\hat{\mathbb{R}})$, supported by $[0, \infty)$, for which*

$$f(z) = L(F)(z), \qquad z = t + iy \quad \text{and} \quad t > 0.$$

The converse is true, and is not difficult to prove.

The L^p-version of this result is due to Doetsch (1936), using results of Hille and Tamarkin (1935). Weighted versions were initiated by Rooney in the 1960s; and a general theory for weighted L^p and H^p spaces was formulated by the author, with Heinig and Johnson, in the 1980s, *e.g.*, [6] for references.

§4. Frames and Exact Frames

Definition 13. a. A sequence $\{g_n\} \subseteq H$, a separable Hilbert space, is a *frame* if there exist $A, B > 0$ such that

$$\forall f \in H, \quad A\|f\|^2 \leq \sum |\langle f, g_n \rangle|^2 \leq B\|f\|^2,$$

where $\langle , \rangle$ is the inner product on H and the norm of $f \in H$ is $\|f\| = \langle f, f \rangle^{\frac{1}{2}}$. For example, if $H = L^2(\mathbb{R})$ and $f, g \in H$ then $\langle f, g \rangle = \int f(t)\overline{g}(t)\, dt$. A and B are the *frame bounds*, and a frame $\{g_n\}$ is *tight* if $A = B$.

A frame $\{g_n\}$ is *exact* if it is no longer a frame when any one of its elements is removed.

Clearly, if $\{g_n\}$ is an orthonormal basis of H then it is a tight exact frame with $A = B = 1$.

b. The *frame operator* of the frame $\{g_n\}$ is the function: $S : H \to H$ defined as $Sf = \sum \langle f, g_n \rangle\, g_n$. In Theorem 14a, the first expansion is the *frame expansion*, and the second is the *dual frame expansion*.

The theory of frames is due to Duffin and Schaeffer [24] in 1952. Expositions include [64] and [28], the former presented in the context of non-harmonic Fourier series and the latter in the setting of wavelet theory.

Theorem 14. *Let $\{g_n\} \subseteq H$ be a frame with frame bounds A and B.*

a. *S is a topological isomorphism with inverse $S^{-1} : H \to H$. $\{S^{-1}g_n\}$ is a frame with frame bounds B^{-1} and A^{-1}, and*

$$\forall f \in H, \quad f = \sum \langle f, S^{-1}g_n \rangle\, g_n = \sum \langle f, g_n \rangle\, S^{-1}\, g_n \quad \text{in } H.$$

b. *If $\{g_n\}$ is tight, $\|g_n\| = 1$ for all n, and $A = B = 1$, then $\{g_n\}$ is an orthonormal basis of H.*

c. *If $\{g_n\}$ is exact, then $\{g_n\}$ and $\{S^{-1}g_n\}$ are biorthonormal, i.e.,*

$$\forall m, n \quad \langle g_m, S^{-1}g_n \rangle = \delta_{mn},$$

and $\{S^{-1}g_n\}$ is the unique sequence in H which is biorthonormal to $\{g_n\}$.

d. *If $\{g_n\}$ is exact, then the sequence resulting from the removal of any one element is not complete in H, i.e., the linear span of this resulting sequence is not dense in H.*

Discussion 15. *Vitali's theorem.*

We comment on Theorem 14a because it is surprisingly useful and because of a stronger result by Vitali (1921).

To prove part b we first use tightness and $A = 1$ to write,

$$\|g_m\|^2 = \|g_m\|^4 + \sum_{n \neq m} |\langle g_m, g_n\rangle|^2;$$

and obtain that $\{g_n\}$ is orthonormal since each $\|g_n\| = 1$. To conclude the proof we then invoke the well-known result: if $\{g_n\} \subseteq H$ is orthonormal then it is an orthonormal basis of H if and only if

$$\forall f \in H, \quad \|f\|^2 = \sum |\langle f, g_n\rangle|^2.$$

In 1921, Vitali proved that an orthonormal sequence $\{g_n\} \subseteq L^2[a,b]$ is complete, and so $\{g_n\}$ is an orthonormal basis, if and only if

$$\forall t \in [a,b], \quad \sum \left| \int_a^t g_n(u)\, du \right|^2 = t - a. \tag{17}$$

For the case $H = L^2[a,b]$, Vitali's result is stronger than part b since (17) is tightness with $A = 1$ for functions $f = \mathbf{1}_{[a,t]}$.

Other remarkable and important contributions by Vitali are highlighted in [2].

Definition 16. Let H be a separable Hilbert space. A sequence $\{g_n\} \subseteq H$ is a *Schauder basis* or *basis* of H if each $f \in H$ has a unique decomposition $f = \sum c_n(f) g_n$. A basis g_n is an *unconditional basis* if

$$\exists C \quad \text{such that} \quad \forall F \subseteq \mathbb{Z}, \quad \text{where} \quad \mathit{card}\, F < \infty, \quad \text{and}$$

$$\forall b_n, c_n \in \mathbb{C}, \quad \text{where} \quad n \in F \quad \text{and} \quad |b_n| \leq |c_n|,$$

$$\| \sum_{n \in F} b_n g_n \| \leq C \| \sum_{n \in F} c_n g_n \|.$$

An unconditional basis $\{g_n\}$ is *bounded* if

$$\exists A, B > 0 \quad \text{such that} \quad \forall n, \ A \leq \|g_n\| \leq B.$$

Separable Hilbert spaces have orthonormal bases, and orthonormal bases are bounded unconditional bases.

Köthe (1936) proved the implication, *b* implies *c*, of the following theorem. The implication, *c* implies *b*, is straightforward; and the equivalence of a and c is found in [64, pages 188-189].

Theorem 17. *Let H be a separable Hilbert space and let $\{g_n\} \subseteq H$ be a given sequence. The following are equivalent:*

a. *$\{g_n\}$ is an exact frame of H;*

b. *$\{g_n\}$ is a bounded unconditional basis of H;*

c. $\{g_n\}$ *is a Riesz basis, i.e., there is an orthonormal basis* $\{u_n\}$ *and a topological isomorphism* $L: H \to H$ *such that* $Lg_n = u_n$ *for each* n.

Definition/Discussion 18. *Weyl-Heisenberg (Gabor) and Fourier frames.*

a. Let $g \in L^2(\mathbb{R})$ and suppose we are given sequences $\{a_n\}$, $\{b_n\} \subseteq \mathbb{R}$. Recall that translation is defined by $(\tau_a g)(t) = g(t-a)$ and, notationally, we write $e_b(t) = e^{2\pi i t b}$. If $\{e_{b_m}\tau_{a_n} g\}$ is a frame for $L^2(\mathbb{R})$ it is called a *Gabor* or *Weyl-Heisenberg* frame. *Fourier frames* $\{e_{b_m}\}$ were defined in [24] for $L^2[-T,T]$. Precisely, if $\{e_{b_m}\}$ is a frame for $L^2[-T,T]$ it is called a *Fourier frame* for $L^2[-T,T]$.

b. Gabor's seminal paper [26] (1946) deals with "regularly latticed" systems $\{e_{mb}\tau_{na}g\}$, where g is the Gaussian; and it turns out that the Heisenberg group is fundamental in analyzing the structure of modulations and translations such as $\{e_{mb}\tau_{na}g\}$ for $g \in L^2(\mathbb{R})$. This explains our terminology in part a. Gabor won the Nobel prize for his conception and analysis of holography (1947), which is a method for photographically recording a three-dimensional image; and, as demonstrated by Schempp, the Heisenberg group also plays a role in this setting. The special kind of light required to demonstrate the capability of holography is a single frequency form called *coherent* light; and it became readily available after the laser was developed in 1960. Further, the *coherent states* of quantum mechanics are the elements of $\{e_{mb}\tau_{na}g\}$ in the case g is the Gaussian and $ab = 1$. For the Gaussian g and $ab = 1$, $\{e_{mb}\tau_{na}g\}$ is not a frame, cf. [58] for $ab < 1$.

Duffin and Schaeffer's Fourier frames were also part of a larger picture, dealing with problems in non-harmonic Fourier series and complex analysis and the work of Paley-Wiener, Levinson, Plancherel-Pólya, and Boas, cf. §7.

c. $\{e_{b_m}\tau_{a_n}g\}$ is a frame for $L^2(\mathbb{R})$ if and only if $\{\tau_{a_n}(e_{b_m}g)\}$ is a frame for $L^2(\mathbb{R})$.

Our Weyl-Heisenberg frames will often be defined for $L^2(\hat{\mathbb{R}})$. As such, we note that

$$\forall g \in L^2(\mathbb{R}), \quad (e_{a_n}\tau_{b_m}\hat{g})^{\vee} = e^{2\pi i a_n b_m}\, e_{b_m}\tau_{-a_n}g,$$

where "$\vee$" designates the inverse Fourier transform.

Theorem 19. *Let* $g \in L^2(\mathbb{R})$ *and suppose* $a, b > 0$. *Define*

$$G(t) = \sum |g(t-na)|^2.$$

Assume that there exist $A, B > 0$ *such that*

$$0 < A \leq G(t) \leq B < \infty \quad \text{a.e. on} \quad \mathbb{R}, \tag{18}$$

and that supp $g \subseteq I$ *where* I *is an interval of length* $1/b$. *Then* $\{e_{mb}\tau_{na}g\}$ *is a frame for* $L^2(\mathbb{R})$, *with frame bounds* $b^{-1}A$ *and* $b^{-1}B$, *and*

$$\forall f \in L^2(\mathbb{R}), \quad S^{-1}f = \frac{bf}{G}. \tag{19}$$

Theorem 20. *Let $g \in L^2(\mathbb{R})$ and suppose $a, b > 0$. Assume $\{e_{na}\tau_{mb}\hat{g}\}$ is a frame for $L^2(\hat{\mathbb{R}})$. Then*

$$S^{-1}(e_{na}\tau_{mb}\,\hat{g}) = e_{na}\tau_{mb}\,S^{-1}\,\hat{g}. \tag{20}$$

Example 21. a. Let $g \in L^2(\mathbb{R})$ and suppose $a, b > 0$ satisfy $ab = 1$. If $\{e_{mb}\tau_{na}g\}$ is a frame then it is an exact frame. This remarkable fact (for $ab = 1$) can be proved using properties of the Zak transform which we now define.

b. The *Zak transform* of $f \in L^2(\mathbb{R})$ is defined as

$$Zf(x,\omega) = a^{1/2}\sum f(xa + ka)\;e^{2\pi ik\omega}$$

for $(x,\omega) \in \mathbb{R}\times\hat{\mathbb{R}}$ and $a > 0$. It turns out that the Zak transform is a unitary map of $L^2(\mathbb{R})$ onto $L^2(Q), Q = [0,1)\times[0,1)$.

c. If $\{e_{mb}\tau_{na}\,g\}$ is a frame for $ab = 1$, it is a bounded unconditional basis (part a and Theorem 17). In particular, the frame decomposition,

$$\forall f \in L^2(\mathbb{R}), \qquad f = \sum c_{m,n}\,e_{mb}\,\tau_{na}\,g,$$

from Theorem 14a is unique; and it is easy to verify that

$$\begin{aligned}\forall m, n, \quad c_{m,n} &= \langle f, S^{-1}e_{mb}\tau_{na}g\rangle \\ &= \int_0^1\int_0^1 \frac{Zf(x,\omega)}{Zg(x,\omega)}e^{-2\pi imx}e^{-2\pi in\omega}\,dx d\omega.\end{aligned}$$

Results in the same spirit as the one stated in Example 21a have been formulated in the coherent state literature for many years; and, in this context, they seem to go back to the analysis of von Neumann found in [60, pages 405-407]. The following is a representative list of such results and the proofs are not difficult, e.g., [5], [28].

Theorem 22. *Let $g \in L^2(\mathbb{R})$ and suppose $a, b > 0$ satisfy $ab = 1$.*

a. *$\{e_{mb}\tau_{na}\,g\}$ is complete in $L^2(\mathbb{R})$ if and only if $|Zf| > 0$ a.e.*

b. *$\{e_{mb}\tau_{na}\,g\}$ is an orthonormal basis of $L^2(\mathbb{R})$ if and only if $|Zf| = 1$ a.e.*

c. *$\{e_{mb}\tau_{na}\,g\}$ is a frame for $L^2(\mathbb{R})$ with frame bounds A and B if and only if $A \le |Zg|^2 \le B$ a.e. In this case, $\{e_{mb}\tau_{na}\,g\}$ is an frame (Example 21a).*

We shall now state the Balian-Low Theorem, which can be proved using the Zak transform.

Theorem 23. *Let $g \in L^2(\mathbb{R})$ and suppose $a, b > 0$ satisfy $ab = 1$. If $\{e_{mb}\tau_{na}g\}$ is a frame then either $tg(t) \notin L^2(\mathbb{R})$ or $\gamma\hat{g}(\gamma) \notin L^2(\hat{\mathbb{R}})$.*

We first learned of the Balian-Low "phenomenon" in an early preprint of [23]. It turns out that this result has been proved at various levels of precision; and [8] contains an analysis of these proofs, as well as a complete, new proof based on some established ideas.

Discussion 24. *Exact frames: oversampling and undersampling.*

a. We have stated Theorem 22 and 23 since a comparison of the exact frame case ($ab = 1$ for regular lattices) and the more general frame case is fundamental in our approach to irregular sampling. For example, exact frames $\{e_{mb}\tau_{na}\, g\}$ (in particular, orthonormal bases) don't have the flexibility to take into account the oversampling occurring naturally in biological processes such as cochlear processing; our contention is that general frames do have such flexibility. The oversampling reflected in Theorem 26 generally involves sampling functions s with more rapid decay in the Fourier domain than the classical sampling function in Theorem 1. The goal is better computational efficiency for low pass filters; the price to be paid is a sampling rate greater than the Nyquist rate. Of course, we can sample greater than the Nyquist rate for the classical sampling function, but the slow decay in the Fourier domain is still a liability.

b. Using the notation from §2, recall that undersampling occurs if $2T\Omega > 1$, where $\{nT\}$ is the sampling sequence and 2Ω is the given frequency bandwidth. If $a = T$ and $b = 2\Omega$ then $ab > 1$, cf. the proof of Theorem 25. It is an expected fact, with a surprisingly abstract proof, due to Rieffel [55], that *if $g \in L^2(\mathbb{R})$ and $a, b > 0$ satisfy $ab > 1$ then $\{e_{mb}\tau_{na}\, g\}$ is complete in $L^2(\mathbb{R})$*, cf. [23].

The quantity $\Delta = \frac{1}{ab}$ is the natural density of the so-called von-Neumann lattices $\{(na, mb)\}$, *e.g.*, Definition 36 and Example 37c.

The most significant contribution in classical analysis related to Rieffel's theorem is due to Landau [43]. Using a theorem of Daubechies [23, Theorem 3.1], he proved that *if g and $\hat{g}$ have sufficient decay and $\{e_{mb}\tau_{na}\, g\}$ is a Gabor frame for $L^2(\mathbb{R})$, then $ab \leq 1$*. This is, of course, weaker than Rieffel's theorem, but the method, although intricate, is constructive. The following are sufficient decay conditions to implement Daubechies' theorem: there are constants $C > 0$ and $\alpha > \frac{1}{2}$ such that

$$\forall (t, \gamma) \in \mathbb{R} \times \hat{\mathbb{R}}, \quad |g(t)| \leq \frac{C}{(1 + t^2)^\alpha} \quad \text{and} \quad |\hat{g}(\gamma)| \leq \frac{C}{(1 + \gamma^2)^\alpha}.$$

Of course, if $\alpha > 3/4$ and $ab = 1$ then, by Theorem 23, there are no such g for which $\{e_{mb}\tau_{na}\, g\}$ is a Gabor frame.

§5. Regular Sampling and Frames

The theme of this section is to deal with classical sampling results by frame methods in the case that the inverse frame operator S^{-1} is a multiplier. The following is Theorem 1 proved by such methods [9, Theorem 3.1].

Theorem 25. *Let $T, \Omega > 0$ be constants for which*

$$0 < 2T\Omega \leq 1.$$

Then

$$\forall f \in PW_\Omega, \quad f = T \sum f(nt)\tau_{nT} d_{2\pi\Omega} \quad \text{in} \quad L^2(\mathbb{R}). \tag{21}$$

(The Dirichlet kernel d_λ was defined in Definition/Notation 8).

Proof: Let $g = (2\Omega)^{-1/2} d_{2\pi\Omega}$, and set $a = T$ and $b = 2\Omega$. Theorem 19 is applicable and so $\{e_{na}\tau_{mb}\hat{g}\}$ is a frame. Consequently, by Theorem 14a and Theorem 20,

$$\forall f \in L^2(\mathbb{R}), \quad \hat{f} = \sum \langle \hat{f}, e_{na}\tau_{mb} S^{-1}\hat{g}\rangle e_{na}\tau_{mb}\hat{g} \quad \text{in} \quad L^2(\hat{\mathbb{R}}). \tag{22}$$

Since $supp\,\hat{g}$ is compact, we have

$$\forall h \in L^2(\hat{\mathbb{R}}), \quad S^{-1}\hat{h} = 2T\Omega\hat{h}$$

by Theorem 20; and, hence, (22) becomes

$$\forall f \in L^2(\mathbb{R}), \quad f = 2T\Omega \sum \langle \hat{f}, e_{na}\tau_{mb}\hat{g}\rangle \tau_{-na} e_{mb} g \quad \textit{in} \quad L^2(\mathbb{R}). \tag{23}$$

If $f \in PW_\Omega$ then

$$\langle \hat{f}, e_{na}\tau_{mb}\hat{g}\rangle = \begin{cases} (2\Omega)^{-1/2}\ f(-nT), & \text{if} \quad m = 0, \\ 0, & \text{if} \quad m \neq 0. \end{cases} \tag{24}$$

The sampling formula (21) follows from (23) and (24). ■

Using the same method from the theory of frames we can prove the following result [9, Theorem 3.3], which we "proved" by Poisson summation in Calculation 3 and which we had in mind in our comment about oversampling in Discussion 24a. The condition, $\hat{g} > 0$ on $(-\frac{1}{2T}, -\Omega] \cup [\Omega, \frac{1}{2T})$, can be relaxed, *e.g.*, [29].

Theorem 26. *Let $T, \Omega > 0$ be constants for which*

$$0 < 2T\Omega \leq 1,$$

and let $g \in \mathcal{S}(\mathbb{R})$ have the following properties:

$$\hat{g} = 1 \quad \textit{on} \quad [-\Omega, \Omega],$$

$$supp\,\hat{g} \subseteq [-\frac{1}{2T}, \frac{1}{2T}],$$

$$\hat{g} > 0 \quad \text{on} \quad (-\frac{1}{2T}, -\Omega] \cup [\Omega, \frac{1}{2T}).$$

Set

$$G(\gamma) = \sum |\hat{g}(\gamma - mb)|^2 \quad \text{and} \quad s(t) = T\Big(\frac{\hat{g}}{G}\Big)^{\vee}(t),$$

where $\Omega + \frac{1}{2T} \le b < \frac{1}{T}$. *Then* $0 < A \le G(\gamma) \le B < \infty$, $s \in \mathcal{S}(I\!R)$, $supp\, \hat{s} = supp\, \hat{g}$, $\hat{s} = T\frac{1}{G}$ *on* $[-\Omega, \Omega]$, *and*

$$\forall f \in PW_\Omega,\ f = \sum f(nT)\tau_{nT}s \quad in \quad L^2(I\!R).$$

($\mathcal{S}(I\!R)$ is the Schwartz space, e.g., [57], and the result is true for g and s having significantly less smoothness.)

Example 27. a. In Theorem 25, $\{e_{na}\tau_{mb}\hat{g}\}$ is a tight frame with frame bounds $A = B = 1$ in the case $2T\Omega = 1$, where $a = T$ and $b = 2\Omega$. From Theorem 14b, $\{e_{na}\tau_{mb}\hat{g}\}$ is an orthonormal basis if and only if $2T\Omega = 1$.

b. To construct functions $g \in \mathcal{S}(\mathbb{R})$ satisfying the hypotheses of Theorem 26, we proceed as follows for $2T\Omega < 1$.

We begin in the standard "distributional way" by defining

$$\psi_\epsilon(\gamma) = \frac{\phi(\epsilon - |\gamma|)}{\int \phi(\epsilon - |\gamma|)\, d\gamma},$$

where $\phi \in C^\infty(\hat{\mathbb{R}})$ vanishes on $(-\infty, 0]$ and equals $e^{-1/\gamma}$ on $[0, \infty)$. Thus, $\psi_\epsilon \in C_c^\infty(\hat{\mathbb{R}})$ is an even function satisfying the conditions, $supp\, \psi_\epsilon = [-\epsilon, \epsilon]$ and $\int \psi_\epsilon(\gamma)\, d\gamma = 1$. Next set

$$\psi_{U,V} = \frac{1}{|V|}\mathbf{1}_V * \mathbf{1}_{U-V}, \quad U, V \subseteq \hat{\mathbb{R}},$$

so that $\psi_{U,V}$ is 1 on U and vanishes off of $U + V - V$. The function g will be defined in terms of $\hat{g}$ as $\hat{g} = \psi_{U,V} * \psi_\epsilon$, where we shall now specify ϵ, U, and V given $2T\Omega < 1$. Let $U = [-u, u]$, where $u \in (\Omega, \frac{1}{2T})$ is arbitrary, and let $\epsilon = u - \Omega$. Choose $V = [-v, v]$ by setting $v = \frac{w-u}{2}$, where $w = \frac{1}{2T} + \epsilon$. These choices are necessitated by a simple geometrical argument, and the resulting function $\hat{g}$ satisfies the desired properties.

Definition/Discussion 28. *Aliasing.*

a. The proof of Theorem 25 required the support condition, $f \in PW_\Omega$, to verify both parts of (24). *When f is not Ω-bandlimited, so that aliasing occurs, phase information contributed by $m \neq 0$ in (23) is required in the frame decomposition of a signal*, cf. Definition/Discussion 6. To quantify this observation, we define the *aliasing pseudomeasure* $\alpha_{t,\Omega}$ on $\mathbb{R}$ (for the low pass filter $d_{2\pi\Omega}$) to be the distributional Fourier transform, $\alpha_{t,\Omega} = A_{t,\Omega}^{\vee}$, where each t is fixed and

$$A_{t,\Omega} \equiv \sum (e^{2\pi i (2m\Omega)t} - 1)\tau_{2m\Omega}\mathbf{1}_{(\Omega)} \in L^\infty(\hat{\mathbb{R}}).$$

The study of pseudomeasures is intimately related to spectral synthesis [1], cf. §9. In equation (25) of part b we shall see the relation between the aliasing pseudomeasure and the manner in which aliasing is manifested in our proof of Theorem 25.

b. Let $f \in L^2(\mathbb{R})$ and assume $2T\Omega = 1$. Then, formally,

$$\begin{aligned} f(t) = {} & T \sum f(nT)\tau_{nT} d_{2\pi\Omega}(t) \\ & + T \sum (f * \alpha_{t,\Omega})(nT)\tau_{nT} d_{2\pi\Omega}(t). \end{aligned} \tag{25}$$

To verify (25), we write (23) as a sum, $\sum_{m=0,n} + \sum_{m\neq 0,n}$, and obtain

$$\begin{aligned} f(t) = {} & 2T\Omega \sum_n \left(\frac{1}{(2\Omega)^{1/2}} \int \hat{f}(\gamma) \mathbf{1}_\Omega(\gamma) e^{-2\pi i n T\gamma} d\gamma \right) g(t+nT) \\ & + 2T\Omega \sum_{m\neq 0} \sum_n \left(\frac{1}{(2\Omega)^{1/2}} \int \hat{f}(\gamma) \mathbf{1}_\Omega(\gamma - 2m\Omega) e^{-2\pi i n T\gamma} d\gamma \right) \times \\ & \qquad e^{-2\pi i (2m\Omega)(t+nT)} g(t+nT), \end{aligned} \tag{26}$$

where $g = (2\Omega)^{-1/2} d_{2\pi\Omega}$. Equation (26) can be written as

$$\begin{aligned} & 2T\Omega \sum_n \frac{1}{(2\Omega)^{1/2}} \left(\int \hat{f}(\gamma) e^{-2\pi i n T\gamma} d\gamma - \int_{[-\Omega,\Omega]^\sim} \hat{f}(\gamma) e^{-2\pi i n T\gamma} d\gamma \right) g(t+nT) \\ & + 2T\Omega \sum_{m\neq 0} \sum_n \left(\frac{1}{(2\Omega)^{1/2}} \int \hat{f}(\gamma)\ \mathbf{1}_\Omega(\gamma - 2m\Omega) e^{-2\pi i n T\gamma} d\gamma \right) \times \\ & \qquad e^{2\pi i (2m\Omega)(t+nT)} g(t+nT), \end{aligned}$$

and so

$$\begin{aligned} f(t) = {} & 2T\Omega \sum f(nT) \frac{1}{(2\Omega)}\ d_{2\pi\Omega}(t-nT) \\ & + 2T\Omega \Bigg\{ \sum_{m\neq 0} \sum_n \left(\frac{1}{(2\Omega)^{1/2}} \int \hat{f}(\gamma) \mathbf{1}_\Omega(\gamma - 2m\Omega) e^{-2\pi i n T\gamma} d\gamma \right) \times \\ & \qquad e^{2\pi i (2m\Omega)(t+nT)} g(t+nT) \\ & - \sum_n \left(\frac{1}{(2\Omega)^{1/2}} \int_{[-\Omega,\Omega]^\sim} \hat{f}(\gamma) e^{-2\pi i n T\gamma} d\gamma \right) g(t+nT) \Bigg\}. \end{aligned} \tag{27}$$

The bracketed term in (27) can be written as

$$\Big\{ \cdots \Big\} = \frac{1}{(2\Omega)^{1/2}} \sum_n g(t+nT) \times$$

$$\Big(\sum_{m\neq 0} e^{2\pi i(2m\Omega)(t+nT)} \int \hat{f}(\gamma)\mathbf{1}_\Omega(\gamma - 2m\Omega)e^{-2\pi i n T\gamma} d\gamma$$

$$- \sum_{m\neq 0} \int \hat{f}(\gamma)\mathbf{1}_\Omega(\gamma - 2m\Omega)e^{-2\pi i n T\gamma} d\gamma\Big)$$

$$= \sum_n \frac{1}{(2\Omega)} \tau_{-nT} d_{2\pi\Omega}(t) \times$$

$$\Big(\sum_{m\neq 0}\Big(\int \hat{f}(\gamma)\mathbf{1}_\Omega(\gamma - 2m\Omega)e^{-2\pi i n T\gamma} d\gamma\Big) \times$$

$$\Big(e^{2\pi i(2m\Omega)(t+nT)} - 1\Big)\Big).$$

At this point we use our hypothesis that $2T\Omega = 1$ so that the bracketed term becomes

$$\Big\{\cdots\Big\} = T \sum_n \tau_{-nT} d_{2\pi\Omega}(t) \sum_{m\neq 0} (e^{2\pi i(2m\Omega)t} - 1) \times$$

$$\int \hat{f}(\gamma)\mathbf{1}_\Omega(\gamma - 2m\Omega)e^{-2\pi i n T\gamma} d\gamma$$

$$= T \sum_n \Big(\int \hat{f}(\gamma) e^{2\pi i n T\gamma}\Big(\sum_{m\neq 0}(e^{2\pi i(2m\Omega)t} - 1)\mathbf{1}_\Omega(\gamma - 2m\Omega)\Big) d\gamma\Big)\tau_{nT} d_{2\pi\Omega}(t)$$

$$= T \sum (f * \alpha_{t,\Omega})(nT)\tau_{nT} d_{2\pi\Omega}(t).$$

Combining this calculation with (27) yields (25).

In general, we must assume $f \in L^2(\mathbb{R}) \cap A(\mathbb{R})$ to expect the convergence of (25) in $L^2(\mathbb{R})$.

c. The *aliasing error* of $f \in L^2(\mathbb{R})$ for the low pass filter $d_{2\pi\Omega}$ with sampling at the Nyquist rate 2Ω is the second term on the right side of (25), viz.,

$$T \sum (f * \alpha_{t,\Omega})(nT)\tau_{nT} d_{2\pi\Omega}(t) \equiv \epsilon_f(t).$$

Formally, standard calculations give

$$\|\epsilon_f\|_\infty \leq 2 \int_{|\gamma|\geq\Omega} |\hat{f}(\gamma)| d\gamma.$$

§6. Irregular Sampling and Exact Frames

The theory of non-harmonic Fourier series was developed by Paley and Wiener [49, Chapters 6 and 7] and Levinson [44, Chapter 4]. Related work preceding

[49] is due to G. D. Birkhoff (1917), J. L. Walsh (1921), and Wiener (1927). The Paley-Wiener and Levinson theory has been reformulated and analyzed in terms of *irregular sampling* by Beutler [16], [17], who obtained completeness results, and Yao and Thomas [63], who obtained an irregular sampling theorem. The Yao and Thomas expansion was discovered independently by Higgins (1976) using reproducing kernels, *e.g.*, [30].

In this section we shall state and prove the Yao-Thomas irregular sampling expansion by frame methods. The coefficients in the expansion are the values of the given signal at the given irregularly spaced sampling points, and, because of our frame methods, the setting is necessarily in terms of exact frames, cf. §8. Whereas we implemented S^{-1} as a multiplier in §5, in this section we shall invoke the formula for S^{-1} contained in the following result, which is a consequence of Theorem 14.

Proposition 29. *Let H be a separable Hilbert space and let $\{g_n\} \subseteq H$ be an exact frame with inverse frame operator S^{-1}. Then*

$$\forall f \in H, \quad S^{-1}f = \sum \langle f, S^{-1}g_n \rangle S^{-1}g_n, \tag{28}$$

and so S^{-1} is the frame operator of the dual frame $\{S^{-1}g_n\}$.

In the case of irregular lattices, we have the following result [9, Theorem 4.1], which is the analogue of Theorem 19 for $\hat{\mathbb{R}}$. Although this result (Theorem 30) and Corollary 31 are concerned with frames and not exact frames, they are required in our proof of Theorem 32.

Theorem 30. *Let $g \in PW_\Omega$ for a given $\Omega > 0$. Assume that $\{a_n\}$, $\{b_m\}$ are real sequences for which*

$$\{e_{a_n}\} \quad \textit{is a frame for} \quad L^2[-\Omega, \Omega],$$

and that there exist $A, B > 0$ such that

$$0 < A \leq G(\gamma) \leq B < \infty \ \text{ a.e. on } \hat{\mathbb{R}},$$

where

$$G(\gamma) = \sum |\hat{g}(\gamma - b_m)|^2.$$

Then $\{e_{a_n}\tau_{b_m}\hat{g}\}$ is a frame for $L^2(\hat{\mathbb{R}})$ with frame operator S; and $\{e_{a_n}\tau_{b_m}\hat{g}\}$ is a tight frame for $L^2(\hat{\mathbb{R}})$ if and only if $\{e_{a_n}\}$ is a tight frame for $L^2[-\Omega, \Omega]$ and G is constant a.e. on $\hat{\mathbb{R}}$.

Corollary 31. *Let us assume the hypotheses and notation of Theorem 30, and set $I_m = [-\Omega, \Omega] + b_m$. Then, for each fixed m, $\{\tau_{b_m}e_{a_n}\}$ is a frame for $L^2(I_m)$ with frame operator S_m, and*

$$\forall h \in L^2(\mathbb{R}),\ S\hat{h} = \sum (\tau_{b_m}\hat{g}) S_m(\hat{h}\tau_{b_m}\bar{\hat{g}}) \quad \text{in} \quad L^2(\hat{\mathbb{R}}).$$

Proof: We compute

$$\begin{aligned} S\hat{h} &= \sum_m \sum_n \langle \hat{h}, e_{a_n}\tau_{b_m}\hat{g}\rangle e_{a_n}\tau_{b_m}\hat{g} \\ &= \sum_m (\tau_{b_m}\hat{g})(\sum_n \langle \hat{h}, e_{a_n}\tau_{b_m}\hat{g}\rangle e_{a_n})\mathbf{1}_{I_m} \\ &= \sum_m (\tau_{b_m}\hat{g})(\sum_n \langle \hat{h}\tau_{b_m}\overline{\hat{g}}, e_{a_n}\rangle_{Im} e_{a_n}\mathbf{1}_{Im}) \\ &= \sum_m (\tau_{b_m}\hat{g}) S_m(\hat{h}\tau_{b_m}\overline{\hat{g}}). \quad \blacksquare \end{aligned}$$

We can now prove the irregular sampling theorem for exact frames. Classical properties of the sampling functions $\{s_n\}$ in Theorem 32 are recorded in the remainder of the section.

Theorem 32. *Let $\{e_{a_n}\}$ be an exact frame for $L^2[-\Omega,\Omega]$ for a given $\Omega > 0$ and real sequence $\{a_n\}$. Define the sampling function s_n in terms of its involution $\tilde{s}_n(t) = \overline{s}_n(-t)$, where*

$$\forall t \in \mathbb{R}, \ \ \tilde{s}_n(t) = \int_{-\Omega}^{\Omega} \overline{h}_n(\gamma) e^{2\pi i t\gamma}\, d\gamma,$$

and where $\{h_n\} \subseteq L^2[-\Omega,\Omega]$ is the unique sequence for which $\{e_{a_n}\}$ and $\{h_n\}$ are biorthonormal. (In particular, $\tilde{s}_n \in PW_\Omega$.) If $t_n = -a_n$ then

$$\forall f \in PW_\Omega, \ \ f = \sum f(t_n) s_n \ \text{ in } \ L^2(\mathbb{R}).$$

Proof: [9, Proof of Theorem 5.2]. Let $g = (2\Omega)^{-1/2} d_{2\pi\Omega}$, and set $b_m = 2m\Omega$. Since $\{e_{a_n}\}$ is a frame we can apply Theorem 30, and, hence, $\{e_{a_n}\tau_{b_m}\hat{g}\}$ is a frame for $L^2(\hat{\mathbb{R}})$ with frame operator S. In particular,

$$\forall h \in L^2(\mathbb{R}), \ \ \hat{h} = \sum \langle \hat{h}, e_{a_n}\tau_{b_m}\hat{g}\rangle S^{-1}(e_{a_n}\tau_{b_m}\hat{g}) \ \text{ in } \ L^2(\hat{\mathbb{R}}). \qquad (29)$$

Similarly to (24), we obtain

$$\langle \hat{f}, e_{a_n}\tau_{b_m}\hat{g}\rangle = \begin{cases} (2\Omega)^{-1/2} f(-a_n), & \text{if} \quad m = 0, \\ 0, & \text{if} \quad m \neq 0 \end{cases} \qquad (30)$$

for $f \in PW_\Omega$.

By means of Corollary 31 we can then verify that

$$S^{-1} = 2\Omega S_0^{-1} \quad \text{on} \quad L^2[-\Omega,\Omega].$$

Thus, since $g \in PW_\Omega$, we compute

$$S^{-1}(e_{a_n}\tau_{b_0}\hat{g}) = (2\Omega)^{1/2}\ S_0^{-1}(e_{a_n}\mathbf{1}_{(\Omega)}),$$

so that, by the exactness hypothesis and Proposition 29, the right side is

$$(2\Omega)^{1/2} \sum_m \langle e_{a_n}, h_m \rangle_{[-\Omega,\Omega]}\ h_m = (2\Omega)^{1/2}\ h_n.$$

Combining these two equalities with (29) and (30) gives the reconstruction,

$$\forall f \in PW_\Omega,\ \hat{f} = \sum_n (2\Omega)^{-1/2} f(-a_n)(2\Omega)^{1/2} h_n \ \text{ in }\ L^2(\mathbb{R}),$$

and the result follows. ■

Definition/Discussion 33. a. For a given $\Omega > 0$, a real sequence $\{a_n\}$ satisfies the *Kadec-Levinson condition* if

$$\sup_n |a_n - \frac{n}{2\Omega}| < \frac{1}{4}(\frac{1}{2\Omega}). \tag{31}$$

b. Levinson [44, Theorem 18] (1940) proved that if (31) is assumed then $\{e_{a_n}\}$ is complete in $L^2[-\Omega,\Omega]$, and there is a unique sequence $\{h_n\}$ for which $\{e_{a_n}\}$ and $\{h_n\}$ are biorthonormal, cf. Theorem 14c. Kadec (1964) provided the direct calculation proving that $\{e_{a_n}\}$ is an exact frame for $L^2[-\Omega,\Omega]$, cf. Theorem 60 and [64, pages 42-44]. Levinson [44, Theorem 19, in particular, page 67] proved that the bound "$\frac{1}{4}$" in (31) is "best possible", *i.e.*, there exists $\{a_n\} \subseteq \mathbb{R}$ for which equality is obtained in (31) and $\{e_{a_n}\}$ is complete in $L^2[-\Omega,\Omega]$, but $\{e_{a_n}\}$ is not an exact frame for $L^2[-\Omega,\Omega]$.

c. There are exact frames of exponentials that do not satisfy the Kadec-Levinson condition, [24, page 362], [29]. On the other hand, it is not known whether there are bases of exponentials which are not exact frames for $L^2[-\Omega,\Omega]$, *e.g.*, [64, page 197].

The explicit formulas in the following result were proved in [49, pages 89-90 and pages 114-116] and [44, pages 48 ff.]. The calculations by Paley and Wiener were refined by Young (1979), *e.g.*, [64, pages 148-150].

Theorem 34. *Let the real sequence $\{a_n\}$ satisfy the Kadec-Levinson condition for a given $\Omega > 0$. Then $\{e_{a_n}\}$ is an exact frame for $L^2[-\Omega,\Omega]$, there is a unique sequence $\{h_n\} \subseteq L^2[-\Omega,\Omega]$ for which $\{e_{a_n}\}$ and $\{h_n\}$ are biorthonormal, and $\tilde{s}_n$, defined in Theorem 32, is of the form*

$$\tilde{s}_n(t) = \frac{\tilde{s}(t)}{\tilde{s}'(a_n)(t-a_n)} \tag{32}$$

where

$$\tilde{s}(t) = (t-a_0)\prod_{n=1}^{\infty}(1-\frac{t}{a_n})(1-\frac{t}{a_{-n}}).$$

Definition/Example 35. a. Let $\{L_n\}$ be a sequence of functions defined on $\mathbb{R}$ and let $\{t_n\}$ be a real sequence. $\{L_n, t_n\}$ is a *Lagrange interpolating system of functions L_n with respect to the sampling sequence $\{t_n\}$* if

$$\forall m, n \in \mathbb{Z}, \quad L_m(t_n) = \delta_{mn}.$$

b. Historically, *Lagrange's interpolation* is the method of defining a specific polynomial L of degree $\deg L \leq N$ such that L interpolates from $N+1$ given points $(t_j, f(t_j)) \in \mathbb{R} \times \mathbb{C}, j = 0, \ldots, N$. L is of the form

$$L(t) = \sum_{m=0}^{N} f(t_m) L_m(t),$$

where

$$L_m(t) = \frac{p(t)}{p'(t_m)(t - t_m)}$$

and

$$p(t) = \prod_{j=0}^{N} (t - t_j),$$

cf. the form of equation (32). Clearly,

$$\forall m, n = 0, \ldots, N, \quad L_m(t_n) = \delta_{mn},$$

since

$$p'(t_m) = \prod_{j \neq m} (t_m - t_j)$$

and

$$\frac{p(t)}{(t - t_m)} = \prod_{j \neq m} (t - t_j).$$

One goal of such interpolation is to provide polynomial approximation for a function f whose values are known at $\{t_j : j = 0, \ldots, N\}$. Such approximation is useful for a variety of reasons, *e.g.*, to estimate $\int f(t)\, dt$ even when we have complete knowledge of f. Unfortunately, the Lagrange polynomials L_m tend to oscillate wildly for large values of N.

To deal with this oscillation, the notion of spline and natural spline were defined on the interval $[t_0, t_N]$ for a given set of points $\{t_j : j = 0, \ldots, N\}$. A function $S : [t_0, t_N] \to \mathbb{C}$ is a *spline* of degree M if $S \in C^{M-1}[t_0, t_N]$ and S, restricted to each interval $[t_{j-1}, t_j]$, is a polynomial of degree $\deg S \leq M$. Because of issues dealing with degrees of freedom and symmetry with respect to endpoints, we deal with $M = 2K - 1$ and suppose

$$\forall j = K,\ K+1, \ldots, 2K-2, \quad S^{(j)}(t_0) = S^{(j)}(t_N) = 0.$$

If a spline satisfies this additional condition, it is a *natural spline*. A fundamental fact about natural splines is that for a given data set $\{(t_j, f(t_j)) : j = 0, \ldots, N\}$, there is a unique natural spline S which interpolates the data in the case $N - 1 \geq K$.

An important type of inequality, which we shall state for $M = 3$ and which addresses the oscillation problem of Lagrange polynomials, is the following. *Let S be the natural spline interpolant for a given data set $\{(t_j, f(t_j)) : j = 0, \ldots, N\}$. If $f : [t_0, t_N] \to \mathbf{C}$ is another interpolant of the data, f' is absolutely continuous, and $f^{(2)} \in L^2[t_0, t_N]$, then*

$$\int_{t_0}^{t_N} |S^{(2)}(t)|^2 \, dt \leq \int_{t_0}^{t_N} |f^{(2)}(t)|^2 \, dt.$$

The curvature of a function f at t is $f^{(2)}(t)/(1 + (f'(t))^2)^{3/2}$. Consequently, if f' is small then $f^{(2)}$ approximates the curvature; and so the natural spline interpolant can be viewed as the interpolant of minimum curvature in this case.

c. Let $\{e_{a_n}\}$ be an exact frame for $L^2[-\Omega, \Omega]$, and set $t_n = -a$. We know there is a unique sequence $\{h_n\}$ for which $\{e_{a_n}\}$ and $\{h_n\}$ are biorthonormal. If we define $\{s_n\}$ as we did in Theorem 32, then $\{s_n, t_n\}$ is a Lagrange interpolating system since

$$\begin{aligned} s_m(t_n) &= \overline{\check{s}}(-t_n) \\ &= \int_{-\Omega}^{\Omega} h_n(\gamma) e^{-2\pi i (-t_n)\gamma} d\gamma \\ &= \langle h_n(\gamma), e^{2\pi i a_n \gamma} \rangle. \end{aligned}$$

d. We shall not discuss Lagrange interpolation any further except to make the following observations:

i. Lagrange interpolating systems not only provide a backdrop for classical considerations with splines (part *b*), but also play a role in expanding on the obvious relations between multiresolution analysis in wavelet theory and the theory of splines, *e.g.*, [22], [46], [47, pages 24-25]. This is all the more interesting since we have used Lagrange interpolating systems in the context of exact Fourier frames associated with coherent states and the Heisenberg group, whereas wavelet theory is characterized by translations and dilations and the $ax + b$ group.

ii. Interpolation problems are a fundamental part of the theory of entire functions, *e.g.*, [19], cf. [63].

iii. Interpolation problems have a dual relationship with the ideas we shall sketch in §9, *e.g.*, [41].

iv. If, instead of PW_Ω, we consider the space of pseudo-measures supported by $[-\Omega, \Omega]$, then Beurling (1959-1960) has characterized the conditions in order that a discrete set $\{(t_j, f(t_j))\} \subseteq \mathbb{R} \times \mathbf{C}$ can be used to interpolate a Fourier transform f on $\mathbb{R}$ of such a pseudo-measure, *e.g.*, [12]; the conditions are in terms of uniformly discrete sequences (§7) and uniform Beurling density

(§9).

At the end of §2 we set our goal of obtaining the sampling formulas (14) and (15). In Theorem 32, accompanied by the well-known formulas of Theorem 34, we have (14), viz.,

$$f = \Sigma f(t_n) s_n.$$

Because of the setting of exact frames and biorthonormal sequences, (14) has a dual frame formulation,

$$f = \Sigma c_n(f) \tau_{t_n} d_{2\pi\Omega},$$

e.g., [29]. We shall now proceed to obtain (15) in §§7 and 8.

§7. The Duffin-Schaeffer Theorem and Frame Conditions

Definition 36. a. A sequence $\{a_n\} \subseteq \mathbb{R}$ is *uniformly discrete* if

$$\exists d > 0, \text{ such that } \forall m \neq n, \ |a_m - a_n| \geq d. \tag{33}$$

A uniformly discrete sequence $\{a_n\} \subseteq \mathbb{R}$ is *uniformly dense* with *uniform density* $\Delta > 0$ if

$$\exists \Delta > 0 \text{ and } \exists L > 0, \text{ such that } \forall n \in \mathbb{Z}, \ |a_n - \frac{n}{\Delta}| \leq L. \tag{34}$$

The description "uniform" is used since (34) has the form

$$\sup_{n \in \mathbb{Z}} |a_n - \frac{n}{\Delta}| < \infty.$$

b. Classically, an increasing sequence $\{a_n\} \subseteq \mathbb{R}$ has *natural density* $\Delta \geq 0$ if

$$\lim_{r \to \infty} \frac{n_0(r)}{r} = \Delta,$$

where

$$n_0(r) = \text{ card}\{a_n \in [-\frac{r}{2}, \frac{r}{2}]\}$$

("card" is cardinality). If a sequence fails to have a natural density it always has (finite or infinite) upper and lower natural density defined in terms of $\overline{\lim}$ and $\underline{\lim}$, resp. *e.g.*, [19, Section 1.5]. In §9 we shall deal with

$$n_I(r) = \text{ card}\{a_n \in I, |I| = r\},$$

where I is an interval, and with *upper* and *lower uniform Beurling* densities.

Example 37. a. Let $a_n = nT$, where $T > 0$ is fixed and $n \in \mathbb{Z}$. Then $nT \notin [-N, N]$ if and only if $nT > N$ or $nT < -N$. Thus, card$\{nT \in [-N, N]\}$

is essentially $2N/T$ and so $\{nT\}$ *has natural density* $1/T$. Clearly, *this example is uniformly dense with uniform density* $1/T$; in fact, $d = T$ in (33), and any $L \geq 0$ can be used in (34) for the case $\Delta = 1/T$, i.e., $\sup|a_n - \frac{n}{\Delta}| = 0$ for $\Delta = 1/T$. If $\{nT\}$ is a sampling sequence then the natural density $\Delta = 1/T$ is the number of samples per second, *e.g.*, §2, Definition 5b.

In this case, if $2T\Omega = 1$ then $\{e_{nT}\}$ is an orthonormal basis of $L^2[-\Omega, \Omega]$, cf. Theorem 38.

b. *Uniform density is more restrictive than natural density.* To see this, let $\{a_n\} \subseteq \mathbb{R}$ be an increasing sequence satisfying (34), and assume $lim_{n\to\infty} a_n = \infty$. This latter assumption is weaker than (33). Suppose $\{a_n\}$ satisfies the convenient, but non-essential, symmetric distribution, $a_{-n} = -a_n < 0$ for each $n \geq 1$. We shall prove that

$$\lim_{r\to\infty} \frac{n_0(r)}{r} = \Delta,$$

with $\Delta > 0$ as in (34).

Because of (34) we have

$$\left|\frac{a_n}{n} - \frac{1}{\Delta}\right| \leq \frac{1}{n},$$

and, hence,

$$\lim_{n\to\infty} \frac{n}{a_n} = \Delta.$$

By definition,

$$\forall n \geq 1, \quad n_0(2a_n) = 2n + 1,$$

and, so,

$$\lim_{n\to\infty} \frac{n_0(2a_n)}{2a_n} = \lim_{n\to\infty} \frac{2n+1}{2a_n} = \Delta$$

by the previous step and the assumption that $\lim_{n\to\infty} a_n = \infty$. Of course, we really want to consider $n_0(r)/r$ instead of $n_0(2a_n)/(2a_n)$. For a given a_n, $n \geq 1$, let a_p be the first element of the sequence for which $a_p > a_n$, i.e., $a_n = a_{n+1} = \ldots = a_{p-1} < a_p$. Then, for any $r \in (a_n, a_p)$,

$$n_0(2r) = 2(p-1) + 1,$$

and so

$$\frac{n_0(2r)}{2r} < \frac{2p-1}{2a_n} = \frac{2(p-1)+1}{2a_{p-1}}$$

since $1/r < 1/a_n$. As indicated above,

$$\lim_{p\to\infty} \frac{2(p-1)+1}{2a_{p-1}} = \Delta$$

and so

$$\overline{\lim_{r\to\infty}} \frac{n_0(r)}{r} \leq \Delta.$$

Similarly,

$$\underline{\lim_{r\to\infty}} \frac{n_0(r)}{r} \geq \Delta,$$

and we have shown that the natural density of $\{a_n\}$ is Δ.

c. If we consider the von Neumann lattice $\{(na, mb) : (n,m) \in \mathbb{Z} \times \mathbb{Z}\}$ for fixed $a, b > 0$, then the expected definition of its natural density is

$$\lim_{r\to\infty} \frac{\text{card}\{(na,mb) \in [-r,r] \times [-r,r]\}}{4r^2} \equiv \Delta.$$

In this case the cardinality in the numerator is essentially $(2r/a)(2r/b)$, and so

$$\Delta = \frac{1}{ab}.$$

Thus, we can phrase the Balian-Low phenomenon (Theorem 23) in terms of the natural density of the von-Neumann lattice. Further, if the natural density Δ of the von Neumann lattice is less than 1, then $\{e_{mb}\tau_{na}g\}$ is not even complete in $L^2(\mathbb{R})$, no matter which $g \in L^2(\mathbb{R})$ is chosen, *e.g.*, Remark 24b.

d. Suppose $\{a_n\} \subseteq \mathbb{R}$ is a sequence satisfying the Kadec-Levinson condition,

$$\sup_{n\in\mathbb{Z}} |a_n - \frac{n}{2\Omega}| < \frac{1}{4}\left(\frac{1}{2\Omega}\right),$$

defined in §6, Definition/Remark 33. We observe that $\{a_n\}$ is uniformly dense with uniform density 2Ω.

To see this we note that (34) is immediate for $\Delta = 2\Omega$ and $L = \frac{1}{4}\left(\frac{1}{2\Omega}\right)$. To verify (33), we compute

$$\begin{aligned}
|a_m - a_n| &= |a_m - \frac{m}{2\Omega} + \frac{m}{2\Omega} - \frac{n}{2\Omega} + \frac{n}{2\Omega} - a_n| \\
&\geq |\frac{m}{2\Omega} - \frac{n}{2\Omega}| - \left(|a_m - \frac{m}{2\Omega}| + |\frac{n}{2\Omega} - a_n|\right) \\
&\geq \frac{1}{2\Omega} - \frac{1}{2}\left(\frac{1}{2\Omega}\right) = \frac{1}{2}\left(\frac{1}{2\Omega}\right) \equiv d
\end{aligned}$$

for $m \neq n$.

Theorem 38. (Duffin and Schaeffer, [24, Theorem I]). *Let $\{a_n\} \subseteq \mathbb{R}$ be a uniformly dense sequence with uniform density Δ. If $0 < 2\Omega < \Delta$ then $\{e_{a_n}\}$ is a Fourier frame for $L^2[-\Omega, \Omega]$.*

After 40 years, Theorem 38 is still difficult to prove, and, among other notions and estimates, its proof involves fundamental properties of entire functions of exponential type.

We shall prove the following component of Theorem 38 since it is useful in our strengthening of this theorem, since it is associated with related work by Plancherel and Pólya (1937-1938) and Boas (1940), and since it is not too difficult to verify.

Lemma 39. *Let $\{e_{a_n}\}$ be a Fourier frame for $L^2[-\Omega,\Omega]$ for a given sequence $\{a_n\} \subseteq I\!R$ and a given $\Omega > 0$, and let $\{b_n\} \subseteq I\!R$ satisfy the condition,*

$$\sup_{n\in\mathbb{Z}} |a_n - b_n| \equiv M < \infty. \tag{35}$$

Then

$$\begin{aligned} &\exists C = C(\{a_n\},\Omega,M) \quad \text{such that} \\ \forall f \in PW_\Omega, \qquad &\sum |f(-b_n)|^2 \le C \sum |f(-a_n)|^2. \end{aligned} \tag{36}$$

Proof: a. We first note that

$$\forall f \in PW_\Omega \text{ and } \forall k \ge 0, \qquad \|f^{(k)}\|_2 \le (2\pi\Omega)^k \left(\int_{-\Omega}^{\Omega} |\hat{f}(\gamma)|^2 d\gamma \right)^{1/2}. \tag{37}$$

This is a consequence of Plancherel's theorem and the Ω-bandlimitedness of f, *i.e.*,

$$\|f^{(k)}\|_2^2 = \|(2\pi i\gamma)^k \hat{f}(\gamma)\|_2^2 \le (2\pi\Omega)^{2k} \int_{-\Omega}^{\Omega} |\hat{f}(\gamma)|^2 d\gamma.$$

b. By Taylor's theorem and the fact that $f \in PW_\Omega$ is entire, we have

$$f(-b_n) - f(-a_n) = \sum_{k=1}^{\infty} \frac{f^{(k)}(-a_n)}{k!}(-b_n + a_n)^k$$

for each n. We use Hölder's inequality to obtain the estimate,

$$|f(-b_n) - f(-a_n)|^2 \le \left(\sum_{k=1}^{\infty} \frac{|f^{(k)}(-a_n)|^2}{p^{2k}k!} \right) \left(\sum_{k=1}^{\infty} \frac{(|a_n - b_n|p)^{2k}}{k!} \right),$$

for any positive number p; and hence

$$|f(-b_n) - f(-a_n)|^2 \le (e^{(Mp)^2} - 1) \sum_{k=1}^{\infty} \frac{|f^{(k)}(-a_n)|^2}{p^{2k}k!}. \tag{38}$$

c. Because of (37) and the Fourier pairing $f^{(k)}(t) \leftrightarrow (2\pi i\gamma)^k \hat{f}(\gamma)$, we have $f^{(k)} \in PW_\Omega$ when $f \in PW_\Omega$. Thus, using (37) again and the (upper) frame hypothesis, we compute

$$\forall k \ge 1, \quad \sum_n |f^{(k)}(-a_n)|^2 \le B\|f^{(k)}\|_2^2 \le B(2\pi\Omega)^{2k}\|f\|_2^2, \tag{39}$$

where B is the upper frame bound of $\{e_{a_n}\}$. Consequently, combining (38), (39) and the lower frame estimate with lower frame bound A, we have

$$\begin{aligned}
\forall f \in PW_\Omega, \quad & \sum |f(-b_n) - f(-a_n)|^2 \\
& \leq (e^{(Mp)^2} - 1) \sum_{k=1}^{\infty} \left(\frac{2\pi\Omega}{p}\right)^{2k} \frac{B}{k!} \|f\|_2^2 \\
& \leq \frac{B}{A} \left(e^{(Mp)^2} - 1\right) \left(\sum_{k=1}^{\infty} \left(\frac{2\pi\Omega}{p}\right)^{2k} \frac{1}{k!}\right) \sum |f(-a_n)|^2 \\
& \equiv K^2 \sum |f(-a_n)|^2, \qquad (40)
\end{aligned}$$

which is finite for any fixed $p > 2\pi\Omega$.

d. Because of Minkowski's inequality, (40) allows us to write

$$\left(\sum |f(-b_n)|^2\right)^{1/2} \leq (1+K) \left(\sum |f(-a_n)|^2\right)^{1/2},$$

and (36) follows. ■

Theorem 40. *Let $\{a_n\} = A_1 \cup A_2 \subseteq \mathbb{R}$ be a disjoint union, where A_1 is a uniformly dense sequence $\{a_{1,n}\}$ with uniform density Δ and where $A_2 = \{a_{2,n}\}$ satisfies the condition,*

$$\sup_{n \in \mathbb{Z}} |a_{1,n} - a_{2,n}| \equiv M < \infty.$$

If $0 < 2\Omega < \Delta$ then $\{e_{a_n}\}$ is a Fourier frame for $L^2[-\Omega, \Omega]$.

Proof: Since A_1 is uniformly dense with uniform density Δ and $0 < 2\Omega < \Delta$, we can apply Theorem 38 to assert that $\{e_a : a \in A_1\}$ is a Fourier frame for $L^2[-\Omega, \Omega]$ with frame bounds A and B_1. Thus,

$$\begin{aligned}
\forall f \in PW_\Omega, \quad A\|f\|_2^2 & \leq \sum_{a \in A_1} |f(-a)|^2 \\
& \leq \sum |f(-a_n)|^2 \equiv \sum_{a \in A_1} |f(-a)|^2 + \sum_{a \in A_2} |f(-a)|^2 \\
& \leq (1+C) \sum_{a \in A_1} |f(-a)|^2 \leq (1+C) B_1 \|f\|_2^2, \qquad (41)
\end{aligned}$$

where the inequalities follow since $f(-a) = \langle \hat{f}, e_a \rangle$ for $a \in A_1$, by the inclusion $A_1 \subseteq A_1 \cup A_2$, because of Lemma 39, and by the upper frame condition, respectively.

Clearly,

$$\sum |\langle \hat{f}, e_{a_n}\rangle|^2 = \sum_{a\in A_1} |\langle \hat{f}, e_a\rangle|^2 + \sum_{a\in A_2} |\langle \hat{f}, e_a\rangle|^2;$$

and, hence, from (41) we see that $\{e_{a_n}\}$ is a Fourier frame for $L^2[-\Omega,\Omega]$ with frame bounds A and $B=(1+C)B_1$. ■

Example 41. In light of Theorem 40 and Theorem 44, and the usefulness of such results in applying Theorem 46 and some of the other material in §8, we shall now point out the generality of sampling sequences $\{t_n\}$ determined by the hypotheses of these theorems, letting $a_n = -t_n$.

Suppose $\{a_{1,n}\}$ has uniform density $\Delta = 1$, upper bound $L \in \mathbb{N}$, and lower uniformly discrete bound d. Let $\{a_n\} = \{a_{1,n}\} \cup \{a_{2,n}\}$, where $\{a_{2,n}\}$ satisfies the hypothesis of Theorem 40. There are two phenomena we wish to mention about such sequences.

First, *if d is small enough then $\{a_{1,n}\}$ can have the property that there is a subsequence of $\mathbb{Z}$ each of whose elements can have as many as $2L+1$ elements of $\{a_{1,n}\}$ close to it.* For example, let $L=3$ and $d=1/10$, and, for simplicity of exposition let $\{b_n\} \equiv \{a_{1,n}\}$ be symmetric. Set $b_1 = 1$, $b_2 = 5-3d$, $b_3 = 5-2d$, $b_4 = 5-d$, $b_5 = 5$, $b_6 = 5+d$, $b_7 = 5+2d$, $b_8 = 5+3d$. Hence, $2L+1 = 7$ elements of $\{b_n\}$ are within a distance of $3d = 3/10$ of $n=5$. In this case, of course, it is necessary that $b_9 \geq 6$.

Second, *for properly chosen $\{a_{2,n}\}$, $\{a_n\}$ can have the property that $\lim_{n\to\infty}(a_{n+1}-a_n) = 0$.*

The types of sequences used in the hypotheses of Theorems 40 and 44 are more closely related than they appear, but we shall not pursue that topic here, cf. Formula 45. The main point is that the sampling sequences that can be used in our theory of §8 are much more general that those allowed by the Kadec-Levinson condition and the classical results of §6. Of course, one does have to show a little respect for Theorem 32 *vis a vis* the frame decompositions of §8, since there are exact frames which do not satisfy the Kadec-Levinson condition.

We can prove an important refinement of Theorem 40, by invoking the following result from the Plancherel-Pólya theory [52], cf., [18], [19, pages 97–103].

Lemma 42. a. *For all $f \in PW_\Omega$,*

$$\int |f(t+iy)|^2 dt \leq e^{2\pi\Omega|y|}\|f\|_2^2.$$

b. Let $\{a_n\} \subseteq \mathbb{R}$ be a uniformly discrete sequence with lower bound d (as in (33)). Then

$$\forall f \in PW_\Omega, \quad \sum |f(-a_n)|^2 \leq B\|f\|_2^2, \tag{42}$$

where B can be taken as

$$B \equiv \frac{4}{\pi^2 \Omega d^2}\left(e^{\pi\Omega d}-1\right).$$

Proof: a. We shall omit the classical proof of part a which involves a Phragmén-Lindelöf argument, *e.g.*, [52], [64, pages 94–96]. Instead, in Remark 43 we shall hint at our verification of part a which is appropriate for our multidimensional work.

b. i. Let f be an element of PW_Ω. Then

$$\forall z_0 \in \mathbb{C} \text{ and } \forall r > 0, \qquad |f(z_0)|^2 \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(z_0 + re^{i\theta})|^2 d\theta, \tag{43}$$

i.e., the continuous function f has the property that $|f|^2$ is subharmonic. (Of course, we really know that f is an entire function.) The verification of (43) is standard. First, $\log|f|$ is upper semicontinuous, and one uses Jensen's formula to prove that $u = \log|f|$ satisfies an inequality analogous to (43). Then, since $\phi(t) = e^{2t}$ is an increasing convex function on $\mathbb{R}$, we obtain a similar inequality, *viz.*, (43) for $\phi \circ u$.

b. ii. We multiply both sides of (43) by r and integrate between 0 and $\frac{d}{2}$ to obtain

$$\begin{aligned}\forall z_0 \in \mathbb{C}, \quad |f(z_0)|^2 &\le \frac{4}{\pi d^2}\int_0^{\frac{d}{2}}\int_{-\pi}^{\pi} |f(z_0 + re^{i\theta})|^2 d\theta dr \\ &= \frac{4}{\pi d^2}\iint_{|z-z_0|\le\frac{d}{2}} |f(z)|^2 dtdy, \quad z = t + iy.\end{aligned}$$

Therefore, letting $t_n = -a_n$, we can make the estimate,

$$\begin{aligned}\sum |f(t_n)|^2 &\le \frac{4}{\pi d^2}\sum \iint_{|z|\le\frac{d}{2}} |f(z+t_n)|^2 dtdy \\ &\le \frac{4}{\pi d^2}\int_{-\frac{d}{2}}^{\frac{d}{2}} \sum \int_{-\frac{d}{2}}^{\frac{d}{2}} |f(t+t_n+iy)|^2 dtdy \\ &= \frac{4}{\pi d^2}\int_{-\frac{d}{2}}^{\frac{d}{2}} \sum \int_{t_n-\frac{d}{2}}^{t_n+\frac{d}{2}} |f(t+iy)| dtdy \\ &\le \frac{4}{\pi d^2}\int_{-\frac{d}{2}}^{\frac{d}{2}} \int |f(t+iy)| dtdy, \end{aligned} \tag{44}$$

where the last inequality follows by the definition of d.

b. iii. We apply part a to estimate the right side of (44), and obtain

$$\begin{aligned}\sum |f(-a_n)|^2 &\le \frac{4}{\pi d^2}\int_{-\frac{d}{2}}^{\frac{d}{2}} e^{2\pi\Omega|y|} dy \; \|f\|_2^2 \\ &= \frac{4}{\pi^2 \Omega d^2}\left(e^{\pi\Omega d}-1\right)\|f\|_2^2. \quad \blacksquare\end{aligned}$$

Discussion 43. *The Plancherel-Pólya theory.*

a. The following formal calculation for $f \in PW_\Omega$ can be made rigorous, not only to provide an alternative to the Phragmén-Lindelöf proof of Lemma 42a but to allow a means of dealing with analogous problems in d-dimensions [10]:

$$\begin{aligned}
&\left(\int |f(t+iy)|^2 dt\right)^{1/2} \\
&= \left(\int \left| \int_{-\Omega}^{\Omega} \hat{f}(\gamma) e^{2\pi i t\gamma} e^{-2\pi y\gamma} d\gamma \right|^2 dt\right)^{1/2} \\
&= \left(\int_{-\Omega}^{\Omega}\int_{-\Omega}^{\Omega} \hat{f}(\gamma)\overline{\hat{f}}(\lambda) e^{-2\pi y(\lambda+\gamma)} \left(\int e^{2\pi i t(\gamma-\lambda)} dt\right) d\lambda d\gamma\right)^{1/2} \\
&= \left(\int_{-\Omega}^{\Omega} \overline{\hat{f}}(\lambda) e^{-2\pi y\lambda} \int_{-\Omega}^{\Omega} \delta(\gamma-\lambda)\hat{f}(\gamma) e^{-2\pi y\gamma} d\gamma d\lambda\right)^{1/2} \\
&= \left(\int_{-\Omega}^{\Omega} (|\hat{f}(\lambda)| e^{-2\pi y\lambda})^2 d\lambda\right)^{1/2} \leq e^{2\pi|y|\Omega} \|f\|_2,
\end{aligned}$$

where the manipulations with δ and $\int e^{2\pi i y(\gamma-\lambda)} dt$ can be dealt with properly by approximate identity arguments.

b. Boas [18, Theorems 1,2,3] has provided a proof of Lemma 42 different from the argument of Plancherel-Póyla, cf. [19, pages 100-101] for another exposition of the Plancherel-Pólya proof. An attractive feature of Boas' approach in 1940 is that it leads us to rewrite (42) as a *weighted Fourier transform norm inequality*, viz.,

$$\|f\|_{2,\nu}^2 \leq B\|\hat{f}\|_{2,\mu}^2. \tag{45}$$

In the case of (42), ν is the discrete measure

$$\nu = \sum \delta_{-a_n},$$

μ is the absolutely continuous measure $\mu = \mathbf{1}_{(\Omega)}$ (usually more comfortably written as "$d\mu(\gamma) = \mathbf{1}_{(\Omega)}(\gamma)d\gamma$"), and PW_Ω is the weighted L^2-space, $L^2_\mu(\hat{\mathbb{R}})$. Weights such as $\mu = \mathbf{1}_{(\Omega)}$ play a role in the Bell Labs uncertainty principle theory from the early 1960s, *e.g.*, [42], and measure weights play a significant role in the general uncertainty principle theory, *e.g.*, [6] as well as [19, Chapter 10].

In the same spirit as Theorem 40, Theorem 38 and the ideas of Plancherel-Pólya play a major role in the following recent observation by Jaffard [33].

Theorem 44. *Let $\{a_n\} \subseteq \mathbb{R}$ be the union of a finite number of uniformly discrete sequences and a uniformly dense sequence having uniform density Δ. If $0 < 2\Omega < \Delta$ then $\{e_{a_n}\}$ is a Fourier frame for $L^2[-\Omega, \Omega]$.*

Proof: Let $\{a_n\} = A_1 \cup \ldots \cup A_n$, where A_1 has uniform density Δ and each of the A_j is uniformly discrete. Take any $\Omega > 0$ for which $0 < 2\Omega < \Delta$. Then, by Theorem 38, $\{e_a : a \in A_1\}$ is a Fourier frame for $L^2[-\Omega, \Omega]$. Thus,

$$\forall f \in PW_\Omega, \qquad A\|f\|_2^2 \leq \sum_{a\in A_1} |\langle \hat{f}, e_a\rangle|^2$$
$$\leq \sum |\langle \hat{f}, e_{a_n}\rangle|^2 = \sum |f(-a_n)|^2. \tag{46}$$

For the upper frame bound, we invoke Lemma 42 and obtain

$$\forall j = 1, \ldots, m \quad \text{and} \quad \forall f \in PW_\Omega,$$
$$\sum_{a\in A_j} |\langle \hat{f}, e_a\rangle|^2 \leq B_j\|f\|_2^2.$$

Consequently,

$$\sum |f(-a_n)|^2 = \sum_{j=1}^{n} \sum_{a\in A_j} |\langle \hat{f}, e_a\rangle|^2 \leq B\|f\|_2^2, \tag{47}$$

where $B = B_1 + \ldots + B_n$. Combining (46) and (47), we obtain our result. ■

Formula 45. *The frame radius.*

We have just used the Duffin-Schaeffer theorem (Theorem 38) to prove Theorem 44. Theorem 44 provides a density/discreteness condition on a sequence $\{a_n\} \subseteq \mathbb{R}$ to ensure that $\{e_{a_n}\}$ is a Fourier frame; and it turns out that this condition on $\{a_n\} \subseteq \mathbb{R}$ is not only sufficient but necessary in order that $\{e_{a_n}\}$ be a Fourier frame [33, Lemma 2], cf. Theorem 40.

There is the following closely related problem for a given sequence $\{a_n\} \subseteq \mathbb{R}$: determine

$$\Omega_r \equiv \sup\{\Omega \geq 0 : \{e_{a_n}\} \text{ is a Fourier frame for } L^2[-\Omega, \Omega]\}.$$

The quantity $\Omega_r \in [0, \infty]$ is the *frame radius* of $\{a_n\}$, and the problem has recently been resolved by Jaffard [33, Theorem 3], and requires the Duffin-Schaeffer theorem (Theorem 38), as well as an important result by Landau [41], cf. §9.

Jaffard's solution includes the following formula for the frame radius in the case $\Omega_r \in (0, \infty)$:

$$\Omega_r = \frac{1}{2} \sup_b \Delta(b), \tag{48}$$

where b is a uniformly dense subsequence of $\{a_n\}$ having uniform density $\Delta(b)$.

The proof of (48) uses Theorem 38 to prove the inequality,

$$\Omega_r \geq \frac{1}{2} \sup_b \Delta(b). \tag{49}$$

To verify (49) let us assume for simplicity that $\{a_n\}$ is uniformly dense with uniform density Δ. If $\Omega_r < \frac{1}{2}\Delta$ we can choose $\Omega \in (\Omega_r, \frac{1}{2}\Delta)$ and contradict the definition of Ω_r by applying Theorem 38; therefore $\Omega_r \geq \frac{1}{2}\Delta$. Details to prove the general case, (49), are found in [33] and are not difficult. The opposite inequality not only requires Landau's theorem, mentioned above, but also a more formidable contribution by Jaffard.

§8. Irregular Sampling and Frames

The results of §7 will be used in conjunction with Theorem 46 to formulate sampling formulas such as Equation (50) in terms of given sequences $\{t_n\}$ of sampling points.

Theorem 46 [9]. *Suppose $\Omega > 0$ and $\Omega_1 > \Omega$, and let the sequence $\{t_n\} \subseteq \mathbb{R}$ have the property that $\{e_{-t_n}\}$ is a Fourier frame for $L^2[-\Omega_1, \Omega_1]$ with frame operator S. Further, let $s \in L^2(\mathbb{R})$ have the properties that $\hat{s} \in L^\infty(\hat{\mathbb{R}})$, supp $\hat{s} \subseteq [-\Omega_1, \Omega_1]$, and $\hat{s} = 1$ on $[-\Omega, \Omega]$. Then*

$$\forall f \in PW_\Omega, \ \ f = \sum c_n(f) \tau_{t_n} s \ \text{ in } \ L^2(\mathbb{R}) \tag{50}$$

where

$$c_n(f) = \left\langle S^{-1}(\hat{f}\mathbf{1}_{(\Omega_1)}), e_{-t_n} \right\rangle_{[-\Omega_1, \Omega_1]}.$$

Proof: Since $\{e_{-t_n}\}$ is a frame for $L^2[-\Omega_1, \Omega_1]$ and $supp\, \hat{f} \subseteq [-\Omega, \Omega]$ we have

$$\hat{f} = \hat{f}\mathbf{1}_{(\Omega_1)} = \sum \left\langle S^{-1}(\hat{f}\mathbf{1}_{(\Omega_1)}), e_{-t_n} \right\rangle_{[-\Omega_1, \Omega_1]} e_{t_n} \ \text{ in } \ L^2[-\Omega_1, \Omega_1]. \tag{51}$$

Equation (51) is a direct consequence of Theorem 14a and the fact that S^{-1}, being a positive operator, is self-adjoint. Using the hypothesis, $f \in PW_\Omega$, we can rewrite (51) as

$$\hat{f} = \sum c_n(f)(e_{-t_n}\mathbf{1}_{(\Omega_1)}) \ \text{ in } \ L^2(\hat{\mathbb{R}}). \tag{52}$$

In fact,

$$\begin{aligned} &\|\hat{f} - \sum_M^N c_n(f)(e_{-t_n}\mathbf{1}_{(\Omega_1)})\|_2^2 \\ &\qquad = \int_{-\Omega_1}^{\Omega_1} |\hat{f}(\gamma) - \sum_M^N c_n(f) e_{-t_n}(\gamma)|^2 d\gamma \\ &\qquad = \|\hat{f} - \sum_M^N c_n(f) e_{-t_n}\|^2_{L^2[-\Omega_1, \Omega_1]}, \end{aligned}$$

and so (52) is valid.

Next, we note that $\hat{f} = \hat{f}\hat{s}$, and, hence,

$$\begin{aligned}\|\hat{f} - \sum_M^N c_n(f)(e_{-t_n}\hat{s})\|_2^2 \\ &= \|\hat{f}\hat{s} - \sum_M^N c_n(f)(e_{-t_n}\mathbf{1}_{(\Omega_1)})\hat{s})\|_2^2 \\ &\leq \|\hat{s}\|_\infty^2 \; \|\hat{f} - \sum_M^N c_n(f)(e_{-t_n}\mathbf{1}_{(\Omega_1)})\|_2^2.\end{aligned}$$

Using this estimate, Equation (52), and the hypotheses on s, we obtain

$$\hat{f} = \sum c_n(f)(e_{-t_n}\hat{s}) \text{ in } L^2(\hat{\mathbb{R}}). \tag{53}$$

Finally, we obtain Equation (50) from Equation (53) and the Plancherel theorem. ■

In the following result, an operation must be performed on a given sequence to construct a subsequence $\{t_n\}$ with which we can implement Theorem 46. As such, it doesn't quite fit into the mold of §7, so we state it now as a lemma to be used in the proof of Theorem 48.

Lemma 47 [29]. *Let $\{a_k\} \subseteq \mathbb{R}$ be a strictly increasing sequence for which*

$$\lim_{k\to\pm\infty} a_k = \pm\infty$$

and

$$\sup_k(a_{k+1} - a_k) \equiv T < \infty.$$

Assume $\Omega_1 > 0$ satisfies the condition

$$2T\Omega_1 < 1.$$

There is a constructible subsequence $\{a_{k_n}\}$ of $\{a_k\}$ such that, setting $-t_n = a_{k_n}$, $\{e_{-t_n}\}$ is a frame for $L^2[-\Omega_1, \Omega_1]$.

Proof: Since $2T\Omega_1 < 1$ we can choose $\epsilon > 0$ so that

$$T < \frac{1}{2\Omega_1 + \epsilon}.$$

Next, choose $\delta > 0$ such that

$$T + \delta < \frac{1}{2\Omega_1 + \epsilon},$$

and define intervals I_n as

$$\forall n \geq 1, \quad I_n = \left[\frac{n}{2\Omega_1+\epsilon} - \frac{1}{2}\Big(\frac{1}{2\Omega_1+\epsilon} - \delta\Big), \right.$$
$$\left. \frac{n}{2\Omega_1+\epsilon} + \frac{1}{2}\Big(\frac{1}{2\Omega_1+\epsilon} - \delta\Big)\right].$$

Note that $\{I_n\}$ is a disjoint collection since

$$\Big(\frac{n+1}{2\Omega_1+\epsilon} - \frac{1}{2}\Big(\frac{1}{2\Omega_1+\epsilon} - \delta\Big)\Big)$$
$$- \Big(\frac{n}{2\Omega_1+\epsilon} + \frac{1}{2}\Big(\frac{1}{2\Omega_1+\epsilon} - \delta\Big)\Big) = \delta > 0. \tag{54}$$

The length of each interval I_n is

$$|I_n| = \frac{1}{2\Omega_1+\epsilon} - \delta > T.$$

Therefore, by the definition of T, each I_n must contain elements of $\{a_k\}$. For each n, choose precisely one such element a_{k_n}, e.g., the smallest or largest element of $\{a_k\} \cap I_n$, and designate it as $-t_n = a_{k_n}$.

Writing $a_n \equiv -t_n$, (54) implies that

$$\forall n, \quad (a_{n+1} - a_n) \geq \delta > 0.$$

Further,

$$|a_n - \frac{n}{2\Omega_1+\epsilon}| \leq \frac{1}{2}\Big(\frac{1}{2\Omega_1+\epsilon} - \delta\Big) \equiv L,$$

by the definition of I_n. Consequently, $\{a_k\}$ is a uniformly dense sequence with uniform density $\Delta \equiv 2\Omega_1 + \epsilon > 2\Omega_1$. By the Duffin-Schaeffer theorem, Theorem 38, $\{e_{-t_n}\}$ is a frame for $L^2[-\Omega_1, \Omega_1]$. ■

In general, Lemma 47 is false when $2T\Omega_1 = 1$, e.g., [29].

Combining Theorem 46 with Lemma 47 we obtain —

Theorem 48. *Let $\{a_k\} \subseteq I\!R$ be a strictly increasing sequence for which*

$$\lim_{k\to\pm\infty} a_k = \pm\infty$$

and

$$\sup_k (a_{k+1} - a_k) \equiv T < \infty.$$

For a given $\Omega > 0$, assume $\Omega_1 > \Omega$ satisfies the condition,

$$2T\Omega_1 < 1,$$

and let $s \in L^2(I\!R)$ have the properties that $\hat{s} \in L^\infty(\hat{I\!R})$, $supp\, \hat{s} \subseteq [-\Omega_1, \Omega_1]$, and $\hat{s} = 1$ on $[-\Omega, \Omega]$. There is a constructible subsequence $\{a_{k_n}\}$ of $\{a_k\}$ such that, setting $-t_n = a_{k_n}, \{e_{-t_n}\}$ is a frame for $L^2[-\Omega_1, \Omega_1]$ with frame operator S, and

$$\forall f \in PW_\Omega, \quad f = \sum \left\langle S^{-1}(\hat{f}\mathbf{1}_{(\Omega_1)}), e_{-t_n} \right\rangle_{[-\Omega_1, \Omega_1]} \tau_{t_n} s \quad in \quad L^2(I\!R).$$

Naturally, the implementation of Theorem 46 depends on the computation of the coefficients in Equation (50). For the case of exact frames these coefficients are computed in the following result in terms of the Lagrange interpolating system introduced in Theorem 32. We shall not now pursue the expansions resulting from a combination of Theorem 46 and Theorem 49, and we refer to [9, Theorem 6.3] for a proof of Theorem 49.

Theorem 49 [9]. *Suppose $\Omega > 0$ and $\Omega_1 > \Omega$, and let the sequence $\{t_n\} \subseteq I\!R$ have the property that $\{e_{a_n}\}$ is an exact frame for $L^2[-\Omega_1, \Omega_1]$ with frame operator S, where $a_n = -t_n$. Define the function s_n in terms of its involution $\tilde{s}_n(t) = \overline{s}_n(-t)$, where*

$$\forall t \in I\!R, \quad \tilde{s}_n(t) = \int_{-\Omega_1}^{\Omega_1} \overline{h}_n(\gamma) e^{2\pi i t\gamma} d\gamma,$$

and where $\{h_n\} \subseteq L^2[-\Omega_1, \Omega_1]$ is the unique sequence for which $\{e_{a_n}\}$ and $\{h_n\}$ are biorthonormal. The coefficients of the expansion in Equation (50) are

$$\forall n, \quad \left\langle S^{-1}(\hat{f}\mathbf{1}_{(\Omega_1)}), e_{-t_n} \right\rangle_{[-\Omega_1, \Omega_1]} = \langle f, s_n \rangle, \tag{55}$$

where $f \in PW_\Omega$.

Of course the real purpose of Theorem 46 is to provide sampling formulas with effectively computed coefficients, where the sampling sequences are not constrained by exactness. This is the role of the following algorithm which is expanded upon in [10].

Algorithm 50 [9]. It is possible to estimate the coefficients in Equation (50), and, in the process, to see to what extent the coefficients contain information from the sampled values $f(t_n)$.

a. Let $\{e_{-t_n}\}$ be a frame for $L^2[-\Omega_1, \Omega_1]$ with frame operator S and frame bounds A and B. Recall that if S_1 and S_2 are operators, then $S_1 \geq S_2$ means that

$$\forall f, \quad \langle S_1 f, f \rangle \geq \langle S_2 f, f \rangle.$$

Thus, by our frame hypothesis, we have

$$AI \leq S \leq BI,$$

where I is the identity operator. Consequently, we can compute

$$I - \frac{2}{A+B}S \leq I - \frac{2}{A+B}AI = \Big(\frac{A+B-2A}{A+B}\Big)I$$

and

$$I - \frac{2}{A+B}S \geq I - \frac{2}{A+B}BI = \Big(\frac{A+B-2B}{A+B}\Big)I;$$

and, hence,

$$\|I - \frac{2}{A+B}S\| \leq \frac{B-A}{A+B} < 1, \tag{56}$$

where the norm in (56) is the operator norm.

b. Because of (56) and the Neumann expansion we have

$$S^{-1} = \frac{2}{A+B}\sum_{k=0}^{\infty}\Big(I - \frac{2}{A+B}S\Big)^k, \tag{57}$$

where the convergence criterion in (57) is the operator norm topology on the space of continuous linear operators on $L^2[-\Omega_1, \Omega_1]$ (into itself). In fact, (56) and the Neumann expansion really yield

$$\Big(\frac{2}{A+B}S\Big)^{-1} = \sum_{k=0}^{\infty}\Big(I - \frac{2}{A+B}S\Big)^k; \tag{58}$$

and

$$\Big(\frac{2}{A+B}S\Big)^{-1}(g) = f$$

means

$$\frac{A+B}{2}g = Sf,$$

so that since S^{-1} is linear we have

$$\frac{A+B}{2}S^{-1}g = f. \tag{59}$$

Equations (58) and (59) give Equation (57).

c. Substituting (57) into (50), we have

$$c_n(f) = \sum_{k=0}^{\infty}\frac{2}{A+B}\Big\langle\Big(I - \frac{2}{A+B}S\Big)^k\Big(\hat{f}\mathbf{1}_{(\Omega_1)}\Big), e_{-t_n}\Big\rangle_{[-\Omega_1,\Omega_1]} \tag{60}$$

for $f \in PW_\Omega$, $\Omega < \Omega_1$.

We can easily calculate the terms of Equation (60), *e.g.*,

$$\frac{2}{A+B}\Big\langle\hat{f}\mathbf{1}_{(\Omega_1)}, e_{-t_n}\Big\rangle_{[-\Omega_1,\Omega_1]} = \frac{2}{A+B}\; f(t_n)$$

and

$$\frac{2}{A+B}\left\langle \left(I - \frac{2}{A+B}S\right)\left(\hat{f}\mathbf{1}_{(\Omega_1)}\right), e_{-t_n} \right\rangle_{[-\Omega_1,\Omega_1]}$$
$$= \frac{2}{A+B}\, f(t_n) - \left(\frac{2}{A+B}\right)^2 \left[2\Omega_1 f(t_n) + \sum_{n_1 \neq n} f(t_{n_1}) d_{2\pi\Omega_1}(t_n - t_{n_1})\right]$$

are the cases $k = 0$ and $k = 1$, respectively. Thus, if we truncate the expansion (60) after the $k = 0$ term, then

$$c_n(f) \approx \frac{2}{A+B}\, f(t_n),$$

which is a particularly auspicious estimate when $A = B = 1$.

Motivated by Theorems 46 and 48, and noting the importance of the frame bounds in Algorithm 50 we have —

Theorem 51. *Suppose $\Omega > 0$, and let the sequence $\{a_n\} \subseteq \mathbb{R}$ have the properties that $\{a_n\}$ is strictly increasing, $\lim_{n\to\pm\infty} a_n = \pm\infty$, and*

$$0 < d \leq \inf(a_{n+1} - a_n) \leq \sup(a_{n+1} - a_n) = T < \infty.$$

Assume $2T\Omega < 1$. Then $\{e_{a_n}\}$ is a Fourier frame for $L^2[-\Omega,\Omega]$ with frame bounds A and B satisfying the inequalities

$$A \geq \frac{(1 - 2T\Omega)^2}{T}$$

and

$$B \leq \frac{4}{\pi^2 \Omega d^2}(e^{\pi\Omega d} - 1).$$

Proof: The upper bound for B is a consequence of Lemma 42b.

The lower bound for A depends on the following result by Gröchenig [27, Theorem 1]. *Let $K : PW_\Omega \to PW_\Omega$ be defined as*

$$Kf = d_{2\pi\Omega} * \sum f(t_n)\mathbf{1}_n,$$

where $t_n = -a_n$ and $\mathbf{1}_n$ is the characteristic function of the interval

$$[(a_{n+1} + a_n)/2, (a_{n+2} + a_{n+1})/2).$$

Then

$$\|Kf\|_2 \leq (1 + 2T\Omega)\|f\|_2 \quad \text{and} \quad \|f - Kf\|_2 \leq 2T\Omega\|f\|_2.$$

Consequently, $\|K^{-1}\| \leq 1/(1-2T\Omega)$, so that since $f = K^{-1}Kf$ we have

$$\|f\|_2 \leq \frac{1}{1-2T\Omega}\|\sum f(t_n)\mathbf{1}_n\|_2$$
$$\leq \frac{\sqrt{T}}{1-2T\Omega}\Big(\sum |f(t_n)|^2\Big)^{1/2},$$

for all $f \in PW_\Omega$, e.g., [29, Corollary 4.4.4]. ■

Remark 52. Our "major frame" sampling formula in this section did not deal with frames for $L^2(\mathbb{R})$, but with Fourier frames for $L^2[-\Omega_1, \Omega_1]$. This provides a gain in simplicity and an opportunity to implement the results from §7. On the other hand, we lose the flexibility to consider aliasing as we did in §5. Further, we can prove Theorem 46 in terms of Gabor frames for $L^2(\mathbb{R})$. This will allow us to deal with aliasing, and to obtain better implementation of Algorithm 50, *e.g.*, [10].

The following examples are concerned with Gabor frames for $L^2(\mathbb{R})$ and $L^2(\hat{\mathbb{R}})$. Details are found in [9, §4], and the material plays a role in [10].

Example 53. *Let* $\{a_n\}$, $\{b_m\}$ *be real sequences and let* $\Omega > 0$. *Assume that* $\{e_{a_n}\}$ *is a Fourier frame for* $L^2[-\Omega, \Omega]$, *and that there exist* $d, \Delta > 0$ *such that*

$$\forall m \in \mathbb{Z}, \qquad 0 < d \leq b_{m+1} - b_m \leq \Delta < 2\Omega,$$

where $\lim_{m\to\pm\infty} b_m \pm \infty$. (In particular, $\{b_m\}$ is uniformly discrete.) *Suppose* $g \in PW_\Omega$ *has the properties that* $\hat{g} \in L^\infty(\hat{\mathbb{R}})$ *and* $\inf\{|\hat{g}(\gamma)|^2 : \gamma \in I\} > 0$ *for some interval* $I \subseteq [-\Omega, \Omega]$ *having length* $|I| = \Delta$. *Then* $\{e_{a_n}\tau_{b_m}\hat{g}\}$ *is a frame for* $L^2(\hat{\mathbb{R}})$, cf. Theorem 51. This result follows once we verify the hypotheses of Theorem 30.

Example 54. Suppose we assume the hypotheses of Theorem 30, and that we consider the case $a_n = na$ and $a = \frac{1}{2\Omega}$.

a. Let S be the frame operator for the frame $\{e_{a_n}\tau_{b_m}\hat{g}\}$ (for $L^2(\hat{\mathbb{R}})$) obtained by Theorem 30. A routine calculation allows us to conclude that

$$\forall f \in L^2(\mathbb{R}), \qquad S^{-1}\hat{f} = \frac{1}{2\Omega}\frac{\hat{f}}{G},$$

where $G(\gamma) = \sum |\hat{g}(\gamma - b_m)|^2$.

b. Part a allows us to write

$$S^{-1}(e_{na}\tau_{b_m}\hat{g}) = \frac{1}{2\Omega}\frac{e_{na}\tau_{b_m}\hat{g}}{G},$$

whereas

$$e_{na}\tau_{b_m}S^{-1}\hat{g} = \frac{1}{2\Omega}\frac{e_{na}\tau_{b_m}\hat{g}}{\tau_{b_m}G}.$$

Consequently, *the operators* S^{-1} *and* $e_{na}\tau_{b_m}$ *are not commutative for irregular sequences.* For perspective, recall that our frame approach to the classical sampling theorem utilized the commutativity of these operators when we invoked Theorem 20 in the proof of Theorem 25.

§9. Stability and Uniqueness

For a given sequence $\{a_n\} \subseteq \mathbb{R}$, a major part of the sampling theory we have presented has depended on whether $\{e_{a_n}\}$ is a frame or exact frame for $L^2[-\Omega, \Omega]$. A more basic problem is to determine whether $\{e_{a_n}\}$ is *complete* in $L^2[-\Omega, \Omega]$, *i.e.*, whether the linear span of $\{e_{a_n}\}$ is dense in $L^2[-\Omega, \Omega]$. The ultimate contribution to this problem is due to Beurling and Malliavin [14], [15], cf. [36], [54] for superb expositions of this profound material. Our remarks, although elementary, are geared to this work of Beurling and Malliavin, as well as other important contributions by Beurling [12], [13, Volume 2, pages 341–350] and Landau [40], [41].

Definition 55. Let $\{t_n\}$ be a real sequence, set $a_n = -t_n$ for each n, and let $\Omega > 0$.

a. $\{t_n\}$ is a *uniqueness sequence for* PW_Ω if for all $f \in PW_\Omega$ for which $f(t_n) = 0$ for each n, we can conclude that f is the 0-function.

b. $\{t_n\}$ is an *energy stable sequence for* PW_Ω if there is a constant $A > 0$ such that

$$\forall f \in PW_\Omega, \quad A\|f\|_2^2 \leq \sum |f(t_n)|^2. \tag{61}$$

Thus, for energy stable sequences, small errors in the samples cause a small error in estimating the energy of the original signal.

Theorem 56. *Let $\{t_n\}$ be a real sequence, set $a_n = -t_n$ for each n, and let $\Omega > 0$.*

a. If $\{e_{a_n}\}$ is a Fourier frame for PW_Ω then $\{t_n\}$ is an energy stable sequence for PW_Ω; and if $\{t_n\}$ is an energy stable sequence for PW_Ω then it is a uniqueness sequence for PW_Ω.

b. $\{t_n\}$ is a uniqueness sequence if and only if $\{e_{a_n}\}$ is complete in $L^2[-\Omega, \Omega]$.

c. If $\{t_n\}$ is a uniformly discrete and energy stable sequence for PW_Ω then $\{e_{a_n}\}$ is a Fourier frame for PW_Ω. In general, the converse is false.

Proof: a. The first part follows from the lower frame bound inequality. The second part follows since the hypothesis implies $\|f\|_2 = 0$ when $f \in PW_\Omega$ vanishes on $\{t_n\}$, and since $f = 0$ on $\mathbb{R}$ when $\|f\|_2 = 0$.

b. Suppose $\{t_n\}$ is a uniqueness sequence and $\hat{f}$ annihilates $\{e_{a_n}\}$, where $f \in PW_\Omega$. By the Fourier inversion formula, f vanishes on $\{t_n\}$; and so, by uniqueness, we have that $f = 0$ on $\mathbb{R}$. We conclude that $\{e_{a_n}\}$ is complete by the Hahn-Banach theorem.

For the converse, take any $f \in PW_\Omega$ which vanishes on $\{t_n\}$. Then, by the Fourier inversion formula, $\hat{f}$ annihilates $\{e_{a_n}\}$; and so, by completeness, we have that $\hat{f} = 0$ on $[-\Omega, \Omega]$. We conclude that $f = 0$ on $\mathbb{R}$, and, consequently, $\{t_n\}$ is a uniqueness sequence for PW_Ω.

c. The first part follows from Lemma 42b of §7 and the definition of an energy stable sequence for PW_Ω. The second part is a consequence of Example

41 and Theorem 44, *i.e.*, there are Fourier frames $\{e_{a_n}\}$, where $\{a_n\}$ is not uniformly discrete. ■

Discussion 57. *Stability.*

a. The classical sampling formula tells us that each $f \in PW_\Omega$ is uniquely determined by its values on $\{nT\}$, where $2T\Omega = 1$. Suppose $\Omega_1 > \Omega$ and $\{nT_1\}$ is the sampling sequence for $f \in PW_\Omega$, where $2T_1\Omega_1 = 1$. Then $2T_1\Omega < 1$, and knowledge of $\{f(nT_1)\}$ reflects *oversampling*, which is essential information for many applications, *e.g.*, [62]. (*Undersampling* gives rise to numerical *instability* and aliasing, *e.g.*, §2, Definition/Remark 6 and §11.) On the other hand, if the values of $f \in PW_\Omega$ are known on $\{nT_1 : n < 0\}$ then the Pólya-Carlson theorem uniquely determines f on all of $\mathbb{R}$, cf. [3], [7], [11], [25], [54] for discussions of this type of uniqueness theorem and prediction theory. At first glance, this result would seem to imply that knowledge of an oversampled speech signal $f \in PW_\Omega$ allows us to predict the speaker's words in the future. As Pollak [53, pages 74–75] observed, the weakness in this argument is due to the effect of instability. In fact, it can be proved that an arbitrarily small error in $f(-T_1)$ can produce an arbitrarily large error in $f(t)$, for arbitrarily small $t > 0$, cf. [19, Chapter 9.2].

b. Let $\{t_n\}$ be a real sequence. Suppose Y is a metric space of continuous functions on $\mathbb{R}$, and let X be a metric space of sequences $\{f(t_n) : f \in Y\}$; we denote the corresponding metrics by ρ_Y and ρ_X. Assume the mapping,

$$\begin{aligned} L : X &\to Y \\ \{f(t_n)\} &\mapsto f, \end{aligned}$$

is well-defined. Then the sampling process, $f \mapsto \{f(t_n)\}$, is *stable* for the metrics ρ_Y and ρ_X if the linear map L is uniformly continuous, *i.e.*, if

$$\forall \epsilon > 0,\ \exists \delta > 0 \quad \text{such that} \quad \rho_X(\{f(t_n)\}, \{g(t_n)\}) < \delta \quad \text{implies} \quad \rho_Y(f, g) < \epsilon.$$

Recall that if metric spaces X and Y are also topological vector spaces then linear maps $L : X \to Y$ which are continuous at the origin are uniformly continuous, *i.e.*, for every neighborhood W of $0 \in Y$ there is a neighborhood V of $0 \in X$ such that

$$\forall x_f, x_g \in X, \quad x_f - x_g \in V \quad \text{implies} \quad L(x_f - x_g) \in W.$$

In the case of (61), be have the situation that $Y = PW_\Omega$, $X = \{f(t_n) : f \in Y\}$, and ρ_Y and ρ_X are the metrics defined from the usual norms on $L^2(\mathbb{R})$ and $l^2(\mathbb{Z})$, respectively. The mapping L is well-defined if the Bessel map

$$\begin{aligned} B : PW_\Omega &\to l^2(\mathbb{Z}) \\ f &\mapsto \{f(t_n)\} \end{aligned}$$

is well-defined and injective, cf. [7, §2.5].

An aspect of stability concerns conditions to ensure the preservation of expansion properties under small perturbations. A typical problem is to find conditions so that a sequence $\{g_n\}$ of functions which is close to a basis $\{f_n\}$ is itself a basis, *e.g.*, Definition 16. The solution to this problem is due to Paley and Wiener [49, pages 100–106] for orthonormal bases $\{f_n\}$ on PW_Ω, and the generalization to bases is due to Boas (1940), cf. the treatment in [64]. "Paley-Wiener stability" in the sense of the following theorem has even become a topic in the theory of locally convex topological vector spaces. We shall state the result at the more down to earth level of Hilbert spaces.

Theorem 58. *Let H be a separable Hilbert space and let $\{f_n\} \subseteq H$ be a basis of H. Suppose that $\{g_n\} \subseteq H$ and $\theta \in [0,1)$ satisfy the condition that*

$$\forall F \subseteq \mathbb{Z}, \quad \textit{where card } F < \infty, \quad \textit{and} \quad \forall c_n \in \mathbf{C}, \quad \textit{where} \quad n \in F,$$

$$\| \sum_{n\in F} c_n(f_n - g_n)\| \leq \theta \| \sum_{n \in F} c_n f_n\|. \tag{62}$$

Then $\{g_n\}$ is a basis, and, even more, there is a topological isomorphism $L : H \to H$ such that $Lf_n = g_n$ for each n, cf. Theorem 17c.

Proof: The hypothesis (62) implies the continuity of the linear map $\tilde{L}$ defined by

$$\tilde{L}\left(\sum c_n f_n\right) = \sum c_n(g_n - f_n),$$

where $\sum c_n f_n \in H$; and, further, $\tilde{L}$ is bounded by $\theta < 1$ in the operator norm, cf. (56). Thus, $L \equiv I - \tilde{L}$ is a topological isomorphism and $Lf_n = f_n - (f_n - g_n) = g_n$. ∎

Example 59. In the case $\{f_n\}$ is an orthonormal basis, or even an exact frame, Theorem 58 allows us to conclude that $\{g_n\}$ is an exact frame when (62) holds.

The following is a well-known calculation for the setting $H = L^2[-\Omega, \Omega]$. It provides an elementary proof for a special case of Kadec's theorem quoted in Definition/Discussion 33b.

Theorem 60. *Let $\{a_n\}$ be a real sequence and let $\Omega > 0$. Suppose*

$$\sup_n |a_n - \frac{n}{2\Omega}| \equiv L < \frac{\log 2}{\pi}\left(\frac{1}{2\Omega}\right). \tag{63}$$

Then $\{e_{a_n}\}$ is an exact frame for $L^2[-\Omega, \Omega]$.

Proof: Let $H = L^2[-\Omega,\Omega]$ and denote the norm on $L^2[-\Omega,\Omega]$ by $\|\ldots\|$. Recall that $\{e_{a_{n/(2\Omega)}}\}$ is an orthonormal basis of H. Clearly,

$$\begin{aligned} e_{a_n}(\gamma) - e_{n/(2\Omega)}(\gamma) &= e^{\pi i n\gamma/\Omega}\left(e^{2\pi i\gamma(a_n - \frac{n}{2\Omega})} - 1\right) \\ &= e^{\pi i n \gamma/\Omega} \sum_{k=1}^{\infty} \frac{[2\pi i \gamma(a_n - \frac{n}{2\Omega})]^k}{k!}, \end{aligned} \tag{64}$$

where the right side is defined on $[-\Omega,\Omega]$ and extended 2Ω-periodically on $\hat{\mathbb{R}}$. Also, we have the estimate

$$\|\gamma^k f(\gamma)\| = \left(\frac{1}{2\Omega}\int_{-\Omega}^{\Omega} |\gamma^k f(\gamma)|^2 d\gamma\right)^{1/2} \leq \Omega^k \|f\| \tag{65}$$

for all $f \in H$ and $k \geq 0$.

For a given finite sequence $\{c_n : n \in F\}$ we use (64), (65), and the orthonormality of $\{e_{n/(2\Omega)}\}$ to make the estimate,

$$\begin{aligned}
&\|\sum_{n\in F} c_n(e_{a_n} - e_{n/(2\Omega)})\| \\
&\qquad = \|\sum_{n\in F} c_n e_{n/(2\Omega)}(\gamma)\sum_{k=1}^{\infty}\frac{[2\pi i\gamma(a_n-\frac{n}{2\Omega})]^k}{k!}\| \\
&\qquad \leq \|\sum_{k=1}^{\infty}\frac{(2\pi i)^k}{k!}\gamma^k(\sum_{n\in F} c_n(a_n-\frac{n}{2\Omega})^k e_{n/(2\Omega)}(\gamma))\| \\
&\qquad \leq \sum_{k=1}^{\infty}\frac{(2\pi\Omega)^k}{k!}\|\sum_{n\in F} c_n(a_n-\frac{n}{2\Omega})^k e_{n/(2\Omega)}\| \\
&\qquad = \sum_{k=1}^{\infty}\frac{(2\pi\Omega)}{k!}(\sum_{n\in F}|c_n|^2|a_n-\frac{n}{2\Omega}|^{2k})^{1/2} \\
&\qquad \leq \sum_{k=1}^{\infty}\frac{(2\pi L\Omega)^k}{k!}\|\sum_{n\in F} c_n e_{n/(2\Omega)}\| \\
&\qquad = (e^{2\pi L\Omega}-1)\|\sum_{n\in F} c_n e_{n/(2\Omega)}\|.
\end{aligned}$$

This estimate allows us to implement (62) if $e^{2\pi L\Omega}-1<1$, *i.e.*, if $2\pi L\Omega < \log 2$; but this is precisely our hypothesis (63). The result follows from Theorem 58 and our observation in Example 59. ∎

In light of the Kadec-Levinson bound of "1/4", we should note that $\frac{\log 2}{\pi} < \frac{1}{4}$ since $\log 16 < \pi = 3.14\ldots$. In fact, $\log 16 = 2.77259\ldots$.

Discussion 61. Two elegant pieces of analysis should be mentioned with regard to the Paley-Wiener stability condition (62). Harry Pollard proved that, when dealing with completeness, (62) can be replaced by

$$\|\sum_{n\in F} c_n(f_n - g_n)\|^2 \leq \theta_f^2\|\sum_{n\in F} c_n f_n\|^2 + \theta_g^2\|\sum_{n\in F} c_n g_n\|^2, \tag{66}$$

for some $\theta_f, \theta_g \in [0,1)$ (Annals of Math., 45 (1944), 738–739). Béla Sz. Nagy proved that, when dealing with exact frames, (62) can be replaced by a condition even weaker than (66) involving the additional mixed term

$$\|\sum_{n\in F} c_n f_n\|\|\sum_{n\in F} c_n g_n\|$$

(Duke Math. J., 14 (1947), 975–978).

Example 62. a. Let $\{a_n\}$ be a real sequence satisfying the density condition (34), viz.,

$$|a_n - \frac{n}{\Delta}| \leq L. \tag{67}$$

Also, suppose $\Omega > 0$ satisfies

$$2\Omega < \Delta. \tag{68}$$

For this calculation, assume $n \in \mathbb{N}$ and $a_n > 0$. From (67) we obtain $\Delta L \geq \Delta a_n - n$ so that $(\frac{L}{n} + \frac{1}{\Delta}) \geq \frac{a_n}{n}$, and hence

$$\frac{n}{a_n}\left(\frac{\Delta L}{n} + 1\right) \geq \Delta.$$

Consequently, (68) allows us to conclude that

$$\overline{\lim_{n\to\infty}} \frac{n}{a_n} > 2\Omega. \tag{69}$$

b. The point of part a is that uniformly discrete sequences satisfying (67) and (68) give rise to Fourier frames $\{e_{a_n}\}$ for $L^2[-\Omega, \Omega]$ by the Duffin-Schaeffer theorem (Theorem 38), and condition (69) is weaker than (67) and (68). The hope is that (69) would lead to completeness which, of course, is weaker than the frame property. Such is the case as we see in the following result (Theorem 63) of Paley and Wiener (1934).

c. This particular theorem of Paley and Wiener is a significant extension of a completeness theorem due to Pólya and Szász (Jahresbericht der Deutschen Mathematiker Vereinigung, 43 (1933), 20), whose density condition is (69) with "$\underline{\lim}$" instead of "$\overline{\lim}$". The added depth of this Paley and Wiener result is due to the use of their fundamental theorem on quasi-analytic functions, *viz.* [49, Theorem XII], which, in turn, is closely related to the Beurling-Malliavin theory and has many other applications, *e.g.*, [3], [4].

Theorem 63. (Paley-Wiener, [49, Theorem XXVIII]). *Let $\{a_n\} \subseteq \mathbb{R}$ be a strictly increasing sequence of positive numbers, and assume the density condition*

$$\overline{\lim_{n\to\infty}} \frac{n}{a_n} > 2\Omega.$$

Then $\{e_{\pm a_n}\}$ is complete in $L^2[-\Omega, \Omega]$.

Discussion 64. Theorems 60, 38 (and 44), and 63 have density hypotheses and conclude with exact frame, frame, and completeness properties, respectively. Theorem 60, depending as it does on Theorem 58, shows how its density hypothesis is tied-in with Paley-Wiener stability.

At the completeness level, a celebrated example by Kahane [35] exhibits sparse non-uniformly distributed sequences $\{a_n\} \subseteq \mathbb{R}$ for which $\{e_{a_n}\}$ is complete in $L^2[-\Omega, \Omega]$, for arbitrarily large Ω. The Beurling-Malliavin theory referenced at the beginning of this section places such examples in the same theory as the seemingly more regular sequences of Theorem 63 (or Theorem 60).

We shall not go into the Beurling-Malliavin completeness theory. Instead, we shall define the lower and upper uniform Beurling densities mentioned in §7, Definition 36b. The lower uniform Beurling density is the correct condition to formulate a converse of the frame result, Theorem 38.

Definition 65. a. Let $\{a_n\} \subseteq \mathbb{R}$ be a strictly increasing sequence for which $\lim_{n \to \pm\infty} a_n = \pm\infty$, and define

$$n^-(r) = \inf_I n_I(r) \quad \text{and} \quad n^+(r) = \sup_I n_I(r),$$

where $I \subseteq \mathbb{R}$ is an interval, $r > 0$, and

$$n_I(r) = card\{a_n \in I : |I| = r\}.$$

Clearly, there are sequences $\{a_n\}$ for which

$$\forall r > 0, \quad n^+(r) = \infty.$$

The *lower* and *upper uniform Beurling densities* of $\{a_n\}$ are

$$\Delta^- \equiv \Delta^-(\{a_n\}) = \lim_{r \to \infty} \frac{n^-(r)}{r}$$

and

$$\Delta^+ \equiv \Delta^+(\{a_n\}) = \lim_{r \to \infty} \frac{n^+(r)}{r},$$

respectively. These limits exist since n^- is superadditive and n^+ is subadditive, cf. [37, §§10.6–10.11].

b. Suppose $\Delta^-(\{a_n\}) = \Delta^+(\{a_n\}) \equiv \Delta \in [0, \infty)$. Then the natural density of $\{a_n\}$ (Definition 36b),

$$\lim_{r \to \infty} \frac{n_0(r)}{r},$$

exists and equals Δ since

$$n^-(r) \leq card\{a_n \in [\frac{-r}{2}, \frac{r}{2}]\} \leq n^+(r).$$

c. Uniform Beurling density is closely related to the more classical notion defined by the right side of (69). For example, $\text{card}\{a_j \in I : |I| = a_n\} = n$ if $I = [0, a_n]$ in the case $a_0 = 0$ and $a_j > 0$ if and only if $j \geq 1$. Thus,

$$\frac{n^-(a_n)}{a_n} \leq \frac{n}{a_n} \leq \frac{n^+(a_n)}{a_n}.$$

In fact, the Duffin-Schaeffer theorem (Theorem 38) is valid if the uniformly discrete sequence $\{a_n\}$ satisfies the condition, $\Delta^-(\{a_n\}) > 2\Omega$.

We can now state the converse of the Duffin-Schaeffer theorem alluded to before Definition 65, cf., §7, Formula 45.

Theorem 66. (Landau [41]). *Let $\Omega > 0$ and let $\{a_n\} \subseteq I\!R$ be a uniformly discrete energy stable sequence for PW_Ω. (Thus, $\{e_{a_n}\}$ is a Fourier frame for $L^2[-\Omega, \Omega]$.) Then*

$$\Delta^-(\{a_n\}) \geq 2\Omega.$$

Discussion 67. *Frames and completeness.*

Theorem 66 is a special case of a deep contribution by Landau [41]. Landau's results extend to higher dimensions and multiband signals. This latter notion means that N-frequency bands are analyzed instead of $[-\Omega, \Omega]$, cf. the subband coding inherent in the analysis of decimation and aliasing in §10.

A critical insight, which one can extract from Landau's work, is that irregular sampling reconstruction formulas such as those in §8 necessarily depend on a close relation between the bandwidth and the lower uniform Beurling density of the sampling set. This relationship is not fundamental in dealing with the analogous problem for obtaining completeness, thereby adding a level of complexity to this latter problem, *e.g.*, Discussion 64.

Definition 68. a. Let $D \subseteq \mathbb{R}$ and $E \subseteq \hat{\mathbb{R}}$ be closed sets. *Balayage is possible for* (D, E) if

$$\forall \mu \in M_b(\mathbb{R}), \ \exists \nu_\mu \in M_b(D) \quad \text{such that} \quad \forall \gamma \in E,$$

$$\int e^{2\pi i t\gamma} d\mu(t) = \int e^{2\pi i t\gamma} d\nu_\mu(t).$$

b. This notion of balayage stems from Poincaré's balayage process in potential theory; and, historically, E was a collection of potential theoretic kernels instead of group characters or frequencies.

A dual formulation of studying balayage problems has the flavor of classical spectral synthesis, *e.g.*, [1, Equation (3.2.52)]. Further, Beurling's theorem, Theorem 69, is a special case of his balayage theory which depends on specific systhesizable sets [12], [13, Volume 2, pages 341–350].

Theorem 69. (Beurling 1959–1960). *Let $D \subseteq I\!R$ be a closed set and let $E = [-\Omega, \Omega]$ for some $\Omega > 0$. Balayage is possible for (D, E) if and only if D contains a uniformly discrete sequence $\{a_n\}$ whose lower uniform Beurling density satisfies the inequality,*

$$\Delta^-(\{a_n\}) > 2\Omega.$$

Problem 70. Discussion 43b and the definition of (energy) stable sequences lead us to the following problem: *find those sequences $\{t_n\}$ such that the formula,*

$$\int f(t)dt = \sum f(t_n), \tag{70}$$

or similar "continuous analogues of series" formulas are valid for all $f \in PW_\Omega$ for which $\int f(t)dt$ exists. The existence of the integral can be considered either in the ordinary sense or as a principal value integral. The general question is quite difficult, and, in light of the weights mentioned in Discussion 43b, there are a number of related questions.

In the case $t_n = nT$, we shall see that the analogue of (70) itself is

$$\int f(t)dt = T \sum f(nT). \tag{71}$$

Even in this case there are questions to be answered if $\{f(nT)\} \in l^2(\mathbb{Z}) \backslash l^1(\mathbb{Z})$.

The following elementary result gives some positive information in the case of regular sampling sequences.

Theorem 71. *Let $T, \Omega > 0$ and let $f \in PW_\Omega$.*

a. *If $\{f(nT)\} \in l^1(\mathbb{Z})$ and $2T\Omega \leq 1$ then $\int f(t)dt$ exists as a principal value integral and (71) is valid.*

b. *Assume $\hat{f} \in C_c^\infty(\hat{\mathbb{R}})$ or that*

$$\exists C \quad \text{such that} \quad \forall t \in \mathbb{R},\ |f(t)| + |f^{(1)}(t)| + |f^{(2)}(t)| \leq \frac{C}{1+t^2}. \tag{72}$$

If $T\Omega \leq 1$ then (71) is valid.

c. *If $2T\Omega = 1$ then $\{f(nT)\} \in l^2(\mathbb{Z})$ and*

$$\int |f(t)|^2 dt = T \sum |f(nT)|^2.$$

Proof: a. By the classical sampling theorem, the right side of

$$f = T \sum f(nT) \tau_{nT} d_{2\pi\Omega}$$

converges uniformly on $\mathbb{R}$. Thus,

$$\begin{aligned} \forall S > 0, \quad \int_{-S}^{S} f(t)dt &= T \sum f(nT) \int_{-S}^{S} \tau_{nT} d_{2\pi\Omega}(t)dt \\ &\equiv T \sum f(nT) - \sum f(nT)\epsilon(S,n); \end{aligned} \tag{73}$$

and, now, a standard argument, first recorded in [34] (as far as I know) and using the hypothesis $\{f(nT)\} \in l^1(\mathbb{Z})$, allows us to compute

$$\lim_{S \to \infty} \sum f(nT)\epsilon(S,n) = 0.$$

Hence, the principal value integral $\int f(t)dt$ exists and (71) is valid by (73).

b. The function-form of the Poisson summation formula recorded in §2, Equation (7) is

$$\sum \hat{f}\left(\frac{n}{T}\right) = T \sum f(nT). \tag{74}$$

If $f \in PW_\Omega$, $\hat{f}$ is continuous, and $\Omega \le 1/T$ then (74) becomes

$$\hat{f}(0) = T \sum f(nT)$$

and this is (71). Our hypothesis, $\hat{f} \in C_c^\infty(\hat{\mathbb{R}})$ or (72), implies the continuity of $\hat{f}$ and the validity of (74), *e.g.*, [38].

c. By hypothesis, even for $2T\Omega \le 1$, $f = T\sum f(nT)\tau_{nT}d_{2\pi\Omega}$ in $L^2(\mathbb{R})$, and so $\hat{f} = T\sum f(nT)e_{-nT}\mathbf{1}_\Omega$ in $L^2(\hat{\mathbb{R}})$. If $2T\Omega = 1$ and we extend $\hat{f}$ $\frac{1}{T}$-periodically on $\hat{\mathbb{R}}$, then we can conclude that $\{f(nT)\} \in l^2(\mathbb{Z})$ by the square integrability of $\hat{f}$ on $[-\Omega, \Omega]$.

Since

$$\int_{-\Omega}^{\Omega} e^{-2\pi i(m-n)T\gamma} d\gamma = \frac{\sin(2\pi(m-n)T\Omega)}{\pi(m-n)T}$$

and $\{f(nT)\} \in l^2(\mathbb{Z})$, we have

$$\|f\|_2 = T\left(\sum_{m,n} f(mT)\overline{f}(nT)\frac{\sin(2\pi(m-n)T\Omega)}{\pi(m-n)T}\right)^{1/2}$$

by the Plancherel theorem, and the result follows. ■

Discussion 72. There are several interesting historical remarks concerning (71) in [31, page 63] and some wonderful examples in [20].

In the proof of Theorem 71a, we cannot integrate the classical sampling formula term by term over $\mathbb{R}$, even though the series converges uniformly. There are counterexamples, and we mention the point since there has been a little confusion on this issue in some of the literature. Of course, formally, (71) is an immediate consequence of the classical sampling formula.

In the proof of Theorem 71b, we only required $T\Omega \le 1$ instead of $2T\Omega \le 1$. Also, our hypotheses for the validity of the Poisson summation formula can be weakened, *but* the formula cannot be used indiscriminately. One of the difficulties in Problem 70 is that "Poisson summation formulas" require some regularity, and there are deep open questions here concerning fundamental problems in number theory.

Because of the proof of Theorem 71c, we note that *if* $2T\Omega \le 1$, *then the mapping,*

$$\begin{aligned} PW_\Omega &\to l^2(\mathbb{Z}) \\ f &\mapsto \{Tf(nT)\}, \end{aligned}$$

is a well-defined, continuous linear injection, cf. the discussion on Bessel sequences in [7, §2]. To verify this claim, consider the function G and sequence $\{c_n\}$ of Fourier coefficients defined in Theorem 1. By the Plancherel theorem for Fourier series, we can assert that $\sum |c_n|^2 < \infty$; and each $c_n = Tf(nT)$ by

Proposition 9. Thus, the mapping is well-defined, and the fact it is an injection is clear. The continuity is a consequence of the inequality,

$$\left(\sum |Tf(nT)|^2\right)^{1/2} = \sqrt{T}\|f\|_2,$$

or (for more general situations) of the uniform boundedness principle. Similar easy calculations yield the *surjectivity* of the above mapping in the case $2T\Omega = 1$.

Example 73. a. Our proof of Theorem 71b combined the Poisson summation formula and bandlimitedness to obtain (71). Although this is an entirely standard procedure, we do mention that Wiener used it in a calculation for a work by Bhatia and Krishnan on light scattering (Proc. Royal Soc., London, Ser. A, 192 (1948), 181–194). Wiener verified that if $2T\Omega \leq 1$ then

$$\forall t \in \mathbb{R}, \quad \frac{1}{2T\Omega} = \sum \left(\frac{\sin 2\pi\Omega(t-nT)}{2\pi\Omega(t-nT)}\right)^2, \tag{75}$$

cf. the classical sampling formula. Equation (75) is an immediate consequence of the Poisson summation formula applied to

$$f(u) = 4\pi\Omega w(2\Omega u - 4\Omega\pi t)$$

where $t \in \mathbb{R}$ is fixed and w is the *Fejér kernel*

$$w(u) = \frac{1}{2\pi}\left(\frac{\sin u/2}{u/2}\right)^2,$$

whose Fourier transform is $\max(1 - 2\pi|\gamma|, 0)$. (We use "$w$" for Fejér since we cannot use "f", and Fejér's original surname was Weiss.) (75) is not unexpected since $f(t) \equiv 1$, although not in $L^2(\mathbb{R})$, has distributional Fourier transform δ which has the required support property, viz., $supp\,\delta = \{0\} \subseteq [-\Omega, \Omega]$, and the sampling function $\hat{w}$ is 1 on the support of $\hat{f}$, *e.g.*, Theorem 46.

b. The proof of the Poisson summation formula involves a periodization procedure. Thus, if $f \in L^1(\mathbb{R})$ then

$$F(t) \equiv \sum f(t+nT), \quad t \in \mathbb{R},$$

is T-periodic and $F \in L^1(\mathbb{R}/T\mathbb{Z})$, *i.e.*, F defined on $\mathbb{R}$ is Lebesgue integrable on every interval of length nT. With this in mind we shall now take another look at part a but with some of the material of §10 in mind. For convenience, we let $T = 1$ and, because of §10, we switch to the frequency axis.

First, let $g \in L^2(\mathbb{R})$, and note that $\{\tau_n g\}$ is orthonormal if and only if

$$\sum |\hat{g}(\gamma+n)|^2 = 1 \text{ a.e.} \tag{76}$$

Now let $g \equiv \mathbf{1}_{[0,1)}$, so that $\{\tau_n g\}$ is an orthonormal basis of some subspace V_0 of $L^2(\mathbb{R})$. For this function, the method of multiresolution analysis in wavelet theory generates the classical Haar orthonormal basis of $L^2(\mathbb{R})$, *e.g.*, [47].

Note that

$$\hat{g}(\gamma) = e^{-\pi i\gamma} d_\pi(\gamma), \tag{77}$$

and, by (76),

$$G(\gamma) \equiv \sum |\hat{g}(\gamma+n)|^2 = 1 \text{ a.e.} \tag{78}$$

$G \in L^1(\mathbb{R}/\mathbb{Z})$ since $\hat{g} \in L^2(\mathbb{R})$; and, because of (78), the nth Fourier coefficient of G is δ_{0n}. Lest the equivalence of orthonormality and (76) cause trepidation, the Fourier coefficients of G can be computed directly as

$$g * \tilde{g}(-n) = \int_{-n}^{1-n} g(t)dt = \delta_{0n}.$$

Combining (77) and (78), we obtain

$$1 = \sum \left(\frac{\sin \pi(\gamma - n)}{\pi(\gamma - n)}\right)^2.$$

In particular,

$$\frac{\pi^2}{\sin^2 \pi\gamma} = \sum \frac{1}{(\gamma+n)^2}, \quad \gamma \notin \mathbb{Z}, \tag{79}$$

so that

$$\pi^2 = \sum \frac{1}{(n+\frac{1}{2})^2};$$

and we can even obtain Euler's formula,

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

by taking limits correctly in (79).

§10. Irregular Sampling - Approaches and Topics

Recent Contributions 74. We begin our final section by listing contributions by others on the topic of irregular sampling, including new closely related methods. Our caveat about this list was made in the Introduction; and, generally, we shall only give a "sampling" of works by some authors who publish "regularly" in the field. We shall not list work referenced in [21], [31], except to point out that Kluvánek's fundamental article [39] gets better with time. Further, there is a new book by Marks [45] and a forthcoming venture by the authors of [21] and [31]. We emphasize again that many contributions are not cited.

Cenker, C., Feichtinger, H., and Herrmann, M., Iterative algorithms in irregular sampling, a first comparison of methods, 1991, preprint.

Cetin, A., Reconstruction of signals from Fourier transform samples, *Signal Processing*, 16 (1989), 129-148.

Donoho, D. and Logan, B., Signal recovery and the large sieve, preprint.

Feichtinger, H. and Gröchenig, K., Error analysis in regular and irregular sampling theory, April 1990, preprint.

Feichtinger, H., Gröchenig, K., and Herrmann, M., Iterative methods in irregular sampling theory, numerical results, ASST 1990, Proc. Conf. Aachen, September 1990.

Frazier, M. and Torres, R., The sampling theorem, ϕ-transform, and Shannon wavelets for R, Z, T, and Z_N, 1991, preprint.

Gröchenig, K., Reconstruction algorithms in irregular sampling, 1991, preprint.

Houdré, C., On path reconstruction of processes from irregular samples, *Probabilistic and stochastic advances in analysis, with applications*, (J. S. and J. L. Byrnes, eds.), NATO-ASI, 1991, Series C, Kluwer Academic Publishers, to appear.

Luthra, A. Extension of Parseval's relation to nonuniform sampling, *IEEE Trans. ASSP*, 36 (1988), 1909-1911.

Marvasti, F. and Analoui, M., Recovery of signals and nonuniform samples using iterative methods, Proc. Int. Symp. CAS, Portland, OR, 1989.

Rawn, M., A stable nonuniform sampling expansion involving derivatives, *IEEE Trans. IT*, 35 (1989), 1223-1227.

Sauer, K. and Allebach, J., Iterative reconstruction of band-limited images from nonuniformly spaced samples, *IEEE Trans. CAS*, 34 (1987), 1497-1506.

Seip, K., An irregular sampling theorem for functions bandlimited in a generalized sense, *SIAM J. Appl. Math.*, 47 (1987), 1112-1116.

Torres, R., Spaces of sequences, sampling theorem, and functions of exponential type, 1990, preprint.

Walter, G., Some wavelet sampling theorems, 1990, preprint.

Walter, G., Nonuniform sampling of bandlimited functions of polynomial growth, 1990, preprint.

Zeevi, Y. and Shlomot, E., Pyramidal image representation in nonuniform systems, SPIE, Vol. 1001, Visual Communications and Image Processing, (1988), 563-570.

Now let's turn to the second part of §10, where we shall discuss aliasing and the transition from coherent states, which has been the topic of this chapter, to formulating a backdrop for a similar treatment of wavelets and wavelet packets.

Background 75. Let $T > 0$ be a sampling period. The measure (on $\mathbb{R}$),

$$p = \sum \delta_{nT},$$

is the *sampling impulse train* where δ_{nT} is the Dirac measure at nT. The measure p is tempered (but not bounded); and, by the Poisson summation formula (Equation (7)), $\hat{p}$ is the tempered measure,

$$\hat{p} = \frac{1}{T} \sum \delta_{n/T}.$$

If f is continuous on $\mathbb{R}$ then the product fp exists as a measure, and

$$fp = \sum f(nT)\delta_{nT} \tag{80}$$

is the *sampled signal*. The "distributional" verification of (80) is immediate:

$$\forall g \in C_c(\mathbb{R}), \quad fp(g) = p(fg) = \sum f(nT)g(nT),$$

and

$$\left(\sum f(nT)\delta_{nT}\right)(g) = \sum f(nT)g(nT).$$

We write $f_p = fp$, and assume f is well-behaved enough to ensure that the exchange formula,

$$\hat{f}_p = \hat{f} * \hat{p},$$

is valid, *e.g.*, [57] provides conditions for validity. Thus

$$\hat{f}_p = \frac{1}{T} \sum \tau_{n/T} \hat{f}.$$

Let us consider the sampling frequency $\frac{1}{T}$ and the low pass filter $h = Td_{\pi/T}$ having frequency response $\hat{h} = T\mathbf{1}_{1/(2T)}$. The formal output of the sampled signal f_p in this linear time-invariant system is

$$h * f_p = T \sum f(nT)\tau_{nT} d_{\pi/T}. \tag{81}$$

(If we define a frequency bandwidth 2Ω by $2T\Omega = 1$, then the right side of (81) is the classical sampling formula.) Formally, then,

$$(h * f_p)^\wedge = \mathbf{1}_{(\frac{1}{2T})} \sum \tau_{n/T} \hat{f}. \tag{82}$$

Example 76. Let $f(t) = \sin 2\pi t\gamma_0$, for some $\gamma_0 > 0$. Distributionally, we calculate

$$\hat{f} = \frac{1}{2i}(\tau_{\gamma_0}\delta - \tau_{-\gamma_0}\delta), \tag{83}$$

since, for example,

$$(\tau_{\gamma_0}\delta)^\wedge(t) = (\tau_{\gamma_0}\delta(\gamma))(e^{2\pi it\gamma}) = \delta_{\gamma_0}(\gamma)(e^{2\pi it\gamma}) = e^{2\pi it\gamma_0}.$$

Now let us choose a sampling period T, with corresponding sampling frequency $1/T$, so that

$$\gamma_0 \in (\frac{1}{2T}, \frac{1}{T}).$$

Consider the linear time-invariant system L defined by (81) where $h = Td_{\pi/T}$. Because of (83), the right side of (82) consists of the two non-zero terms,

$$\mathbf{1}_{(\frac{1}{2T})}\tau_{1/T}\hat{f} = -\frac{1}{2i}\ \delta_{-\gamma_0+\frac{1}{T}}$$

and

$$\mathbf{1}_{(\frac{1}{2T})}\tau_{-1/T}\hat{f} = -\frac{1}{2i}\ \delta_{\gamma_0-\frac{1}{T}}.$$

Note that $-\gamma_0 + \frac{1}{T} > 0$ and $\gamma_0 - \frac{1}{T} < 0$. In any case,

$$(h * f_p)^\wedge = -\frac{1}{2i}(\delta_{\frac{1}{T}-\gamma_0} - \delta_{-(\frac{1}{T}-\gamma_0)}), \quad \frac{1}{T} - \gamma_0 > 0.$$

Thus, if $\gamma_0 \in (\frac{1}{2T}, \frac{1}{T})$, the original frequency γ_0 takes on the identity or *alias* of the lower (and,, incidentally, negative in this example) frequency $\gamma_0 - \frac{1}{T}$, *i.e.*, the output of the sampled signal is

$$h * f_p(t) = \sin\Big(2\pi t(\gamma_0 - \frac{1}{T})\Big)$$

instead of $\sin 2\pi t\gamma_0$. We *undersampled* the original signal f, and consequently were *not* able to reconstruct it by the right side of (81). Undersampling means that the sampling period is big so that the sampling frequency is small; in our case, this meant that $\frac{1}{2T} < \gamma_0$, cf. Definition/Remark 6. (The condition $\gamma_0 < \frac{1}{T}$ was not essential to make this point about undersampling and aliasing, but serves to pick out certain elements on the right side of (82).)

Definition 77. a. Let f be a discrete signal, *i.e.*, f is a complex-valued function defined on $\mathbb{Z}$. For example, f could be the sampled signal for some continuous function in the case $T = 1$. The *decimation* or *downsampling* of f is the discrete signal $f_2 : \mathbb{Z} \to \mathbb{C}$ defined as

$$\forall n \in \mathbb{Z}, \quad f_2[n] = f[2n].$$

We hasten to point out that this terminology is in the spirit but not the letter of the word "decimation". Also, we could just as easily have defined f_k as $f_k[n] = f[kn]$.

b. Formally, the *Fourier series* of $f : \mathbb{Z} \to \mathbb{C}$ is the 1-periodic function,

$$F(\gamma) \equiv \sum f[n]e^{-2\pi i n\gamma}, \tag{84}$$

where the *Fourier coefficients* are

$$f[n] = \int_0^1 F(\gamma)e^{2\pi i n\gamma}\, d\gamma, \tag{85}$$

cf. the proof of Theorem 1. [38] contains a rigorous treatment of the theory of Fourier series. The most natural and difficult (to prove) convergence result in the theory is Carleson's theorem (1966): *if* $F \in L^2(\mathbb{R}/\mathbb{Z})$ *and* f *is defined by* (85) *then the series in (84) converges to* F *a.e.*

c. Let the discrete signals f and f_2 have Fourier series F and F_2, respectively. Formally, it is easy to see that F_2 is a 1-periodic function and

$$\forall \gamma \in \hat{\mathbb{R}}, \quad F_2(\gamma) = \frac{1}{2}\Big(F(\frac{\gamma}{2}) + F(\frac{\gamma}{2} + \frac{1}{2})\Big). \tag{86}$$

d. In Equation (80) we defined the sampled signal of a given continuous function. We shall now define the *sampled signal* of a given discrete signal f. With Equation (80) in mind, we define

$$\delta[m-n] = \delta_{mn},$$

and consider the *sampling period* $T = 2$. Thus, the *sampled signal* f_p of f is

$$\forall j \in \mathbb{Z}, \quad f_p[j] = \sum_n f[2n]\delta[j-2n].$$

If F_p is the Fourier series of f_p, then F_p is a $\frac{1}{2}$- periodic function and

$$\forall \gamma \in \hat{\mathbb{R}}, \quad F_p(\gamma) = \frac{1}{2}\Big(F(\gamma) + F(\gamma + \frac{1}{2})\Big).$$

e. The process of decimation can be viewed as taking place in two steps, viz., forming f_p from f and then forming f_2 from f_p. In terms of Fourier series, this process is reflected by the two equations,

$$F_p(\gamma) = \frac{1}{2}\Big(F(\gamma) + F(\gamma + \frac{1}{2})\Big)$$

and

$$F_2(\gamma) = F_p(\frac{\gamma}{2}).$$

Definition/Discussion 78. *Aliasing.*

In characterizing aliasing in terms of undersampling, we say that *aliasing occurs in the decimation process* if

$$supp\, F \cap supp\, \tau_{-1/2} F \neq \phi \tag{87}$$

in the first step of decimation, when forming f_p from f. (Technically, we should deal with the interiors of the supports in (87).) In this case, F is the *unaliased component* and $\tau_{-1/2}F$ is the *aliased component*. The reason for this terminology is that there is γ_0 for which $F(\gamma_0) = F(\gamma_0 + \frac{1}{2})$. Consequently, we have the expected expansion,

$$F(\gamma_0) = \sum f[n] e^{-2\pi i n \gamma_0},$$

and the "aliased" expansion,

$$F(\gamma_0) = \sum (-1)^n f[n] e^{-2\pi i n \gamma_0},$$

cf. Definition/Discussion 6.

The decimation process and accompanying aliasing phenomenon logically lead us to the method of subband coding, to the notion of a *quadrature mirror filter* (QMF), and to our closing material, which can be viewed as an introduction to wavelets and wavelet packets.

Before defining a QMF (Definition 79), we shall make the following remarks about *subband coding*. First, a subband coding procedure filters a spectrum, *e.g.*, a speech spectrum, into separate frequency subbands, and downsamples the signals corresponding to the given subbands. In the case of two subbands we have the "2-decimations" of Definition 77a and Theorem 80; and, hence, we preserve, and don't increase, the number of data points from the original signal. The resulting signals are encoded for transmission and transmitted. Finally, they are decoded, interpolated, *e.g.*, by inserting zeros, and reconstructed as a single message at the receiver. There are important reasons for this procedure. These include limiting noise from encoding and decoding to the relevant subbands, allocating bits based on perceptual criteria for a given subband, providing a structure to deal with compression problems, and sending "images" at various levels of resolution to control "expense". The early work on subband coding is well documented in the IEEE ICASSP and information sciences publications from the 1970s and early 1980s.

QMFs are used in subband coding to remove aliasing, *e.g.*, Theorem 81.

Definition 79. Let $H \in L^2(\mathbb{R}/\mathbb{Z})$ be the Fourier series of $h \in l^2(\mathbb{Z})$. The discrete signal h is a *quadrature mirror filter* with frequency response H if

$$|H(\gamma)|^2 + |H(\gamma + \frac{1}{2})|^2 = 2 \qquad a.e. \tag{88}$$

In particular, $\|H\|_{L^2(\mathbf{T})} = 1$.

Theorem 80. *Let g_i, h_i, g_o, h_o be filters corresponding to linear time-invariant systems, and let f be a discrete signal. ("i" is for "in" and "o" is for "out".) Consider the filterings/decimations*

$$f_{g_i}[n] \equiv \sum \overline{g}_i[k-2n]f[k]$$

$$f_{h_i}[n] \equiv \sum \overline{h}_i[k-2n]f[k]$$

and the inserting zeros/filterings

$$g_o * f_{0,g_i} \qquad \text{and} \qquad h_o * f_{0,h_i},$$

where

$$f_{0,g_i}[2n] \equiv f_{g_i}[n] \qquad \text{and} \qquad f_{0,g_i}[2n+1] \equiv 0.$$

Define

$$f_{g,h} \equiv g_o * f_{0,g_i} + h_o * f_{0,h_i},$$

and let the Fourier series of the various sequences be denoted by the corresponding capitalized letter. Then

$$\begin{aligned} F_{g,h} &= \frac{1}{2}F(G_o\overline{G}_i + H_o\overline{H}_i) \\ &+ \frac{1}{2}\ \tau_{-1/2}F(G_o\tau_{-1/2}\overline{G}_i + H_o\tau_{-1/2}\overline{H}_i). \end{aligned} \tag{89}$$

By Definition/Discussion 78, the second term on the right side of (89) is the *aliased component*. The following result shows how QMFs give rise to alias-free reconstruction; and, together with Theorem 80, it is an essential component of the wavelet packet point of view.

Theorem 81. *Consider the setup of Theorem 80, and suppose $g = g_i = g_o$, $h = h_i = h_o$, and*

$$G(\gamma) = -e^{-2\pi i\gamma}\ \overline{H}(\gamma + \frac{1}{2}). \tag{90}$$

Assume h is a QMF. Then the aliased component of (89) vanishes, and

$$F_{g,h} = F.$$

If h is a low-pass filter, then g defined by (90) is a high-pass filter; and the filterings/decimations f_g and f_h defined in Theorem 80 give rise to relatively disjoint subbands. In the case of (90),

$$g[n] = (-1)^n\ \overline{h}[-n+1]. \tag{91}$$

We mention this because of Theorem 83, where Equation (91) appears again, cf. [47] for the development of Definition 82, the proof of Theorem 83, and historical comments.

Definition 82. A *multiresolution analysis* $\{V_j, \theta\}$ of $L^2(\mathbb{R})$ is an increasing sequence of closed linear subspaces $V_j \subseteq L^2(\mathbb{R})$ and an element $\theta \in V_0$ for which the following properties hold:

i. $\bigcap V_j = \{0\}$ and $\overline{\bigcup V_j} = L^2(\mathbb{R})$,
ii. $f(t) \in V_j$ if and only if $f(2t) \in V_{j+1}$,
iii. $f \in V_0$ and $k \in \mathbb{Z}$ imply $\tau_k f \in V_0$,
iv. $\{\tau_k \theta\}$ is an exact frame for the Hilbert space V_0.

Let $\{V_j, \theta\}$ be a multiresolution analysis of $L^2(\mathbb{R})$. If $\varphi \in V_0$ is defined by

$$\hat{\varphi}(\gamma) = \theta(\gamma)\Big(\sum |\hat{\theta}(\gamma+n)|^2\Big)^{-1/2},$$

then $\{\tau_k \varphi\}$ is an orthonormal basis of V_0, cf. Equation (76).

Theorem 83. *Let $\{V_j, \varphi\}$ be a multiresolution analysis of $L^2(\mathbb{R})$, where $\{\tau_k\varphi\}$ is an orthonormal basis of V_0.*

a. Let W_j be the orthogonal complement of V_j in V_{j+1}, i.e.,

$$V_{j+1} = V_j \oplus W_j.$$

There is $\psi \in W_0$ such that $\{\tau_k\psi\}$ is an orthonormal basis of W_0, and $\{\psi_{m,n}\}$ is an orthonormal basis of $L^2(\mathbb{R})$, where

$$\psi_{m,n}(t) = 2^{m/2}\ \psi(2^m t - n).$$

$\{\psi_{m,n}\}$ is called a wavelet basis.

b. The functions ϕ and ψ satisfy the following properties:

i. $\varphi(t) = \sqrt{2}\sum h[n]\varphi(2t-n)$

and

$$\hat{\varphi}(2\gamma) = \tfrac{1}{\sqrt{2}} H(\gamma)\hat{\varphi}(\gamma),$$

where h is a QMF having Fourier series H;

ii. $\psi(t) = \sqrt{2}\sum g[n]\varphi(2t-n)$

and

$$\hat{\psi}(2\gamma) = -e^{-2\pi i\gamma}\big(\tfrac{1}{\sqrt{2}}\big)\overline{H}(\gamma+\tfrac{1}{2})\hat{\varphi}(\gamma),$$

where

$$g[n] = (-1)^n \overline{h}[-n+1].$$

§11. Notation

$\mathbb{R}$ is the real line and $\mathbb{C}$ is the set of complex numbers. $L^p(\mathbb{R})$, $1 \leq p < \infty$, is the space of Lebesgue measurable functions $f: \mathbb{R} \to \mathbb{C}$ for which

$$\|f\|_p \equiv \left(\int |f(t)|^p dt \right)^{1/p} < \infty,$$

where "$\int$" indicates integration over $\mathbb{R}$. The usual adjustment is made to define $L^\infty(\mathbb{R})$ and its norm $\|\cdots\|_\infty$. The Fourier transform of $f \in L^1(\mathbb{R})$ is the function $\hat{f}$ defined by

$$\hat{f}(\gamma) \equiv \int f(t) e^{-2\pi i t\gamma} dt,$$

where $\gamma \in \hat{\mathbb{R}}(=\mathbb{R})$; and a similar definition is made for $f \in L^2(\mathbb{R})$. If F is defined on $\hat{\mathbb{R}}$ then

$$F^\vee(t) \equiv \int F(\gamma) e^{2\pi i t\gamma} d\gamma.$$

The pairing between f and $\hat{f}$ is designated by $f \leftrightarrow \hat{f}$, and the space of absolutely convergent Fourier transforms is $A(\hat{\mathbb{R}})$ with norm $\|\hat{f}\|_A \equiv \|f\|_1$.

$C^\infty(\mathbb{R})$ is the space of infinitely differentiable functions on $\mathbb{R}$, $\mathcal{S}(\mathbb{R})$ is the Schwartz space of "rapidly decreasing" elements of $C^\infty(\mathbb{R})$, and $C_c^\infty(\mathbb{R})$ is the space of compactly supported elements of $C^\infty(\mathbb{R})$. $C(\mathbb{R})$ is the space of continuous functions on $\mathbb{R}$, and $C_c(\mathbb{R})$ is the space of compactly supported elements of $C(\mathbb{R})$. $C^m[a,b]$ is the space of m-times continuously differentiable functions on $[a,b]$. The space of pseudo-measures is the Banach space dual of $A(\mathbb{R})$ with norm $\|\cdots\|_A$; and the space $M_b(\mathbb{R})$ of bounded Radon measures is the Banach space dual of $C_c(\mathbb{R})$ (or $C_c^\infty(\mathbb{R})$ or $\mathcal{S}(\mathbb{R})$) taken with the norm $\|\cdots\|_\infty$. δ_a is the Dirac measure supported by the point $a \in \mathbb{R}$, and $\delta_0 \equiv \delta$ is the unit under convoluion in the Banach algebra $M_b(\mathbb{R})$.

$\mathbb{Z}$ is the set of integers, $\mathbb{N}$ is the set of positive integers and $\mathbf{T} = \hat{\mathbb{R}}/\mathbb{Z}$ is the circle group identified with any interval $[\alpha, \alpha+1) \subseteq \hat{\mathbb{R}}$. $l^p(\mathbb{Z})$, $1 \leq p < \infty$, is the space of sequences $f: \mathbb{Z} \to \mathbb{C}$ for which $\left(\sum |f(n)|^p\right)^{1/p} < \infty$, where "$\sum$" indicates summation over $\mathbb{Z}$. Just as $\hat{\mathbb{R}}$ is the dual group of $\mathbb{R}$, so $\mathbf{T}$ is the dual group of $\mathbb{Z}$. $L^p(\mathbf{T})$, $1 \leq p < \infty$, is the space of 1-periodic Lebesgue measurable functions F on $\hat{\mathbb{R}}$ for which

$$\|F\|_{L^p(\mathbf{T})} \equiv \left(\int_0^1 |F(\gamma)|^p d\gamma \right)^{1/p} < \infty.$$

If f is defined on $\mathbb{R}$ and μ is a positive measure then

$$\|f\|_{2,\mu} \equiv \left(\int |f(t)|^2 d\mu(t) \right)^{1/2}.$$

The following special symbols are used throughout the chapter.

$$(\tau_a g)(t) \equiv g(t-a),$$

$$e_b(t) \equiv e^{2\pi i t b},$$

$$\mathbf{1}_S(t) \equiv \begin{cases} 1, & \text{if} \quad t \in S \\ 0, & \text{if} \quad t \notin S, \end{cases}$$

$$\mathbf{1}_{(\Omega)}(t) \equiv \mathbf{1}_{[-\Omega,\Omega]}(t),$$

$$d(t) \equiv \frac{\sin t}{\pi t}, \quad \text{“}d\text{” for Dirichlet},$$

$$f_\lambda(t) \equiv \lambda f(t\lambda),$$

$$\delta_{mn} = \begin{cases} 1, & \text{if} \quad m = n \\ 0, & \text{if} \quad m \neq n, \end{cases}$$

card S is the cardinality of S, $|S|$ is the Lebesgue measure of S, ϕ is the empty set, and *supp* F is the support of F.

References

1. Benedetto, J., *Spectral Synthesis,*, Academic Press, New York, 1975.
2. Benedetto, J., *Real Variable and Integration*, B. G. Teubner, Stuttgart, 1976.
3. Benedetto, J., Fourier uniqueness criteria and spectrum estimation theorems, *Fourier Techniques and Applications*, J. F. Price (ed.), Plenum Plublishing, 1985, 149-170.
4. Benedetto, J., A quantitative maximum entropy theorem for the real line, Integral Equations and Operator Theory, **10** (1987), 761-779.
5. Benedetto, J., Gabor representations and wavelets, *Commulative Harmonic Analysis*, D. Colella (ed.), Contemp. Math., **19** American Mathematical Society, Providence, 1989, 9-27.
6. Benedetto, J., Uncertainty principle inequalities and spectrum estimation, *Recent Advances in Fourier Analysis and its Applications,* J. S. and J. L. Byrnes (eds.), NATO-ASI Series C, Kluwer Academic Publishers, **315** (1990), 143-182.
7. Benedetto, J., Stationary frames and spectral estimation, *Probabilistic and Stochastic Advances in Analysis, with Applications,* J. S. and J. L. Byrnes (eds.), NATO-ASI, 1991 Series C, Kluwer Academic Publishers, to appear.
8. Benedetto, J., C. Heil, and D. Walnut, Remarks on the proof of the Balian-Low theorem, Canad. J. Math.
9. Benedetto, J. and W. Heller, Irregular sampling and the theory of frames, I, Note Math, 1991.

10. Benedetto, J. and W. Heller, Irregular sampling and the theory of frames, II, preprint.
11. Benedicks, M., Supports of Fourier transform pairs and related problems on positive harmonic functions, TRITA-MAT, **10**, 1980, Stockholm.
12. Beurling, A. Local harmonic analysis with some applications to differential operators, Belfer Graduate School of Science, Annual Science Conference Proceedings, (1966), 109-125.
13. Beurling, A. *The Collected Works of Arne Beurling,* L. Carleson, P. Malliavin, J. Neuberger, and J. Wermer (eds.), Birkhäuser, Boston, 1989.
14. Beurling, A. and P. Malliavin, On Fourier transforms of measures with compact support, Acta Math., **107** (1962), 291-309.
15. Beurling, A. and P. Malliavin, On the closure of characters and the zeros of entire functions, Acta Math., **118** (1967), 79-93.
16. Beutler, F., Sampling theorems and bases in Hilbert space, Inform. Contr., **4** (1961), 97-117.
17. Beutler, F., Error-free recovery of signals from irregularly spaced samples, SIAM Review, **8** (1966), 328-335.
18. Boas, R., Entire functions bounded on a line, Duke J. Math., **6** (1940), 148-169.
19. Boas, R., *Entire Functions,* Academic Press, New York, 1954.
20. Boas, R. and H. Pollard, Continuous analogues of series, Amer. Math. Monthly, **80** (1973), 18-25.
21. Butzer, P., W. Splettstösser, and R. Stens, The sampling theorem and linear prediction in signal analysis, Jber. d. Dt. Math.-Verein., **90** (1988), 1–70.
22. Chui, C. and J.-Z. Wang, A cardinal spline approach to wavelets, preprint.
23. Daubechies, I., The wavelet transform, time-frequency localization and signal analysis, IEEE Trans. Inform. Theory, **36** (1990), 961-1005.
24. Duffin, R. and A. Schaeffer, A class of nonharmonic Fourier series, Trans. Amer. Math. Soc. **72** (1952), 341-366.
25. Gabardo, J.-P., Tempered distributions with spectral gaps, Math. Proc. Camb. Phil. Soc., **106** (1989), 143-162.
26. Gabor, D., Theory of communication, J. IEE, London, **93** (1946), 429-457.
27. Gröchenig, K., Reconstruction algorithms in irregular sampling, preprint.
28. Heil, C. and D. Walnut, Continuous and discrete wavelet transforms, SIAM Review, **31** (1989), 628-666.
29. Heller, W., Frames of exponentials and applications, Ph.D. thesis, University of Maryland, College Park, MD, 1991.
30. Higgins, J., *Completeness and basis properties of sets of special functions*, Cambridge University Press, 1977.
31. Higgins, J., Five short stories about the cardinal series, Bull. Amer. Math. Soc., **12** (1985), 45-89.
32. Hörmander, L., *The analysis of linear partial differential operators*, I and II, Springer-Verlag, New York, 1983.

33. Jaffard, S., A density criterion for frames of complex exponentials, preprint.
34. Jagerman, D., Bounds for truncation error of the sampling expansion, SIAM J. Appl. Math., **14** (1966), 714-723.
35. Kahane, J.-P., Sur la totalité des suites d'exponentielles imaginaires, Ann. Inst. Fourier, **8** (1958), 273-275.
36. Kahane, J.-P., Travaux de Beurling et Malliavin, Séminaire Bourbaki, 1961/62, No. 255. (Reprinted and updated by Benjamin Publishers, New York, 1966.)
37. Kahane, J.-P., *Séries de Fourier Absolument Convergentes,* Springer-Verlag, New York, 1970.
38. Katznelson, Y., *An Introduction to Harmonic Analysis (1968),* Dover Publications, New York, 1976.
39. Kluvánek, I., Sampling theorem in abstract harmonic analysis, Mat. Fyz. Casopsis Sloven. Akad. Vied., **15** (1965), 43-48.
40. Landau, H., Sampling, data transmission, and the Nyquist rate, Proc. IEEE, **55** (1967), 1701-1706.
41. Landau, H., Necessary density condition for sampling and interpolation of certain entire functions, Acta Math., **117** (1967), 37-52.
42. Landau, H., An overview of time and frequency limiting, *Fourier Techniques and Applications,* J. Price (ed.), Plenum Press, New York, 1985, 201-220.
43. Landau, H., On the density of phase-space expansions, preprint.
44. Levinson, N., *Gap and density theorems,* Am. Math. Soc. Colloq. Publ., vol. 26, Am. Math. Soc., 1940.
45. Marks, R., *Introduction to Shannon Sampling and Interpolation Theory,* Springer-Verlag, New York, 1991.
46. Meyer, Y., Wavelets and Operators, CEREMADE, Université de Paris Dauphine, 1989.
47. Meyer, Y., *Ondelettes et opérateurs,* I, Hermann, Paris , 1990.
48. Oppenheim, A. and A. Willsky, *Signals and Systems,* Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1983.
49. Paley, R. and Wiener, N. *Fourier transforms in the complex domain,* Am. Math. Soc. Colloq. Publ., vol. 19, Am. Math. Soc., 1934.
50. Papoulis, A., Error analysis in sampling theory, Proc. IEEE, **54** (1966), 947-955.
51. Papoulis, A., *Signal Analysis,* McGraw-Hill Co., New York, 1977.
52. Plancherel, M., Intégrales de Fourier et fonctions entières, Colloque sur l'analyse harmonique, Nancy (1947), **15** (1949), 32-43.
53. Pollak, H., *New directions in mathematics,* Dartmouth Mathematics Conference, J. Kemeny, R. Robinson, and R. Ritchie (eds.), Prentice-Hall, New Jersey, 1963.
54. Redheffer, R., Completeness of sets of complex exponentials, Advances in Math., **24** (1977), 1-63.

55. Rieffel, M., Von Neumann algebras associated with pairs of lattices in Lie groups, Math. Ann., **257** (1981), 403-418.
56. Rudin, W., *Real and Complex Analysis*, McGraw-Hill Co., New York, 1966.
57. Schwartz, L., *Théorie des Distributions,* Hermann, Paris, 1966.
58. Seip, K. and R. Wallstén, Sampling and interpolation in the Bargmann-Fock space, preprint, 1990-1991.
59. Stein, E. and G. Weiss, *An Introduction to Fourier Analysis on Euclidean Spaces,* Princeton University Press, 1971.
60. von Neumann, J., *Mathematical Foundations of Quantum Mechanics,* Princeton University Press, 1932, 1949, and 1955.
61. Weinert, H. (ed.), *Reproducing Kernel Hilbert Spaces,* Hutchinson Ross Publishing.
62. Wexler, J. and S. Raz, Discrete Gabor expansions, Signal Processing, **21** (1990), 207-220.
63. Yao, K. and J. Thomas, On some stability and interpolatory properties of nonuniform sampling expansions, IEEE Trans. Circuit Theory, **14** (1967), 404-408.
64. Young, R., *An Introduction to Nonharmonic Fourier Series,* Academic Press, New York, 1980.

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