PREFACE

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The formula used to define Fourier frames is written explicitly by Paley and Wiener, [7] page 115, in the context of one of their fundamental theorems dealing with non-harmonic Fourier series. Fledgling forms of Fourier frames go back to Dini (1880) and then G. D. Birkhoff (1917), and leap to the profound results of Beurling and H. J. Landau in the 1960s, [2], [6]. Fourier frames lead naturally to non-uniform sampling formulas, and this is far from a complete story, e.g., [1].

Notwithstanding the importance of Fourier frames, and research such as that by Paley, Wiener, Beurling, and Landau, frames are with us today because of the celebrated work of Duffin and Schaeffer [5] in 1952. They explicitly found and featured the mathematical power of Fourier frames, and did the *right thing* mathematically by formulating such frames for Hilbert spaces, extracting central features of frames such as the decomposition of functions in terms of frames and understanding the role of overcomplete systems such as frames as opposed to orthonormal bases. It should also be pointed out that parallel to this development, a major analysis of bases was under way by the likes of Bari and Köthe; and their results could be rewritten in terms of frames, see, e.g., [8].

And then wavelet theory came along! More precisely, with regard to frames, there was the important work of Daubechies, Grossmann, and Meyer (1986) [4]; and there was the subsequent wonderful mix of mathematics and engineering and physics providing new insights as regards the value of frames. We now understood a basic role for frames with regard to noise reduction, stable decompositions, and robust representations of signals. In retrospect, frame research in the 1990s, besides its emerging prominence in wavelet theory and time-frequency analysis and their applications, was an *analytic* incubator, with all the accompanying excitement, that led to *finite* frames!

By the late 1990s and continuing today as an expanding mysterious universe, finite frame theory has become a dominant, intricate, relevant, and vital field. There were specific topics such as frame potential energy theory, $\Sigma - \Delta$ quantization, quantum detection, and periodic approximants in ambiguity function behavior, all with important applications. This has brought to bear a whole new vista of advanced technologies to understand frames and to unify ideas. The power of harmonic analysis and engineering brilliance are still part and parcel of frames, whether finite or not, but now we also use geometry and algebraic geometry, combinatorics, number theory, representation theory, and advanced linear and abstract algebra. There are major influences from compressive sampling, graph theory, and finite uncertainty principle inequalities.

The time was right just a few years ago *to stop and smell the roses*, and the volume on finite frames, edited by Casazza and Kutyniok [3] appeared (2013). Amazingly and not surprisingly, given the talent pool of researchers, the intrigue and intricacies of the problems, and the applicability of the subject, the time is *still* right. Kasso Okoudjou's 2015 AMS Short Course on Finite Frame Theory was perfectly conceived. He assembled the leading experts

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in the field, not least of whom in my opinion was Okoudjou himself, to explain the latest and deepest results. This book is the best step possible towards the future. Enjoy!

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