LOCAL SAMPLING FOR REGULAR WAVELET AND GABOR EXPANSIONS

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Abstract. The local behavior of regular wavelet sampling expansions is quantified. The term "regular" refers to the decay properties of scaling functions φ of a given multiresolution analysis. The regularity of the sampling function corresponding to φ is proved. This regularity is used to determine small intervals of sampling points so that the sampled values of a signal f at this finite set of points gives rise to a sampling expansion approximating f to within a predetermined margin of error.

1. Introduction

1.1. **Background.** In 1990, two of the authors began discussing local sampling for Gabor expansions in the spirit of local sampling results by Helms and Thomas [HT62] and Jagerman [Jag66] from the 1960s, cf., Logan [Log84]. These discussions and results were formulated in terms of the Classical Sampling Theorem, see Theorem 1.1.

Since then there has been a plethora of activity in the sampling and wavelet communities. For example, there are biennial Sampling Theory and Applications conferences, as well as edited volumes on sampling, e.g., [BF01], [Mar01], [BZ03]. Also in recent years classical ideas from interpolation theory, e.g., [Sch46], [Wal94], have coalesced with the concept of multiresolution analysis from wavelet theory in a natural structurally appealling way, e.g., see Theorem 2.6 which is formulated in terms of the mixed norm space $C^{1,\infty}(\mathbb{Z}^d, \mathbb{T}^d)$ defined in Definition 2.2. As such, our original idea from 1990 has focused on integrating these two directions.

The main result of this paper, Theorem 4.2, is a local sampling theorem in the setting of a subspace of $C^{1,\infty}(\mathbb{Z}, \mathbb{T})$, in which natural decay conditions are exploited.

Before describing the local sampling problem, we shall state the Classical Sampling Theorem (Theorem 1.1). In order to do this we formally define the *Fourier transform* $\widehat{f}$ of a complex valued function f defined on the real line $\mathbb{R}$ to be

$$\forall \gamma \in \widehat{\mathbb{R}},\ \widehat{f}(\gamma) = \int f(t) e^{-2\pi i t \gamma} dt,$$

where integration is over $\mathbb{R}$ and $\widehat{\mathbb{R}}$ designates $\mathbb{R}$ considered as a frequency domain. We shall be dealing with the usual $L^p(\mathbb{R})$ spaces as defined, for example, in [Ben97], [Kat68]; and

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PW_Ω in Theorem 1.1 is the *Paley-Wiener space*

$$PW_\Omega = \{f \in L^2(\mathbb{R}) : \operatorname{supp} \widehat{f} \subseteq [-\Omega, \Omega]\},$$

where $\operatorname{supp} \widehat{f}$ denotes the *support* of $\widehat{f}$. τ_y is the translation operator defined as $(\tau_y f)(x) = f(x-y)$.

Theorem 1.1. *Let $T, \Omega > 0$ satisfy the condition that $0 < 2T\Omega \leq 1$, and let $s \in PW_{1/(2T)}$ satisfy the condition that $\widehat{s} \equiv S = 1$ on $[-\Omega, \Omega]$ and $S \in L^\infty(\widehat{\mathbb{R}})$. Then*

$$\forall f \in PW_\Omega,\ f = T \sum_{n=-\infty}^{\infty} f(nT) \tau_{nT} s, \tag{1.1}$$

where the convergence in (1.1) is in $L^2(\mathbb{R})$ norm and uniformly in $\mathbb{R}$.

See [Ben97] (Theorem 3.10.10) and [BF01] (Chapter 1) for a proof and history, respectively.

1.2. **Problem and outline.** We shall examine the local behavior of the sampling series (1.1). In other words, and in the more general case of a scaled version of (1.1), we shall try to solve the following problem : given $m \in \mathbb{N}$, an interval I, and an $\varepsilon > 0$, find the least possible length of the interval $[N_b 2^{-m}, N_c 2^{-m}]$ containing I, such that

$$\sup_{x \in I} \left| f(x) - \sum_{n=N_b}^{N_c} f(n/2^m) s(2^m x - n) \right| < \varepsilon, \tag{1.2}$$

where $N_b, N_c \in \mathbb{Z}$.

Our setting is restricted to sampling functions s derived in a natural way from a given multiresolution analysis and scaling function φ, see Section 2. In Theorem 4.2 of Section 4 we give an estimate of the error formulated in (1.2) for sufficiently regular sampling functions s. Refinements of this error for a special case is the content of Corollary 4.3. Theorem 4.4 provides the error estimate (1.2) using the method of Theorem 4.2, but only assuming decay of a scaling function φ of a given multiresolution analysis. The relation between the decay of φ and s is established in Theorem 3.6 of Section 3. Finally, in Section 5, we present a local sampling theorem based on the Gabor transform. The results in Sections 3 - 5 are proved in the setting of the real-line $\mathbb{R}$. It is meaningful to prove analogous results on $\mathbb{R}^d$, but the present methods would have to be extended and combined with an intricate geometric analysis.

In a sense the error estimated in Section 4 is a truncation error. This error has been extensively studied, especially by Jagerman in [Jag66], for the case of bandlimited functions f when $N_c = -N_b = N$. In [AK00] two of us extended Jagerman's result to the setting of translation invariant sampling spaces. It is natural to examine further the local error for non-symmetric intervals as in (1.2).

For perspective, we note the close relation of this material with the uncertainty principle, see [BHW95] and [BF94]. In fact, some forms of the uncertainty principle can be viewed in terms of an interplay between localization on a time space and localization on a corresponding frequency space. In a work such as this, dealing with MRAs, frequency localization can be considered as a projection on a wavelet space, cf., the Bell Labs inequality for wavelet spaces [KP99]. Moreover, on certain sampling spaces we can easily examine local effects of the uncertainty principle, e.g., the Gibbs phenomenon on wavelet and sampling spaces [AK99], [Kar98]. In this context and as far as we know, it seems that there is an absence of systematic work dealing with the effects of the uncertainty principle on sampling spaces; we hope to address this issue on another occasion.

1.3. **Notation.** We shall use the standard notation from harmonic analysis as found in [Kat68]. We also incorporate the following notation.

$\mathbb{Z}$ is the additive group of integers and $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ is the d-dimensional torus. $A(\mathbb{T}^d)$ is the Banach algebra of absolutely convergent Fourier series

$$\Psi(\gamma) = \sum_{m\in\mathbb{Z}^d} a_m e^{-2\pi i\langle m,\gamma\rangle}$$

with norm $\|\Psi\|_{A(\mathbb{T}^d)} = \sum_{m\in\mathbb{Z}^d} |a_m|$. If $\Psi \in A(\mathbb{T}^d)$ then the Fourier coefficients a_m are of the form

$$\Psi^\vee[m] = \int_{\mathbb{T}^d} \Psi(\gamma) e^{2\pi i\langle m,\gamma\rangle} d\gamma.$$

$C(\mathbb{T}^d)$ is the space of continuous functions on T^d, i.e., the 1-periodic functions on $\mathbb{R}^d$. Clearly, $A(\mathbb{T}^d) \subseteq C(\mathbb{T}^d)$. ONB denotes *orthonormal basis*. Finally, $a_m = 0(A_m),\ |m| \to \infty$, where $A_m > 0$, is the classical convention to indicate that

$$\exists C > 0 \text{ such that } \forall m \in \mathbb{Z},\ |a_m| \leq CA_m.$$

2. The sampling function of an MRA

The following concept is fundamental in wavelet theory, e.g., see [Dau92], [Mal98], [Mey90].

Definition 2.1. *a.* A *dyadic multiresolution analysis* (MRA) of $L^2(\mathbb{R}^d)$ is a sequence $\{V_m\ :\ m \in \mathbb{Z}\}$ of closed linear subspaces of $L^2(\mathbb{R}^d)$ with the following properties:

i. (Inclusion) $\forall m \in \mathbb{Z},\ V_m \subseteq V_{m+1}$,
ii. (Separation) $\bigcap_m V_m = \{0\}$,
iii. (Density) $\overline{\bigcup_m V_m} = L^2(\mathbb{R}^d)$,
iv. (Scaling) $f(x) \in V_m \iff f(2^{-m}x) \in V_0$,
v. (Orthonormality) $\exists\varphi \in V_0$, a *scaling function*, such that

$$\{\tau_n\varphi : n \in \mathbb{Z}^d\} \text{ is an ONB for} V_0.$$

b. We shall use the standard notation $\varphi_{m,n}(x) = 2^{md/2}\varphi(2^m x - n)$ for $m \in \mathbb{Z}$, $n \in \mathbb{Z}^d$.

In this section we shall give conditions on the scaling function φ of an MRA so that a sampling theorem similar to Theorem 1.1 holds. The main result is Theorem 2.6, and the sampling formula there asserts that any $f \in V_m$ can be written as

$$f = \sum_{n\in\mathbb{Z}^d} f(n/2^m)s(2^m. - n), \tag{2.1}$$

where the convergence is in the $L^2(\mathbb{R}^d)$ sense and where s is the unique *sampling function* of V_0, i.e., it satisfies $s(n) = \delta_{0,n},\ n \in \mathbb{Z}^d$, see also [Wal94], [Zay93]. Theorem 2.6 is best formulated in the setting of the following function space.

Definition 2.2. *a.* Let $C^{1,\infty}(\mathbb{Z}^d, \mathbb{T}^d)$ be the subspace of bounded continuous functions $f \in C_b(\mathbb{R}^d)$ with the property that

$$\|f\|_{C^{1,\infty}(\mathbb{Z}^d,\mathbb{T}^d)} = \sup_{x\in\mathbb{T}^d} \sum_{n\in\mathbb{Z}^d} |f(x+n)| < \infty.$$

If $f \in C^{1,\infty}(\mathbb{Z}^d, \mathbb{T}^d)$, then $f \in L^1(\mathbb{R}^d)$ and f is uniformly continuous on $\mathbb{R}^d$.

b. Spaces of the type $C^{1,\infty}(\mathbb{Z}^d, \mathbb{T}^d)$ are mixed norm spaces and they have a long history in analysis. The norm for $f \in C^{1,\infty}(\mathbb{Z}^d, \mathbb{T}^d)$ is first global (over $\mathbb{Z}^d$), then local (over $\mathbb{T}^d$); and it is defined in the tradition of the *Benedek-Panzone spaces* [BP61]. Even earlier, Wiener, e.g., [Wie33], defined what we call *Wiener amalgam spaces* as a natural setting in which to prove Tauberian theorems, see [FS85] for a survey and [BBE89] for applications. The mixed norm for Wiener amalgam spaces is first local, then global.

More recently than [BP61] but in order to deal with refinement equations and the construction of pre-wavelets, Micchelli [Mic91] introduced the analogue of $C^{1,\infty}(\mathbb{Z}^d, \mathbb{T}^d)$ for measurable functions, cf., [JM91]. Also a generalization of $C^{1,\infty}(\mathbb{Z}^d, \mathbb{T}^d)$ was introduced in [BZ97] to prove a class of Poisson summation formulas (PSFs), one of which is stated below in Theorem 2.5. Finally, Fischer [Fis95] used precisely $C^{1,\infty}(\mathbb{Z}^d, \mathbb{T}^d)$ in her study of MRA and approximation theory.

Example 2.3. *a.* The space $C_c(\mathbb{R}^d)$ of compactly supported functions in $\mathbb{R}^d$ is contained in $C^{1,\infty}(\mathbb{Z}^d, \mathbb{T}^d)$.

b. Let f be continuous on $\mathbb{R}^d$ and assume there is $\varepsilon > 0$ such that $f(x) = 0(|x|^{-(d+\varepsilon)}),\ \ |x| \to \infty$. Then $f \in C^{1,\infty}(\mathbb{Z}^d, \mathbb{T}^d)$.

c. $L^{p,q}(\mathbb{Z}^d, \mathbb{T}^d)$ is the space of measurable functions f on $\mathbb{R}^d$ with the property that

$$\|f\|_{L^{p,q}(\mathbb{Z}^d,\mathbb{T}^d)} = \left(\int_{\mathbb{T}^d} \left(\sum_{n\in\mathbb{Z}^d} |f(x+n)|^p \right)^{q/p} dx \right)^{1/q},$$

where $p, q \in [1,\infty]$ and with the usual modification in the case p or q is ∞. Similarly we can define $C^{p,q}(\mathbb{Z}^d, \mathbb{T}^d)$.

d. Clearly $L^{1,\infty}(\mathbb{Z}^d, \mathbb{T}^d) \subseteq L^{1,q}(\mathbb{Z}^d, \mathbb{T}^d) \subseteq L^{1,1}(\mathbb{Z}^d, \mathbb{T}^d) = L^1(\mathbb{R}^d)$. Also, $C^{1,\infty}(\mathbb{Z}^d, \mathbb{T}^d) \subseteq C^{2,\infty}(\mathbb{Z}^d, \mathbb{T}^d) \subseteq L^2(\mathbb{R}^d)$.

The following theorem is a consequence of definitions and the inequality,

$$\left| \sum_{n\in\mathbb{Z}^d} \langle f, \tau_n\varphi\rangle \tau_n\varphi(x) \right| \leq \left(\|\varphi\|_{L^\infty(\mathbb{R}^d)} \|\varphi\|_{C^{1,\infty}(\mathbb{Z}^d,\mathbb{T}^d)} \right)^{\frac{1}{2}} \|f\|_{L^2(\mathbb{R}^d)}, \tag{2.2}$$

which itself is a consequence of Hölder's inequality in the case $\{\tau_n\varphi\}$ is an ONB for its closed linear span.

Theorem 2.4. *Let $\{V_m\}$ be an MRA of $L^2(\mathbb{R}^d)$ with scaling function $\varphi \in C^{1,\infty}(\mathbb{Z}^d, \mathbb{T}^d)$. (In particular, φ is a uniformly continuous element of $L^1(\mathbb{R}^d) \bigcap C^{2,\infty}(\mathbb{Z}^d, \mathbb{T}^d) \subseteq L^1(\mathbb{R}^d) \bigcap L^2(\mathbb{R}^d)$.)*

a. $\forall m \in \mathbb{Z},\ V_m \subseteq C_b(\mathbb{R}^d) \bigcap L^2(\mathbb{R}^d)$.

b. $\forall m \in \mathbb{Z}$ *and* $\forall f \in V_m$,

$$f = \sum_{n \in \mathbb{Z}^d} \langle f, \varphi_{m,n} \rangle \varphi_{m,n}$$

in $L^2(\mathbb{R}^d)$ *and uniformly on* $\mathbb{R}^d$*, cf., the conclusion of Theorem 1.1.*

c. Each V_m *is a reproducing kernel Hilbert space. In fact, for each* $x \in \mathbb{R}^d$*, the point-evaluation functional,* $V_m \to \mathbb{C}$*, defined by* $f \mapsto f(x)$*, not only satisfies the norm inequality* $|f(x)| \leq C\|f\|_{L^2(\mathbb{R}^d)}$ *but* C *is independent of* x*, e.g., (2.2).*

d. *For each* m*, the reproducing kernel* K_m *associated with part c is*

$$K_m(x, y) = \sum_{n \in \mathbb{Z}^d} \varphi_{m,n}(x) \overline{\phi_{m,n}(y)},$$

and, hence,

$$\forall f \in V_m, \;\; f(y) = \langle f(x), K_m(x, y) \rangle.$$

The validity of a uniform sampling expansion is often equivalent to a corresponding Poisson summation formula (PSF), e.g., [Ben97]. The appropriate PSF for $C^{1,\infty}(\mathbb{Z}, \mathbb{T})$ is found in [BZ97] (Theorem 5).

Theorem 2.5. *Let* $\{k_\lambda : \lambda = 1, 2, ...\} \subseteq A(\mathbb{T})$ *be an approximate identity on* $\mathbb{T}$*. For any* $f \in C^{1,\infty}(\mathbb{Z}, \mathbb{T})$,

$$\sum_{n \in \mathbb{Z}} f(x + n) = \lim_{\lambda \to \infty} \sum_{n \in \mathbb{Z}} k_\lambda^\vee[n] \widehat{f}(n) e^{2\pi i n x},$$

with convergence being uniform convergence on compact sets.

The following result is well known, not too difficult to prove, and central to our interest. The construction we know goes back to Schoenberg [Sch73] from 1946. Walter [Wal94] used the methods in the context of wavelets and sampling, cf., [Fis95]. Before stating the theorem, note that

$$\varphi \in C^{1,\infty}(\mathbb{Z}^d, \mathbb{T}^d) \;\text{ implies }\; \Phi(\gamma) = \sum_{m \in \mathbb{Z}^d} \varphi(m) e^{-2\pi i \langle m, \gamma \rangle} \in A(\mathbb{T}^d). \tag{2.3}$$

We shall make use of Wiener's lemma on the inversion of absolutely convergent Fourier series, see [Ben97] (page 201): *if* $\Psi \in A(\mathbb{T}^d)$ *is non-vanishing on* $\mathbb{T}^d$*, then* $1/\Psi \in A(\mathbb{T}^d)$. For the case of Φ defined in (2.3) by $\varphi \in C^{1,\infty}(\mathbb{Z}^d, \mathbb{T}^d)$ we write

$$\frac{1}{\Phi(\gamma)} \equiv \sum_{k \in \mathbb{Z}^d} c_k e^{-2\pi i \langle k, \gamma \rangle}. \tag{2.4}$$

The nonvanishing of Φ is a nonvacuous hypothesis in what follows, as is illustrated in the case of the quadratic $B-$spline.

Theorem 2.6. *Let* $\{V_m\}$ *be an MRA of* $L^2(\mathbb{R}^d)$ *with scaling function* $\varphi \in C^{1,\infty}(\mathbb{Z}^d, \mathbb{T}^d)$*, and assume the corresponding Fourier series* $\Phi \in A(\mathbb{T}^d)$ *is non-vanishing on* $\mathbb{T}^d$*.*

a. There is a unique function $s \in V_0$ *with the property that*

$$\forall m \in \mathbb{Z} \;\text{ and }\; \forall f \in V_m \subseteq C_b(\mathbb{R}^d) \cap L^2(\mathbb{R}^d), \quad f = \frac{1}{2^{dm/2}} \sum_{n \in \mathbb{Z}^d} f\left(\frac{n}{2^m}\right) s_{m,n}$$

in $L^2(\mathbb{R}^d)$ *and absolutely and, therefore, uniformly on* $\mathbb{R}^d$. *The function* s *is the* sampling function *of* $\{V_m\}$.

b. Further, $s \in C^{1,\infty}(\mathbb{Z}^d, \mathbb{T}^d)$,

$$s(x) = \sum_{k\in\mathbb{Z}^d} c_k \varphi(x-k) \quad \text{and} \quad \widehat{s}(\gamma) = \frac{\widehat{\varphi}(\gamma)}{\Phi(\gamma)},$$

and

$$\forall k \in \mathbb{Z}^d, \quad s(k) = \begin{cases} 1, & \textit{if } \ k = 0, \\ 0, & \textit{if } \ k \neq 0. \end{cases} \tag{2.5}$$

c. $\{\tau_n s : n \in \mathbb{Z}^d\}$ *is an exact frame (Riesz basis) for* V_0. *In fact, from part b, we have*

$$\inf_{\gamma\in\mathbb{T}^d} \frac{1}{|\Phi(\gamma)|^2} \leq \sum_{n\in\mathbb{Z}^d} |\widehat{s}(\gamma+n)|^2 \leq \sup_{\gamma\in\mathbb{T}^d} \frac{1}{|\Phi(\gamma)|^2}.$$

Example 2.7. Let $\varphi \in C^{1,\infty}(\mathbb{Z}^d, \mathbb{T}^d)$ and let $\{\tau_n\varphi\}$ be an ONB for its closed linear span V_0.

a. If $f \in V_0$ and the sampled values $\{f(n)\}$ of f are known, then it is easy to compute $\{\langle f, \tau_n\varphi\rangle\}$ by the following method which is also used in Theorem 2.6. We know that

$$f(n) = \sum_k \langle f, \tau_k\varphi\rangle \varphi(n-k),$$

and, hence,

$$\sum_{n\in\mathbb{Z}^d} f(n) e^{-2\pi i\langle n,\gamma\rangle} = \Phi(\gamma) \sum_{k\in\mathbb{Z}^d} \langle f, \tau_k\varphi\rangle e^{-2\pi i\langle k,\gamma\rangle}.$$

If $\Phi \in A(\mathbb{T}^d)$ never vanishes, then we divide by Φ and obtain

$$\langle f, \tau_k\varphi\rangle = \sum_{n\in\mathbb{Z}^d} f(n) c_{k-n}.$$

Thus, as noted in [Dau92] (page 156), knowledge of sampled values $f(n)$ accompanied by MRA algorithms lead to the computation of $\{\langle f, \psi_{m,n}\rangle\}$, where ψ is an MRA wavelet associated with φ.

b. In determining finer properties of the sampling function s of $\{V_m\}$, it becomes relevant to estimate the coefficients $\{c_k\}$ in (2.4), assuming $\Phi \in A(\mathbb{T}^d)$ is non-vanishing. In general, this is difficult to do without further conditions, see Theorem 3.5. On the other hand, in an elementary calculation, Wiener showed that if $2|\varphi(0)| > \|\Phi\|_{A(\mathbb{T})}$, then $1/\Phi \in A(\mathbb{T})$, e.g., [Ben97] (Section 3.4). With this assumption, one can make some quantitative observations concerning the c_k.

c. This material, and subsequent results for Lipschitz spaces contained in $C^{1,\infty}(\mathbb{Z}, \mathbb{T})$, require functions $\varphi \in C^{1,\infty}(\mathbb{Z}, \mathbb{T})$ for which $|\Phi| > 0$ on $\mathbb{T}$ and $\{\tau_n\varphi\}$ is an ONB, or even a Riesz basis, of its closed span. Such examples do exist, e.g., Meyer's wavelets [Mey90], [Dau92], [AK00]; but more systematic constructions are required, cf., [BSW93], [HWW96].

3. Estimating the decay of the sampling function

Recall from Theorem 2.6 that, with natural hypotheses, if a scaling function φ is in $C^{1,\infty}(\mathbb{Z}^d, \mathbb{T}^d)$, then its corresponding sampling function $s \in C^{1,\infty}(\mathbb{Z}^d, \mathbb{T}^d)$. Further, we know that if f is a continuous function on $\mathbb{R}^d$ satisfying the decay condition $f(x) = 0(|x|^{-(d+\varepsilon)}), |x| \to \infty$, then $f \in C^{1,\infty}(\mathbb{Z}^d, \mathbb{T}^d)$. In this section we let $d = 1$, and we shall prove (Theorem 3.6) that, with natural hypotheses, if a scaling function $\varphi \in C^{1,\infty}(\mathbb{Z}, \mathbb{T})$ satisfies the decay condition $\varphi(x) = 0(|x|^{-(\eta+a)})$, $|x| \to \infty$, for an integer $\eta \geq 2$ and $\alpha \in [0, 1)$, then its sampling function satisfies the same condition. We should point out that the hypothesis $\eta \geq 2$ is an artifact of our techniques.

Definition 3.1. Let $\alpha \geq 0$. A function $F : \mathbb{T} \to \mathbb{C}$ satisfies a *Lipschitz condition of order* α if

$$\exists C > 0 \text{ such that } \forall \gamma, \lambda \in \mathbb{T},\ |F(\gamma) - F(\lambda)| \leq C|\gamma - \lambda|^{\alpha}.$$

In this case we write $F \in Lip\, \alpha$.

We shall need the following well known properties of $Lip\, \alpha$, whose proofs are found in [Lor44], [Kat68] (page 25), and [Bar64], (pages 17, 38, and 71).

Proposition 3.2. *a. If $F \in L^1(\mathbb{T})$ and $0 < \alpha < 1$, then*

$$F^{\vee}[m] = 0(|m|^{-(1+\alpha)}),\ |m| \to \infty, \quad \text{implies} \quad F \in Lip\, \alpha. \tag{3.1}$$

b. If $F \in C(\mathbb{T})$ and the modulus of continuity $\omega(F, \delta)$ is defined as

$$\omega(F, \delta) = \sup_{\gamma, 0 \leq \lambda \leq \delta} |F(\gamma + \lambda) - F(\gamma)|,$$

then

$$F \in Lip\, \beta \text{ if and only if } \omega(F, \delta) = 0(\delta^{\beta}), \quad \delta \to 0, \tag{3.2}$$

and

$$\forall m \in \mathbb{Z}\backslash\{0\}, \qquad |F^{\vee}[m]| \leq \frac{1}{2}\omega\left(F, \frac{1}{2|m|}\right). \tag{3.3}$$

Lemma 3.3. *Let $\{u_m : m \in \mathbb{Z}\}$, $\{v_m : m \in \mathbb{Z}\}$, $\{w_m : m \in \mathbb{Z}\} \subseteq \mathbb{C}$ and let $\mu, \nu > 0$ satisfy the following conditions: $\{w_m\} \in l^p(\mathbb{Z})$, $p \in (1, \infty]$,*

$$|u_m| = 0(|m|^{-\mu}) \text{ and } |v_m| = 0(|m|^{-\nu}|w_m|),\ |m| \to \infty, \tag{3.4}$$

$$\nu \geq \mu, \tag{3.5}$$

and

$$\mu > \frac{p-1}{p} = \frac{1}{p'}, \quad p \in (1, \infty),\ \text{resp.},\ \mu > 1,\ p = \infty. \tag{3.6}$$

Then

$$\sum_{j \in \mathbb{Z}} u_j v_{m-j} = 0(|m|^{-\mu}),\ |m| \to \infty. \tag{3.7}$$

Proof. By (3.4) assume there are $C_\mu, C_\nu > 0$ such that

$$|u_m| \le C_\mu |m|^{-\mu} \text{ and } |v_m| \le C_\nu |m|^{-\nu} |w_m|.$$

Pick $m \neq 0$ and let λ_m be the non-negative integer for which $\lambda_m \le |m|/2 < \lambda_m + 1$. Then

$$\left| \sum_{j\in\mathbb{Z}} u_j v_{m-j} \right| \le C_\nu \sum_{|j|\le\lambda_m} |u_j||w_{m-j}| \frac{1}{|m-j|^\nu} + C_\mu \sum_{|j|>\lambda_m} \frac{1}{|j|^\mu} |v_{m-j}|, \tag{3.8}$$

where $|\omega_j| \le C$ for all $j \in \mathbb{Z}$. If $|j| \le \lambda_m$, then $|m-j| \ge |m| - |j| \ge |m| - \lambda_m \ge |m|/2$; and if $|j| > \lambda_m$ then $|j|^{-\mu} \le (\lambda_m + 1)^{-\mu} \le (|m|/2)^{-\mu}$. Thus the right side of (3.8) is bounded by

$$C_\nu \left(\frac{2}{|m|}\right)^\nu \sum_{|j|\le\lambda_m} |u_j||w_{m-j}| + CC_\mu \left(\frac{2}{|m|}\right)^\mu \sum_{|j|>\lambda_m} |v_{m-j}|$$

$$\le \left(\frac{2}{|m|}\right)^\mu \left[C_\nu \|\{u_j\}\|_{l^{p'}(\mathbb{Z})} \|\{w_j\}\|_{l^p(\mathbb{Z})} + CC_\mu \|\{v_j\}\|_{l^1(\mathbb{Z})} \right]. \tag{3.9}$$

Note that $\{v_j\} \in l^1(\mathbb{Z})$ by (3.5) and since $\{w_j\} \in l^p(\mathbb{Z})$; and $\{u_j\} \in l^{p'}(\mathbb{Z})$ by (3.6). (3.7) follows by combining (3.8) and (3.9). □

Lemma 3.4. *Let $\eta \ge 2$ be an integer and let $\alpha \in [0, 1)$. Define $\Phi(\gamma) = \sum_{m\in\mathbb{Z}} \varphi(m) e^{-2\pi i m\gamma}$, where $\{\varphi(m) : m \in \mathbb{Z}\} \subseteq \mathbb{C}$ satisfies the condition*

$$\varphi(m) = 0(|m|^{-(\eta+\alpha)}), \qquad |m| \to \infty. \tag{3.10}$$

Assume $|\Phi| > 0$ on $\mathbb{T}$. Then Φ, $1/\Phi^j \in A(\mathbb{T})$ for all $j \ge 0$; and, if

$$1/\Phi^2(\gamma) = \sum_{m\in\mathbb{Z}} d(m) e^{-2\pi i m\gamma},$$

then

$$d_m = 0(|m|^{-(\eta+\alpha)+1}), \qquad |m| \to \infty.$$

Proof. *i.* Let $0 < \alpha < 1$. By Wiener's lemma on the inversion on non-vanishing absolutely convergent Fourier series, we have $1/\Phi \in A(\mathbb{T})$; and since $A(\mathbb{T})$ is a Banach algebra under pointwise multiplication, we have $1/\Phi^j \in A(\mathbb{T})$ for all $j \ge 0$. Also, for fixed $0 \le k \le \eta - 1$

$$\Phi^{(k)}(\gamma) = \sum_{m\in\mathbb{Z}^d} b_m^{(k)} e^{-2\pi i m\gamma} \in A(\mathbb{T}), \tag{3.11}$$

where $b_m^{(k)} = (-2\pi i m)^k \varphi(m)$, since

$$\forall m \neq 0, \ |b_m^{(k)}| \le K|m|^{-(\eta+\alpha)+k} \le K|m|^{-1-\alpha}. \tag{3.12}$$

In fact, (3.11) is a consequence of (3.12) and the fact that the derived series converges uniformly, cf., [Ben97] (page 181).

ii. With $0 < \alpha < 1$, Lorentz' result (Proposition 3.2a) coupled with (3.12) allow us to conclude that $\Phi^{(k)} \in Lip\,\alpha$ for $0 \leq k \leq \eta - 1$. Also note that products of $Lip\,\alpha$ functions on $\mathbb{T}$ are $Lip\,\alpha$, and that

$$\forall j \geq 1,\ (\Phi^{-j})^{(1)} = -j\Phi^{-(j+1)}\Phi^{(1)} \in A(\mathbb{T}).$$

Further, the product of $F \in Lip\,\alpha$ and Φ^{-j} is also in $Lip\,\alpha$ on $\mathbb{T}$. This is a consequence of the fact that $\alpha \leq 1$ and that $(\Phi^{-j})^{(1)} \in L^\infty(\mathbb{T})$, which allows us to find a uniform constant K in the following calculation:

$$|(F\Phi^{-j})(\gamma) - (F\Phi^{-j})(\lambda)| \leq C|\gamma - \lambda|^\alpha \|\Phi^{-j}\|_{L^\infty(\mathbb{T})} + K|\gamma - \lambda|\|F\|_{L^\infty(\mathbb{T})} \leq C_1|\gamma - \lambda|^\alpha.$$

iii. We now write d_m, $m \neq 0$, as

$$d_m = \int_0^1 \Phi(\gamma)^{-2} e^{2\pi i m\gamma} d\gamma = \frac{2}{2\pi i m}\int_0^1 \Phi(\gamma)^{-3}\Phi^{(1)}(\gamma)e^{2\pi i m\gamma}d\gamma,$$

$$= \frac{-2}{(2\pi i m)^2}\int_0^1 \left[-3\Phi(\gamma)^{-4}\Phi^{(1)}(\gamma)^2 + \Phi(\gamma)^{-3}\Phi^{(2)}(\gamma)\right] e^{2\pi i m\gamma}d\gamma$$

$$= \frac{-2(-1)^{\eta-1}}{(2\pi i m)^{\eta-1}}\int_0^1 \Psi(\gamma)e^{2\pi i m\gamma}d\gamma, \tag{3.13}$$

where $\Psi \in Lip\,\alpha$ by the discussion in part *ii.* Since $\Psi \in Lip\,\alpha$, we invoke Proposition 3.2, viz., (3.2) and (3.3), in conjunction with (3.13) to assert that

$$|d_m| \leq C|m|^{-(\eta-1)}\omega(\Psi, (2|m|)^{-\alpha}) = 0(|m|^{-(\eta-1)-\alpha}), \quad |m| \to \infty.$$

This is the desired result for $\eta \geq 2$ and $0 < \alpha < 1$.

iv. Now let $\alpha = 0$. In this case we have

$$d_m = \frac{C}{(2\pi i m)^{\eta-2}}\int_0^1 \Psi_1(\gamma)e^{2\pi i m\gamma}d\gamma,$$

where $\Psi_1 \in A(\mathbb{T})$ by the discussion in part *ii.* We observe that $\Phi^{(\eta-1)}$ is in $L^2(\mathbb{T})$, noting that $|b_m^{(\eta-1)}| = 0(|m|^{-1})$ as $|m| \to \infty$. Thus, it is easy to see that the derivative of Ψ_1 exists and is in $L^2(\mathbb{T})$, and that

$$d_m = \frac{C}{(2\pi i m)^{\eta-1}}\int_0^1 \Psi_1^{(1)}(\gamma)e^{2\pi i m\gamma}d\gamma.$$

Thus, $|d_m| = 0(|m|^{-(\eta-1)})$ as $|m| \to \infty$. □

Theorem 3.5. *Let $\eta \geq 2$ be an integer and let $\alpha \in [0,1)$. Define $\Phi(\gamma) = \sum_{m\in\mathbb{Z}} \varphi(m)e^{-2\pi i m\gamma}$, where $\{\varphi(m) : m \in \mathbb{Z}\} \subseteq \mathbb{C}$ satisfies condition (3.10). Assume $|\Phi| > 0$ on $\mathbb{T}$ and set*

$$\forall m \in \mathbb{Z}, \qquad c_m = \int_0^1 \frac{1}{\Phi(\gamma)}e^{2\pi i m\gamma}d\gamma$$

as in (2.4). Then

$$c_m = 0(|m|^{-(\eta+\alpha)}),\ |m| \to \infty. \tag{3.14}$$

Proof. *i.* First we consider the case $0 < \alpha < 1$. Clearly $\Phi \in A(\mathbb{T})$ by (3.10), and hence $1/\Phi \in A(\mathbb{T})$ since $|\Phi| > 0$ on $\mathbb{T}$. Using the notation from Lemma 3.4 we have

$$\Phi^{(1)}(\gamma) = \sum_{m\in\mathbb{Z}} b_m^{(1)} e^{-2\pi i m\gamma} \in A(\mathbb{T}),$$

where

$$|b_m^{(1)}| = |-2\pi i m\varphi(m)| \leq K|m|^{-(\eta+\alpha)+1}, \quad m \neq 0.$$

For $m \neq 0$ we compute

$$c_m = \frac{1}{2\pi i m}\int_0^1 \Phi(\gamma)^{-2}\Phi^{(1)}(\gamma)e^{2\pi i m\gamma}d\gamma.$$

Thus, by Parseval's theorem, e.g., [Ben97] (Theorem 3.4.12), we have

$$c_m = \frac{1}{2\pi i m}\sum_{j\in\mathbb{Z}} b_j^{(1)} d_{m-j}, \tag{3.15}$$

where $\{d_m\}$ is defined in Lemma 3.4, and where

$$|b_j^{(1)}| = 0(|j|^{-(\eta+\alpha)+1}) \text{ and } |d_j| = 0(|j|^{-(\eta+\alpha)+1}),\ |j| \to \infty, \tag{3.16}$$

by Lemma 3.4 and the aforementioned bound on $\{b_j^{(1)}\}$. We now apply Lemma 3.3 directly to the data in (3.16): $\mu = \eta + \alpha - 1$, $\nu = \eta + \alpha - 1$, $p = \infty$, $w_j = 1$, $u_j = b_j^{(1)}$, and $v_j = d_j$ for each j. Then $\mu = \nu \geq 1 + a > 1$ and so (by Lemma 3.3)

$$\sum_{j\in\mathbb{Z}} b_j^{(1)} d_{m-j} = 0(|m|^{-(\eta+\alpha)+1}),\ |m| \to \infty.$$

Combining this estimate with (3.15) we obtain the desired result.

ii. Now let $\eta \geq 2$ and $\alpha = 0$. The case $\eta > 2$ is an immediate consequence of part *i* by setting $\alpha = 0$, so only the case $\eta = 2$ and $\alpha = 0$ remains to be checked. For this case we follow the proof in part *i* and obtain (3.15), i.e.,

$$c_m = \frac{1}{2\pi i m}\sum_{j\in\mathbb{Z}} b_j^{(1)} d_{m-j},$$

where $b_j^{(1)} = 0(|j|^{-1}),\ |j| \to \infty$, and

$$|d_j| = \left|\int_0^1 \Phi(\gamma)^{-2}e^{-2\pi i j\gamma}d\gamma\right| = \frac{1}{\pi|j|}\left|\int_0^1 \Phi(\gamma)^{-3}\Phi^{(1)}(\gamma)e^{-2\pi i j\gamma}d\gamma\right| = \frac{1}{|j|\pi}|\delta_j|,\ j \neq 0.$$

Clearly, the δ_j designate the Fourier coefficients of the function $\Phi^{(1)}\Phi^{-3}$, which, in turn, is an element of $L^2(\mathbb{T})$. This latter assertion is a consequence of the facts that $\Phi^{-3} \in C(\mathbb{T})$ and that the Fourier coefficients $b_j^{(1)}$ of $\Phi^{(1)}$ are dominated by $|j|^{-1}, j \neq 0$. We can now apply Lemma 3.3 by setting $\mu = 1$, $\nu = 1$, $p = 2$, $w_j = \delta_j$, $u_j = b_j^{(1)}$, and $v_j = d_j$ for each j; and we obtain

$$\sum_{j\in\mathbb{Z}} b_j^{(1)} d_{m-j} = 0(|m|^{-1}), \quad |m| \to \infty.$$

Combining this with (3.15) we obtain $|c_m| = 0(|m|^{-2}),\ |m| \to \infty$. □

Theorem 3.6. *Let $\eta \geq 2$ be an integer and let $\alpha \in [0, 1)$. Further, let $\{V_m\}$ be an MRA of $L^2(\mathbb{R})$ with scaling function $\varphi \in C^{1,\infty}(\mathbb{Z}, \mathbb{T})$ satisfying the condition*

$$\varphi(x) = 0(|x|^{-(\eta+\alpha)}),\ |x| \to \infty. \tag{3.17}$$

Assume the corresponding Fourier series $\Phi \in A(\mathbb{T})$, with Fourier coefficients $\{\varphi(m)\}$, is non-vanishing on $\mathbb{T}$. Then the sampling function s of $\{V_m\}$ satisfies the condition

$$s(x) = 0(|x|^{-(\eta+\alpha)}),\ |x| \to \infty. \tag{3.18}$$

Proof. From Theorem 2.6 we know that $s(x) = \sum_{m\in\mathbb{Z}} c_m\varphi(x - m)$, and because of (3.17) and Theorem 3.5, we know that $c_m = 0(|m|^{-(\eta+\alpha)}), |m| \to \infty$. For any $x \in \mathbb{R}\backslash\{0\}$, let $m_x \in \mathbb{N} \cup \{0\}$ satisfy $0 \leq m_x \leq |x|/2 < m_x + 1$. Then

$$s(x) = \sum_{|m|\leq m_x} c_m\varphi(x - m) + \sum_{|m|>m_x} c_m\varphi(x - m) = s_1(x) + s_2(x), \tag{3.19}$$

where s_1, resp., s_2, is the first, resp., second, sum in (3.19).

If $|m| \leq m_x$, then

$$|x - m| \geq ||x| - |m|| = |x| - |m| \geq |x| - m_x \geq |x|/2,$$

and so

$$|s_1(x)| \leq C_\varphi \left(\frac{|x|}{2}\right)^{-(\eta+\alpha)} \sum_{|m|\leq m_x} |c_m|, \tag{3.20}$$

where C_φ is a bound from (3.17) independent of x. Further, because of Theorem 3.5 we also have the estimate

$$|s_2(x)| \leq C_c(m_x + 1)^{-(\eta+\alpha)} \sum_{m\in\mathbb{Z}} |\varphi(x - m)| < C_c \left(\frac{|x|}{2}\right)^{-(\eta+\alpha)} \sup_{t\in\mathbb{R}} \sum_{m\in\mathbb{Z}} |\varphi(t - m)|, \tag{3.21}$$

where C_c is a bound from (3.14) (Theorem 3.5) independent of x. The *sup* term in (3.21) is the $C^{1,\infty}(\mathbb{Z}, \mathbb{T})$ norm of φ.

Combining (3.20) and (3.21), we obtain (3.18) with a bound

$$C_s = 2^{\eta+\alpha} \left(C_\varphi\|1/\Phi\|_{A(\mathbb{T})} + C_c\|\varphi\|_{C^{1,\infty}(\mathbb{Z},\mathbb{T})}\right).$$

□

The bound C_s at the end of the proof of Theorem 3.6 grows with η, and this is meaningful in that it reveals the trade-off between the bound implicit in (3.18) and the rate of decay asserted by (3.18). The trade-off can be made more technically explicit with the goal of applying Theorem 4.2 (see below), even when ϕ has compact support and, therefore, s has fast decay.

4. A local error for wavelet expansions

In this section we shall estimate the local error for wavelet expansions.

Definition 4.1. Let $N_1, N_2 \in \mathbb{Z}$ for $N_1 < N_2$, let $h = 2^{-m}$ for $m \in \mathbb{Z}$, and let $d = 1$. The *local error* of the sampling formula (2.1), i.e.,

$$f = \frac{1}{2^{m/2}} \sum_{n \in \mathbb{Z}} f\left(\frac{n}{2^m}\right) s_{m,n},$$

is

$$E_{N_1,N_2} f(x) \equiv f(x) - \sum_{n=N_1}^{N_2} f(nh) s(h^{-1}x - n) \ , \tag{4.1}$$

where $x \in [N_1 h, N_2 h]$.

We consider an arbitrary interval $I \subseteq \mathbb{R}$, and let $a = \inf\{x \in I\},\ b = \sup\{x \in I\}$, and

$$M_a = \sup\{n \in \mathbb{Z} : nh \le a\} \text{ and } M_b = \inf\{n \in \mathbb{Z} : nh \ge b\}. \tag{4.2}$$

We define

$$P_a = M_a - N_a \text{ and } P_b = N_b - M_b, \tag{4.3}$$

where $N_a < M_a$ and $N_b > M_b$ are integers. Notationally, the left side of (1.2) can be written as

$$\sup_{x \in I} |E_{N_a,N_b} f(x)| = \sup_{x \in I} \left| f(x) - \sum_{n=N_a}^{N_b} f(n/2^m) s(2^m x - n) \right|.$$

The following result does not depend on the results of Section 3, nor does it depend essentially on the MRA setup except to determine a sampling function; but it does depend significantly on the methods in [HT62], [Jag66].

Theorem 4.2. *Let $\{V_m\}$ be an MRA of $L^2(\mathbb{R})$ with scaling function $\varphi \in C^{1,\infty}(\mathbb{Z}, \mathbb{T})$. Assume the corresponding Fourier series $\Phi \in A(\mathbb{T})$ is non-vanishing on $\mathbb{T}$, and let $s \in V_0 \subseteq C_b(\mathbb{R}) \cap L^2(\mathbb{R})$ be the corresponding uniquely determined sampling function, see Theorem 2.6. Let $I \subseteq \mathbb{R}$ be a bounded interval with associated constants a, b, M_a, M_b as in (4.2), where $h = 2^{-m}$, and let P_a, P_b be as in (4.3) for $N_a < M_a$ and $N_b > M_b$. Assume*

$$s(x) = 0(|x|^{-\beta}), \quad |x| \to \infty \tag{4.4}$$

for some fixed $\beta > 1/2$, and set

$$A_N(h^{-1}x) = \left(\sum_{n > N} |n - h^{-1}x|^{-2\beta} \right)^{\frac{1}{2}}.$$

For each $m \in Z$ and $f \in V_m$ define

$$K_N = K_N(f) = \left(h \sum_{n > N} |f(nh)|^2 \right)^{\frac{1}{2}} \text{ and } \ L_N = L_N(f) = \left(h \sum_{n < N} |f(nh)|^2 \right)^{\frac{1}{2}}.$$

a. There is a constant C, arising from (4.4) and independent of $f \in V_m$, such that

$$\forall x \in [N_a h, N_b h],\ |E_{N_a,N_b} f(x)| \le C h^{-\frac{1}{2}} \left[K_{N_b}(f) A_{N_b}(h^{-1}x) + L_{N_a}(f) A_{-N_a}(-h^{-1}x) \right],$$

noting $I \subseteq [M_a h, M_b h] \subseteq [N_a h, N_b h]$.

b. Let C be the constant of part a. Then

$$\sup_{x\in I}|E_{N_a,N_b}f(x)| \le C\frac{h^{\beta-1}}{\sqrt{2\beta-1}}\left[\frac{K_{N_b}(f)}{(P_bh+r_b)^{\beta-1/2}}+\frac{L_{N_a}(f)}{(P_ah+r_a)^{\beta-1/2}}\right],$$

where $r_a = a - M_ah \le h$ and $r_b = M_bh - b \le h$ are non-negative.

Proof. *a.* If $x = N_ah$ or $x = N_bh$ then the local error (4.1) is 0. For $x \in (N_ah, N_bh)$ we have

$$|E_{N_a,N_b}f(x)| \le C\left(\sum_{n>N_b}|f(nh)|(n-h^{-1}x)^{-\beta}+\sum_{n<N_a}|f(nh)|(h^{-1}x-n)^{-\beta}\right)$$

by (4.1), where C is the constant arising from (4.4). Thus, part a is a consequence of the Cauchy-Schwarz inequality, since, for example, if $x \le N_bh$ and $n > N_b$ then $n - xh^{-1} > 0$.

b. We first note that if $x \in I$ then

$$A_{N_b}(h^{-1}x) \le \left(\int_{N_b}^{\infty}\frac{1}{(y-h^{-1}x)^{2\beta}}dy\right)^{\frac{1}{2}} = \frac{h^{\beta-1/2}}{\sqrt{2\beta-1}}\frac{1}{(N_bh-x)^{\beta-1/2}},$$

with a similar estimate for $A_{-N_a}(-h^{-1}x)$. Combining this with part a, we have

$$|E_{N_a,N_b}f(x)| \le \frac{Ch^{\beta-1}}{\sqrt{2\beta-1}}\left[\frac{K_{N_b}(f)}{(N_bh-x)^{\beta-1/2}}+\frac{L_{N_a}(f)}{(x-N_ah)^{\beta-1/2}}\right] \tag{4.5}$$

if $x \in I$. Now, note that if $x \in I$ then

$$N_bh - x = N_bh - M_bh + M_bh - x \ge P_bh + M_bh - b = P_bh + r_b$$

and

$$x - N_ah \ge r_a + P_ah.$$

The function $g_{N,h,\beta}(x) = (Nh-x)^{-\beta+1/2}$ is strictly increasing for $x < Nh$, and $g_{N,h,\beta}(-x)$ is strictly decreasing for $x > -Nh$. Thus, for $a \le x \le b$, we have

$$g_{N_b,h,\beta}(x) \le g_{N_b,h,\beta}(b) = g_{P_b,h,\beta}(-r_b)$$

and

$$g_{N_a,h,\beta}(x) \le g_{N_a,h,\beta}(-a) = g_{P_a,h,\beta}(-r_a).$$

Substituting these inequalities into (4.5) yields part b. □

Corollary 4.3. *Assume the setup, hypotheses, and notation of Theorem 4.2. Let $P = P_a = P_b \ge 1$, and, for each m, let K_m be the reproducing kernel of V_m, e.g., Theorem 2.4. Then*

$$\sup_{x\in I}|E_{N_a,N_b}f(x)| \le \frac{2C}{\sqrt{2\beta-1}}\frac{1}{P^{\beta-1/2}}\left(\sum_{n\in\mathbb{Z}}|f(nh)|^2\right)^{\frac{1}{2}}$$

$$\le \frac{2C}{\sqrt{2\beta-1}}\frac{1}{P^{\beta-1/2}}\left(\sum_{n\in\mathbb{Z}}K_m(nh,nh)\right)^{\frac{1}{2}}\|f\|_{L^2(\mathbb{R})}.$$

Proof. The first inequality is elementary from Theorem 4.2*b* using the same value of C as in Theorem 4.2. The second inequality is a consequence of the definition of the reproducing kernel:

$$\begin{aligned}\sum_{n\in\mathbb{Z}} |f(nh)|^2 &= \sum_{n\in\mathbb{Z}} |\langle f(x), K_m(x,nh)\rangle|^2 \\ &\leq \|f\|^2_{L^2(\mathbb{R})} \sum_{n\in\mathbb{Z}} \langle K_m(x,nh), K_m(x,nh)\rangle \\ &= \|f\|^2_{L^2(\mathbb{R})} \sum_{n\in\mathbb{Z}} K_m(nh,nh).\end{aligned}$$

□

We now invoke Theorem 3.6 in order to obtain the conclusion of Theorem 4.2, but, because of the MRA setting, to obtain this conclusion by only making a decay hypothesis on the scaling function φ (and not on its associated sampling function s). The proof of Theorem 4.4 is immediate from Theorems 3.6 and 4.2.

Theorem 4.4. *Assume the setup, hypotheses, and notation of Theorem 4.2 exclusive of the assumption (4.4) on the decay of the sampling function. Let* $\eta \geq 2$ *and* $\alpha \in [0,1)$, *and assume*

$$\varphi(x) = 0(|x|^{-(\eta+\alpha)}), \quad |x| \to \infty.$$

There is $C > 0$ *such that*

$$\forall f \in V_m,\ \sup_{x\in I} |E_{N_a,N_b} f(x)| \leq C \frac{h^{(\eta+\alpha)-1}}{\sqrt{2(\eta+\alpha)-1}} \left[\frac{K_{N_b}(f)}{(P_b h + r_b)^{(\eta+\alpha)-1/2}} + \frac{L_{N_a}(f)}{(P_a h + r_a)^{(\eta+\alpha)-1/2}} \right].$$

5. A sampling theorem based on the Gabor transform

In this section we give a sampling theorem for bandlimited functions, using the Gabor transform for $L^2(\mathbb{R})$. For more details see [BF94], as well as [Ben89] for the $L^\infty(\mathbb{R})$ case. For other fully developed aspects of the Gabor theory see [FS98], [Grö00].

Definition 5.1. *a.* Let $g \in L^2(\mathbb{R}) \bigcap L^\infty(\mathbb{R})$. The Gabor transform G_g of $f \in L^2(\mathbb{R})$ is defined to be

$$(G_g f)_x(\omega) = \langle f(t), g(t-x)e^{-2\pi i t\omega}\rangle = \int_{-\infty}^{\infty} f(t)\overline{g(t-x)} e^{2\pi i t\omega} dt.$$

b. If $G_g f$ is the Gabor transform of $f \in L^2(\mathbb{R})$, then the inverse Gabor transform of f is given by

$$((G_g f)_x(\omega))^{-1}(t) = \frac{1}{\|g\|^2_{L^2(\mathbb{R})}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (G_g f)_x(\omega) g(t-x) e^{-2\pi i t\omega} dx d\omega, \tag{5.1}$$

and $f(t) = ((G_g f)_x(\omega))^{-1}(t)$ a.e. on $\mathbb{R}$.

Theorem 5.2. *Let* $f \in PW_\Omega$ *and* $g \in PW_\Gamma$. *If* $P(y) \equiv \int_{-\infty}^{\infty} g(y+x)\overline{g(x)}dx$ *is the positive definite autocorrelation of* g, *then*

$$f(t) = \frac{1}{2\Lambda\|g\|_2^2} \sum_{n\in\mathbb{Z}} f\left(\frac{n}{2\Lambda}\right) P\left(t - \frac{n}{2\Lambda}\right) \frac{\sin[2\pi(t-n/(2\Lambda))]}{\pi(t-n/(2\Lambda))}, \text{ a.e.,} \tag{5.2}$$

where $\Lambda = \Gamma + \Omega$.

Proof. For a fixed $x \in \mathbb{R}$ the Gabor transform $(G_g f)_x(\omega)$ may be considered as the Fourier transform of $f(t)\overline{g(t-x)}$. Using Parseval's formula, we have

$$(G_g f)_x(\omega) = e^{2\pi i x\omega} \int_{|\gamma|\leq\Omega} \widehat{f}(\gamma)\overline{\widehat{g}(\omega-\gamma)} e^{-2\pi i x\gamma} d\gamma e^{2\pi i x\omega}.$$

Obviously, $f(t)\overline{g(t-x)} \in PW_\Lambda$, where $\Lambda = \Gamma + \Omega$. Thus, the Fourier transform $(G_g f)_x(\omega)$ may be developed in a Fourier series, as follows:

$$(G_g f)_x(\omega) = \frac{1}{2\Lambda} \sum_{n\in\mathbb{Z}} \left(\int_{|\gamma|\leq\Lambda} (G_g f_x)(\gamma) e^{-i\pi n\gamma/\Lambda} d\gamma \right) e^{i\pi n\omega/\Lambda} = \frac{1}{2\Lambda} \sum_{n\in\mathbb{Z}} f(\frac{n}{2\Lambda})\overline{g(\frac{n}{2\Lambda}-x)} e^{\pi i n\omega/\Lambda},$$

where $\omega \in [-\Lambda, \Lambda)$. The second equality follows from the definition of $G_g f$ for fixed x and the Fourier inversion theorem. Each series converges a.e. on $\widehat{\mathbb{R}}$. After applying Fubini's theorem, we substitute the second series into (5.1) to obtain

$$f(t) = \frac{1}{\|g\|^2_{L^2(\mathbb{R})}} \int_{-\infty}^{\infty} g(t-x) \int_{-\Lambda}^{\Lambda} (G_g f)_x(\omega) e^{-2\pi i t\omega} d\omega dx \tag{5.3}$$

$$= \frac{1}{2\Lambda\|g\|^2_{L^2(\mathbb{R})}} \int_{-\infty}^{\infty} \sum_{n\in\mathbb{Z}} f(\frac{n}{2\Lambda})\overline{g(\frac{n}{2\Lambda}-x)} g(t-x) \int_{|\omega|\leq\Lambda} e^{-2\pi(t-n/(2\Lambda))\omega} d\omega dx, \ a.e.$$

Interchanging summation and integration in the last term of (5.3), noting that term by term integration of Fourier series of integrable functions is allowed whether they converge or not, we see that (5.3) becomes

$$f(t) = \frac{1}{2\Lambda\|g\|^2_{L^2(\mathbb{R})}} \sum_{n\in\mathbb{Z}} f(\frac{n}{2\Lambda}) P(t-\frac{n}{2\Lambda}) \frac{\sin[2\pi(t-n/(2\Lambda))]}{\pi(t-n/(2\Lambda))}, \ a.e.,$$

and this is the desired result. □

Corollary 5.3. *Let $K(t,x) = g(t-x)$ be the (sinc) reproducing kernel of the space PW_Γ, where $(x,t) \in \mathbb{R}^2$. If $\Lambda = \Gamma + \Omega$, then*

$$\forall f \in PW_\Omega, \ f(t) = \frac{1}{2\Lambda\|g\|^2_{L^2(\mathbb{R})}} \sum_{n\in\mathbb{Z}} f(\frac{n}{2\Lambda}) g(t-\frac{n}{2\Lambda}) \frac{\sin[2\pi\Lambda(t-n/(2\Lambda))]}{\pi(t-n/(2\Lambda))}, \ a.e.$$

Proof. If $h \in PW_\Gamma$, then

$$h(x) = \langle h(\cdot), K(\cdot, x)\rangle = \langle h(\cdot), g(\cdot - x)\rangle = \int_{-\infty}^{\infty} h(y)\overline{g(y-x)} dy.$$

Thus, if P is defined as in Theorem 5.2, then $P(t-n/(2\Lambda)) = g(t-n/(2\Lambda))$, and the result follows from (5.2). □

Corollary 5.4. *Assume the setting and notation of Theorems 4.2 and 5.2. Let*

$$s(t) = \frac{P(t)\sin(2\pi t\Lambda)}{2\pi t\Lambda\|g\|^2_{L^2(\mathbb{R})} t},$$

where $P(y) = \int_{-\infty}^{\infty} g(y+x)\overline{g(x)}dx$, *and suppose* $P(y) = 0(|y|^{-\beta}),\ |y| \to \infty$. *If* $h = (2\Lambda)^{-1}$, *then*

$$\forall f \in PW_\Omega, \quad \sup_{x\in I} |E_{N_a,N_b} f(x)| \leq C \frac{h^\beta - 1}{\sqrt{2\beta - 1}} \left[\frac{K_{N_b}(f)}{(P_b h + r_b)^{\beta - 1/2}} + \frac{L_{N_a}(f)}{(P_a h + r_a)^{\beta - 1/2}} \right].$$

Proof. PW_Ω can be viewed as a V_0. The decay hypothesis on P implies (4.4); and the definition of s yields the sampling formula $f = \sum f(n/(2\Lambda))\tau_{n/2\Lambda} s$ by Theorem 5.2. This formula, coupled with the proof of Theorem 4.2, gives the result. □

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