Contemporary Mathematics

Complex Sigma-Delta quantization algorithms for finite frames

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ABSTRACT. We record a $\mathbb{C}$-alphabet case for $\Sigma\Delta$ quantization for finite frames. The basic theory and error analysis are presented in the case of bounded frame variation for a given sequence $\{F_N\}$ for frames for $\mathbb{C}^d$. If bounded frame variation is not available for the given sequence, then there is still a satisfactory error analysis depending on the correct permutation of each F_N. An algorithm is designed to construct this permutation, and relevant simulations and examples are given.

1. Introduction

1.1. Background and Outline. Sigma-Delta ($\Sigma\Delta$) quantization has a long history going back to mid-20th century. It has become an industry standard along with Pulse Code Modulation (PCM). See [**2**] for a recent analysis and comparison of PCM and $\Sigma\Delta$ quantization.

A mathematical analysis of $\Sigma\Delta$ quantization for bandlimited functions is due to Daubechies and DeVore [**8**]. An analysis of PCM for frames is due to Goyal, Kelner, Kovačević, Vetterli and Thao in [**9**], [**10**], [**11**], Jimenez, L.Wang and Y.Wang in [**15**], and Cvetković in [**7**]. The natural sequel, $\Sigma\Delta$ for frames, was developed by Powell, Yilmaz, and one of the authors [**4**]. The frames in this case were finite unit-norm tight frames (FUNTFs) for $\mathbb{R}^d$.

In this paper we define and develop the theory of $\Sigma\Delta$ quantization for FUNTFs for $\mathbb{C}^d$. We formulate the finite quantization alphabet for $\mathbb{C}^d$ in Section 2. This is well-known to the experts, including Powell, Yilmaz, and Lammers - so well-known that it is difficult to find in the literature. Parroting the theory in [**4**] for $\mathbb{C}^d$ is also routine, but we do it briefly for context in Section 3.

A major theoretical issue in $\Sigma\Delta$ quantization theory is to quantify the error $\|x - \widetilde{x}\|$ between a given signal $x \in \mathbb{C}^d$ and its non-linearly quantized approximant $\widetilde{x}$. This error analysis for $\mathbb{C}^d$ is the topic of Section 4 for sequences $\{F_N\}$ of frames which have so-called bounded frame variation, a notion introduced in [**4**]. It is natural to do the error analysis of $\|x - \widetilde{x}\|$ for more general sequences of frames which arise in applications. Yang Wang brought this issue to our attention, and he had the brilliant observation of showing the relevance of the traveling salesman problem [**17**].

Section 5 and 6 are the main parts of this paper and are inspired by [**17**]. We construct an algorithm in Section 5 which allows us to choose permutations of general frames F_N, so that $\|x - \widetilde{x}\|_N$ is small for x and $\widetilde{x}$ expanded in terms of F_N. In particular, $\lim_{N\to\infty} \|x - \widetilde{x}\|_N = 0$ with explicit rate of convergence. We view this algorithm as extremely effective and useful in light of the examples and simulations in Section 6.

1.2. Overview of Frames.

Definition 1.1. Let H be a separable Hilbert space. A set $F = \{e_j\}_{j\in J} \subseteq H$ is a *frame* for H if

$$\exists A, B > 0 \quad \text{such that} \quad \forall x \in H, \quad A\|x\|^2 \leq \sum_{j\in J} |\langle x, e_j\rangle|^2 \leq B\|x\|^2.$$

A frame F is a *tight frame* if we can choose $A = B$. If, in addition, each e_j is unit-norm, we say that F is a *unit-norm tight frame*. Also,

(1) The linear function $L : H \to \ell^2(J)$ defined by

$$Lx = (\langle x, e_j\rangle)_{j\in J}$$

is the *Bessel map* or the *analysis operator*.

(2) The Hilbert space adjoint of L, L^*, is the *synthesis operator*, and it is given by

$$\forall c = (c_j)_{j\in J} \in \ell^2(J), \quad L^*c = \sum_{j\in J} c_j e_j.$$

(3) $S = L^*L$ is the *frame operator*, and it is given by

$$Sx = \sum_{j\in J} \langle x, e_j\rangle e_j.$$

S is positive definite, and it satisfies $AI \leq S \leq BI$.

(4) Let $\widetilde{e_j} = S^{-1}e_j$. Then, $\widetilde{F} = \{\widetilde{e_j}\}_{j\in J}$ is the *canonical dual frame* of F, and it satisfies

$$\forall x \in H, \quad x = \sum_{j\in J} \langle x, \widetilde{e_j}\rangle e_j.$$

In particular, if $\widetilde{L}$ is the Bessel map of $\widetilde{F}$, then $L^*\widetilde{L} = I$.

(5) $G = LL^*$ is the *Grammian operator*.

Definition 1.2. A frame $F = \{e_j\}_{j=1}^N$ for $\mathbb{F}^d$ with finite number of elements is called a *finite frame*. If F is unit-norm and tight, then it is called a finite unit-norm tight frame (FUNTF).

Theorem 1.3. *a. Any spanning set in $\mathbb{F}^d$ is a frame for $\mathbb{F}^d$.*
b. If $F = \{e_j\}_{j=1}^N$ is a FUNTF for $\mathbb{F}^d$ with frame constant A, then $A = N/d$.

Proof. *a.* Let $\{e_j\}_{j=1}^N$ be a spanning set for $\mathbb{F}^d$. Since $\{x \in \mathbb{F}^d : \|x\| = 1\}$ is compact, there is an x_0, $\|x_0\| = 1$ at which the continuous function $\sum_{j=1}^N |\langle x, e_j\rangle|^2$ attains its minimum value. Let $A = \sum_{j=1}^N |\langle x_0, e_j\rangle|^2$.

$$A = 0 \Rightarrow \forall j = 1, \ldots, N, \quad \langle x_0, e_j\rangle = 0 \Rightarrow x \notin span\{e_j\}_{j=1}^N.$$

Therefore, $A > 0$. Moreover,

$$\forall x \in \mathbb{F}^d, \quad A \leq \sum_{j=1}^{N} |\langle \frac{x}{\|x\|}, e_j \rangle|^2 \Rightarrow A\|x\|^2 \leq \sum_{j=1}^{N} |\langle x, e_j \rangle|^2.$$

On the other hand,

$$\forall x \in \mathbb{F}^d, \quad \sum_{j=1}^{N} |\langle x, e_j \rangle|^2 \leq \|x\|^2 \sum_{j=1}^{N} \|e_j\|^2.$$

We can choose $B = \sum_{j=1}^{N} \|e_j\|^2$.

b. If F is a finite frame, L, S and G can be represented as matrices. In particular, since F is a FUNTF, $S = AI$ and $G = (\langle e_i, e_j \rangle)$. Using a property of traces,

$$Ad = trace(S) = trace(G) = \sum_{j=1}^{N} \|e_j\|^2 = N.$$

□

2. Alphabet in Complex Case

Let $K \in \mathbb{N}$ and $\delta > 0$. Given the quantization alphabet

$$\mathcal{A}_K^{\delta} = \left\{ \left(k + \frac{1}{2}\right)\delta + i\ell\delta : k = -K, \ldots, K-1 \text{ and } \ell = -K, \ldots, K \right\},$$

consisting of $2K(2K+1)$ elements, we define the $2K(2K+1)$-*level uniform scalar quantizer* with stepsize δ by

$$Q(u) = \arg \min_{q \in \mathcal{A}_K^{\delta}} |u - q|_{\max},$$

where $|\cdot|_{\max}$ is defined by $|z|_{\max} = \max\{|Re(z)|, |Im(z)|\}$ for all $z \in \mathbb{C}$. See Figure 1 for $\mathcal{A}_3^{\delta}$. Thus, $Q(u)$ is the element of the alphabet which is closest to u in the norm $|\cdot|_{\max}$. If there are at least two elements of $\mathcal{A}_K^{\delta}$ that are equally close to u, then we let $Q(u)$ be the element with the largest real part. If these equally closest elements have the same real parts, then we let $Q(u)$ be the element with the largest imaginary part.

Remark 2.1. In a way, the complex scalar quantizer here performs PCM quantization for $\mathbb{C}$ identified with $\mathbb{R}^2$. In this regard, and with a view to the bit budget, the combination of two scalar quantizers for real and imaginary parts is also a viable point of view. In fact, each complex inner product space has the structure of a real inner product space by taking the real part of the iner product; and norms are not affected by switching between the two inner products. Consequently, it is possible to investigate the intrinsically complex properties of complex Hilbert spaces in the context of real Sigma-Delta quantization. This is not a topic in this paper, but the contribution herein can be seen as a sine qua non for implementing quantization methods such as tiling [**12, 18, 3**], in conjunction with algebraic and analytic tools from the theory of several complex variables.

Secondly, a general form of the results in this paper still holds if we replace the alphabet $\mathcal{A}_K^{\delta}$ with any other alphabet of the form

$$\{(k\omega_1 + \ell\omega_2)\delta : k = -K, \ldots, K-1 \text{ and } \ell = -K, \ldots, K\},$$

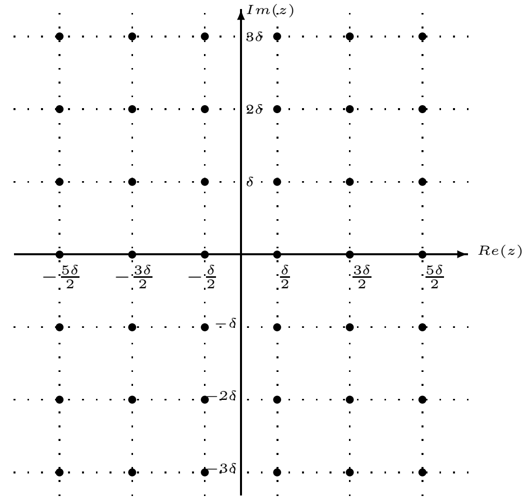

FIGURE 1. The quantization alphabet for $K = 3$.

for fixed $\omega_1, \omega_2 \in \mathbb{C}\backslash\{0\}$, which are linearly independent over $\mathbb{R}$.

DEFINITION 2.2. Given $K \in \mathbb{N}, \delta > 0$, and the corresponding quantization alphabet $\mathcal{A}_K^\delta$ and the scalar quantizer Q with stepsize δ, let $\{x_n\}_{n=1}^N \subseteq \mathbb{C}$, and let p be a permutation of $\{1, \ldots, N\}$. The associated *first order* $\Sigma\Delta$ *quantization* is defined by the iteration

$$\begin{aligned} u_n &= u_{n-1} + x_{p(n)} - q_n, \\ q_n &= Q(u_{n-1} + x_{p(n)}), \end{aligned} \tag{2.1}$$

where u_0 is a specified constant. The first order $\Sigma\Delta$ quantizer produces the *quantized sequence* $\{q_n\}_{n=1}^N$, and an auxiliary sequence $\{u_n\}_{n=0}^N$ of *state variables.*

3. Stability and Error Estimates

PROPOSITION 3.1. *Let K be a positive integer, let $\delta > 0$, and consider the $\Sigma\Delta$ system defined by (2.1). If $|u_0|_{\max} \leq \delta/2$ and for all $n = 1, \ldots, N$,*

$$|x_n|_{\max} \leq \Big(K - \frac{1}{2}\Big)\delta,$$

then for all $n = 1, \ldots, N$,

$$|u_n|_{\max} \leq \frac{\delta}{2}.$$

PROOF. Without loss of generality assume that p is the identity permutation. The proof proceeds by induction. The base case, $|u_0|_{\max} \leq \delta/2$, holds by assumption. Next, suppose that $|u_{j-1}|_{\max} \leq \delta/2$. This implies that $|u_{j-1} + x_j|_{\max} \leq K\delta$, and hence, by (2.1) and the definition of Q,

$$|u_j|_{\max} = |u_{j-1} + x_j - Q(u_{j-1} + x_j)|_{\max} \leq \frac{\delta}{2}.$$

$\square$

Definition 3.2. Let $F = \{e_n\}_{n=1}^N$ be a finite frame for $\mathbb{C}^d$, and let p be a permutation of $\{1, \ldots, N\}$. We define the *variation* of the frame F with respect to p as

$$\sigma(F, p) = \sum_{n=1}^{N-1} \|e_{p(n)} - e_{p(n+1)}\|. \tag{3.1}$$

We now derive error estimates for the $\Sigma\Delta$ scheme in Definition 2.2 for $K \in \mathbb{N}$ and $\delta > 0$. Given a frame $F = \{e_n\}_{n=1}^N$ for $\mathbb{C}^d$, a permutation p of $\{1, \ldots, N\}$, and $x \in \mathbb{C}^d$, we shall calculate how well the quantized expansion

$$\widetilde{x} = \sum_{n=1}^{N} q_n S^{-1} e_{p(n)}$$

approximates the frame expansion

$$x = \sum_{n=1}^{N} x_{p(n)} S^{-1} e_{p(n)}, \quad x_{p(n)} = \langle x, e_{p(n)} \rangle.$$

Here, $\{q_n\}_{n=1}^N$ is the quantized sequence which is calculated using Definition 2.2 and the sequence $\{x_{p(n)}\}_{n=1}^N$ of frame coefficients. We now state our first result on the approximation error, $\|x - \widetilde{x}\|$. We shall use $\|\cdot\|_{\text{op}}$ to denote the operator norm induced by the Euclidean norm, $\|\cdot\|$, for $\mathbb{C}^d$.

Theorem 3.3. *Given the $\Sigma\Delta$ scheme of Definition 2.2. Let $F = \{e_n\}_{n=1}^N$ be a finite unit-norm frame for $\mathbb{C}^d$, let p be a permutation of $\{1, \ldots, N\}$, let $|u_0|_{\max} \leq \delta/2$, and let $x \in \mathbb{C}^d$ satisfy $\|x\| \leq (K - 1/2)\delta$. The approximation error $\|x - \widetilde{x}\|$ satisfies*

$$\|x - \widetilde{x}\| \leq \sqrt{2}\|S^{-1}\|_{\text{op}} \Big(\sigma(F, p)\frac{\delta}{2} + |u_N|_{\max} + |u_0|_{\max} \Big),$$

where S^{-1} is the inverse frame operator for F.

Proof. We have

$$\begin{aligned} x - \widetilde{x} &= \sum_{n=1}^{N} (x_{p(n)} - q_n) S^{-1} e_{p(n)} = \sum_{n=1}^{N} (u_n - u_{n-1}) S^{-1} e_{p(n)} \\ &= \sum_{n=1}^{N-1} u_n S^{-1} (e_{p(n)} - e_{p(n+1)}) + u_N S^{-1} e_{p(N)} - u_0 S^{-1} e_{p(1)}. \end{aligned}$$

Since $\|x\| \leq (K - 1/2)\delta$ it follows that for all $n = 1, \ldots, N$,

$$|x_n|_{\max} = |\langle x, e_n \rangle|_{\max} \leq |\langle x, e_n \rangle| \leq \|x\|\|e_n\| \leq \Big(K - \frac{1}{2}\Big)\delta.$$

Thus, by Proposition 3.1 and the inequality $|\cdot| \leq \sqrt{2}\,|\cdot|_{\max}$,

$$\begin{aligned} \|x - \widetilde{x}\| &\leq \sum_{n=1}^{N-1} \frac{\delta}{2}\sqrt{2}\|S^{-1}\|_{\text{op}}\|e_{p(n)} - e_{p(n+1)}\| + |u_N|\|S^{-1}\|_{\text{op}} + |u_0|\|S^{-1}\|_{\text{op}} \\ &= \|S^{-1}\|_{\text{op}} \Big(\sigma(F, p)\frac{\delta}{2}\sqrt{2} + |u_0| + |u_N| \Big) \\ &\leq \sqrt{2}\|S^{-1}\|_{\text{op}} \Big(\sigma(F, p)\frac{\delta}{2} + |u_N|_{\max} + |u_0|_{\max} \Big). \end{aligned}$$

$\square$

Corollary 3.4. *Given the $\Sigma\Delta$ scheme of Definition 2.2. Let $F = \{e_n\}_{n=1}^N$ be a finite unit-norm tight frame for $\mathbb{C}^d$, let p be a permutation of $\{1, \ldots, N\}$, let $|u_0|_{\max} \leq \delta/2$, and let $x \in \mathbb{C}^d$ satisfy $\|x\| \leq (K - 1/2)\delta$. The approximation error $\|x - \widetilde{x}\|$ satisfies*

$$\|x - \widetilde{x}\| \leq \sqrt{2}\frac{d}{N}\left(\sigma(F,p)\frac{\delta}{2} + |u_N|_{\max} + |u_0|_{\max}\right).$$

Proof. Since F is a unit-norm tight frame for $\mathbb{C}^d$, it follows that F has frame bound $A = N/d$ and $S^{-1} = A^{-1}I$, where I is the identity operator. Hence,

$$\|S^{-1}\|_{\text{op}} = \left\|\frac{d}{N}I\right\|_{\text{op}} = \frac{d}{N}.$$

The result now follows from Theorem 3.3. □

Corollary 3.5. *Given the $\Sigma\Delta$ scheme of Definition 2.2. Let $F = \{e_n\}_{n=1}^N$ be a unit-norm tight frame for $\mathbb{C}^d$, let p be a permutation of $\{1, \ldots, N\}$, let $|u_0|_{\max} \leq \delta/2$, and let $x \in \mathbb{C}^d$ satisfy $\|x\| \leq (K - 1/2)\delta$. The approximation error $\|x - \widetilde{x}\|$ satisfies*

$$\|x - \widetilde{x}\| \leq \sqrt{2}\frac{\delta d}{2N}\left(\sigma(F,p) + 2\right).$$

Proof. Apply Corollary 3.4 and Proposition 3.1. □

Recall that the initial state u_0 in (2.1) can be chosen arbitrarily. It is therefore convenient to take $u_0 = 0$, because this will give a smaller constant in the approximation error given by Theorem 3.3. Likewise, the error constant can be improved if one has more information about the final state variable, $|u_N|_{\max}$. It is somewhat surprising that for *zero sum* frames the value of $|u_N|_{\max}$ is determined by whether the frame has an even or odd number of elements.

Theorem 3.6. *Given the $\Sigma\Delta$ scheme of Definition 2.2. Let $F = \{e_n\}_{n=1}^N$ be a unit-norm tight frame for $\mathbb{C}^d$ and assume that F satisfies the zero sum condition*

$$\sum_{n=1}^N e_n = 0. \tag{3.2}$$

Additionally, set $u_0 = 0$ in (2.1). *Then*

$$|u_N|_{\max} = \begin{cases} 0, & \text{if } N \text{ is even;} \\ \delta/2 \text{ or } 0, & \text{if } N \text{ is odd.} \end{cases} \tag{3.3}$$

Proof. Note that (2.1) implies

$$u_N = u_0 + \sum_{n=1}^N x_n - \sum_{n=1}^N q_n = \sum_{n=1}^N x_n - \sum_{n=1}^N q_n. \tag{3.4}$$

Next, (3.2) implies

$$\sum_{n=1}^N x_n = \sum_{n=1}^N \langle x, e_n\rangle = \langle x, \sum_{n=1}^N e_n\rangle = 0. \tag{3.5}$$

By the definition of the quantization alphabet $\mathcal{A}_K^\delta$ each q_n is of the form $a + bi$ where a is an odd integer multiple of $\delta/2$ and b is an integer multiple of δ.

If N is even it follows that $\sum_{n=1}^{N} q_n$ is of the form $x + yi$ where each x and y is an integer multiple of δ. Thus, by (3.4) and (3.5), $|u_N|_{\max}$ is an integer multiple of δ. However, $|u_N|_{\max} \leq \delta/2$ by Proposition 3.1, so that we have $u_N = 0$.

If N is odd it follows that $\sum_{n=1}^{N} q_n$ is of the form $p + qi$ where p is an odd integer multiple of $\delta/2$ and q is an integer multiple of δ. Thus, by (3.4) and (3.5), $|u_N|_{\max}$ is either an odd integer multiple of $\delta/2$ or an integer multiple of δ. However, $|u_N|_{\max} \leq \delta/2$ by Proposition 3.1, so that we have $|u_N|_{\max} = \delta/2$ or $|u_N|_{\max} = 0$. □

Corollary 3.7. *Given the $\Sigma\Delta$ scheme of Definition 2.2. Let $F = \{e_n\}_{n=1}^{N}$ be a unit-norm tight frame for $\mathbb{C}^d$ and assume that F satisfies the zero sum condition (3.2). Let p be a permutation of $\{1, \ldots, N\}$ and let $x \in \mathbb{C}^d$ satisfy $\|x\| \leq (K-1/2)\delta$. Additionally, set $u_0 = 0$ in (2.1). Then the approximation error $\|x - \widetilde{x}\|$ satisfies*

$$\|x - \widetilde{x}\| \leq \begin{cases} \sqrt{2}\frac{\delta d}{2N}\sigma(F,p), & \text{if } N \text{ is even;} \\ \sqrt{2}\frac{\delta d}{2N}\left(\sigma(F,p) + 1\right), & \text{if } N \text{ is odd.} \end{cases} \tag{3.6}$$

Proof. Apply Corollary 3.4, Theorem 3.6, and Proposition 3.1. □

Corollary 3.7 shows that as a consequence of Theorem 3.6, one has smaller constants in the error estimate for $\|x - \widetilde{x}\|$ when the frame size N is even.

4. Families of Frames with Bounded Variation

One way to obtain arbitrarily small approximation error, $\|x - \widetilde{x}\|$, using the estimates of the previous section is simply to fix a frame and decrease the quantizer step size δ towards zero, while letting $K = \lceil 1/\delta \rceil$. PCM quantization heavily depends on this method. By Corollary 3.5, as δ goes to 0, the approximation error goes to zero. However, this approach is not always desirable. For example, in analog-to-digital (A/D) conversion of bandlimited signals, it can be quite costly to build quantizers with very high resolution, i.e., small δ and large K, e.g., [**8**]. Instead, many practical applications involving A/D and D/A converters make use of oversampling, i.e., redundant frames, and use low resolution quantizers, e.g., [**14**]. To be able to adopt this type of approach for the quantization of finite frame expansions, it is important to be able to construct families of frames with uniformly bounded frame variation.

Let us begin by making the observation that if $F = \{e_n\}_{n=1}^{N}$ is a finite unit-norm frame and p is any permutation of $\{1, 2, \ldots, N\}$, then $\sigma(F,p) \leq 2(N-1)$. However, this bound is too weak to be of much use since substituting it into an error bound such as the even case of (3.6) only gives

$$\|x - \widetilde{x}\| \leq \frac{\delta d(N-1)}{N}.$$

In particular, this bound does not go to zero as N gets large, i.e., as one chooses more redundant frames. On the other hand, if one finds a family of frames and a sequence of permutations, such that the resulting frame variations are uniformly bounded, then one is able to obtain an approximation error of order $1/N$.

The most natural examples of unit-norm tight frames in $\mathbb{C}^d, d \geq 2$ are the harmonic frames, e.g., see [**9**], [**19**], [**11**].

Example 4.1 (Harmonic frames). If we form an $N \times d$ matrix using any d columns of the $N \times N$ Discrete Fourier Transform (DFT) matrix $(e^{2\pi ijk/N})_{j,k=0}^{N-1}$, then the rows of this $N \times d$ matrix, up to a multiplication by a proper constant, constitute a FUNTF for $\mathbb{C}^d$. In other words, if $\{k_1, \ldots, k_d\} \subseteq \{0, 1, \ldots, N-1\}$ and if we let

$$e_j = \frac{1}{\sqrt{d}} \left[e^{-2\pi i j \cdot k_1/N}, e^{-2\pi i j \cdot k_2/N}, \ldots, e^{-2\pi i j \cdot k_d/N} \right], \tag{4.1}$$

then $\{e_j\}_{j=0}^{N-1}$ is a FUNTF for $\mathbb{C}^d$. In fact, for any $x \in \mathbb{C}^d$,

$$\begin{aligned}
\sum_{j=0}^{N-1} |\langle x, e_j \rangle|^2 &= \sum_{j=0}^{N-1} \sum_{l,l'=0}^{N-1} x[l+1]\overline{x[l'+1]e_j[l]}e_j[l'] \\
&= \sum_{l,l'=0}^{N-1} x[l+1]\overline{x[l'+1]}\frac{1}{d} \sum_{j=0}^{N-1} e^{2\pi i j \cdot k_l/N} e^{-2\pi i j \cdot k_{l'}/N} \\
&= \frac{N}{d} \sum_{l=0}^{N-1} |x[l+1]|^2 \\
&= \frac{N}{d} \|x\|^2.
\end{aligned}$$

In particular, when $k_l = l$ for $l = 1, \ldots, d$, we shall follow the notation of [**19**], and denote this particular harmonic frame by H_N^d, although the terminology "harmonic frame" is not specifically used there.

We shall show that harmonic frames have uniformly bounded frame variation. Let $\{k_1, \ldots, k_d\} \subseteq \{0, 1, \ldots, N-1\}$, and let $\{e_j\}_{j=0}^{N-1}$ be as in (4.1). Then, for $j = 0, \ldots, N-2$, we have

$$\begin{aligned}
\|e_j - e_{j+1}\|^2 &= \frac{1}{d} \sum_{l=1}^{d} |e^{-2\pi i k_l j/N} - e^{-2\pi i k_l (j+1)/N}|^2 \\
&= \frac{1}{d} \sum_{l=1}^{d} \left(e^{-2\pi i k_l j/N} - e^{-2\pi i k_l (j+1)/N} \right) \left(e^{2\pi i k_l j/N} - e^{2\pi i k_l (j+1)/N} \right) \\
&= \frac{1}{d} \sum_{l=1}^{d} 2 - \left(e^{2\pi i k_l/N} + e^{-2\pi i k_l/N} \right) \\
&= \frac{2}{d} \sum_{l=1}^{d} 1 - \cos \frac{2\pi k_l}{N} = \frac{4}{d} \sum_{l=1}^{d} \sin^2 \frac{\pi k_l}{N} \\
&\leq \frac{4}{d} \sum_{l=1}^{d} \left(\frac{\pi k_l}{N} \right)^2 = \frac{4\pi^2}{dN^2} \cdot \sum_{l=1}^{d} k_l^2
\end{aligned}$$

Thus, if p is the identity permutation, then

$$\sigma(\{e_j\}_{j=0}^{N-1}, p) = \sum_{j=0}^{N-2} \|e_j - e_{j+1}\| \leq (N-1) \frac{2\pi}{N\sqrt{d}} \left(\sum_{l=1}^{d} k_l^2 \right)^{1/2} \leq \frac{2\pi}{\sqrt{d}} \left(\sum_{l=1}^{d} k_l^2 \right)^{1/2}. \tag{4.2}$$

In particular,

$$\sigma(H_N^d, p) \leq \frac{2\pi}{\sqrt{d}} \left(\sum_{l=1}^{d} l^2 \right)^{1/2} = \frac{2\pi}{\sqrt{d}} \sqrt{\frac{d(d+1)(2d+1)}{6}} \leq \frac{2\pi}{\sqrt{3}}(d+1).$$

Harmonic frames also satisfy (3.2), if none of the k_l is equal to zero. In fact, since $k_l \neq 0$, $e^{-2\pi i k_l/N} \neq 1$ for each $l = 1, \dots, d$. Then,

$$\sum_{j=0}^{N-1} e_j[l] = \sum_{j=0}^{N-1} e^{-2\pi i k_l j/N} = \frac{1 - e^{-2\pi i k_l j}}{1 - e^{-2\pi i k_l/N}} = 0$$

for every $l = 1, \dots, d$.

We can now derive error estimates for $\Sigma\Delta$ quantization of harmonic frames for the identity permutation. If we set $u_0 = 0$ and assume that $x \in \mathbb{C}^d$ satisfies $\|x\| \leq (K - 1/2)\delta$, then combining (4.2), Corollaries 3.1, 3.4, and 3.7, and the fact that harmonic frames satisfy (3.2) gives

$$\|x - \widetilde{x}\| < \begin{cases} \sqrt{2}\frac{\delta d}{2N}\frac{2\pi}{\sqrt{d}}C_d, & \text{if } N \text{ is even,} \\ \sqrt{2}\frac{\delta d}{2N}\left(\frac{2\pi}{\sqrt{d}}C_d + 1\right), & \text{if } N \text{ is odd,} \end{cases}$$

where $C_d = \sqrt{\sum_{l=1}^{d} k_l^2}$

5. Frame Variation and Algorithms

In [**17**], Wang proves the following theorem, and asserts that the proof leads to an algorithm for finding an ordering that reduces the frame variation for frames for $\mathbb{R}^d$.

THEOREM 5.1. *Let $S = \{v_j\}_{j=1}^N \subseteq [-\frac{1}{2}, \frac{1}{2}]^d$ with $d \geq 3$. There exists a permutation p of $\{1, \dots, N\}$ such that*

$$\sum_{j=1}^{N-1} \|v_{p(j)} - v_{p(j+1)}\| \leq 2\sqrt{d+3}N^{1-\frac{1}{d}} - 2\sqrt{d+3}.$$

COROLLARY 5.2. *Let $F = \{e_j\}_{j=1}^N$ be a unit norm frame for $\mathbb{F}^d$, $d \geq 3$. There exists a permutation p of $\{1, \dots, N\}$ such that*

i. *if $\mathbb{F} = \mathbb{R}$, then $\sigma(F, p) \leq 4\sqrt{d+3}\ N^{1-\frac{1}{d}} - 4\sqrt{d+3}$,*
ii. *if $\mathbb{F} = \mathbb{C}$, then $\sigma(F, p) \leq 4\sqrt{2d+3}\ N^{1-\frac{1}{2d}} - 4\sqrt{2d+3}$.*

PROOF. *i.* If $\mathbb{F} = \mathbb{R}$, then $\{\frac{1}{2}e_j\}_{j=1}^N \subseteq [-\frac{1}{2}, \frac{1}{2}]^d$. The result follows from Theorem 5.1.

ii. If $\mathbb{F} = \mathbb{C}$, let $w_j = (Re(e_j), Im(e_j)) \in \mathbb{R}^{2d}$. Then,

(5.1)
$$\sigma(\{w_j\}_{j=1}^N, p) = \sum_{j=1}^{N-1} \|w_{p(j)} - w_{p(j+1)}\| = \sum_{j=1}^{N-1} \|e_{p(j)} - e_{p(j+1)}\| = \sigma(\{e_j\}_{j=1}^N, p),$$

$\|w_j\| = 1$, and $\{\frac{1}{2}w_j\}_{j=1}^N \subseteq [-\frac{1}{2}, \frac{1}{2}]^{2d}$. Thus, by (5.1) and Theorem 5.1, there is a permutation p satisfying

$$\sigma(F, p) = 2\sigma(\{\frac{1}{2}w_j\}_{j=1}^N, p) \leq 4\sqrt{2d+3}N^{1-\frac{1}{2d}} - 4\sqrt{2d+3}.$$

$\square$

Combining the results of Corollary 5.2 and Corollary 3.5, we obtain the following result.

THEOREM 5.3. *Let $F = \{e_n\}_{n=1}^N$ be a FUNTF for $\mathbb{C}^d$, $|u_0|_{\max} \leq \delta/2$, and let $x \in \mathbb{C}^d$ satisfy $\|x\| \leq (K-1/2)\delta$. Then, there exists a permutation p of $\{1,\ldots,N\}$ such that the approximation error $\|x-\widetilde{x}\|$ satisfies*

$$\|x-\widetilde{x}\| \leq \sqrt{2}\delta d\left((\frac{1}{2}-2\sqrt{2d+3})N^{-1}+2\sqrt{2d+3}N^{-\frac{1}{2d}}\right).$$

Wang did not explicitly describe how to compute the permutations that satisfy the upper bound that he gave in Theorem 5.1. However, we can induce the steps from his proof in order to construct the following *Algorithm YW*.

(1) Start with a permutation p. If this permutation satisfies the upper bound given in Theorem 5.1, we are done.

(2) If p does not satisfy the bound, divide $[-\frac{1}{2},\frac{1}{2}]^d$ equally into 2^d subcubes, pick the nonempty subcubes (say $\{C_k\}$), and find permutations (say $\{p_k\}$) in these smaller subcubes. (We shall describe a way to find such permutations in *Algorithm 1* below.)

(3) If a p_k does not satisfy the bound given in Theorem 5.1, divide C_k further into smaller subcubes. Proceed in this way until the bound in Theorem 5.1 is met in each subcube.

(4) Let p be the union of these smaller permutations.

We can use this recipe for a set of real vectors. For a set of complex vectors $\{e_n\}$, we stack the real and imaginary parts of each e_n to form $w_n = (Re(e_n), Im(e_n)) \in \mathbb{R}^{2d}$, and apply the algorithm to the set of real vectors $\{w_n\}$. Once the algorithm finds a suitable p, we reverse the process, and obtain the new complex set of vectors from the new real set of vectors. This makes sense due to (5.1).

For Algorithm YW to run efficiently, we need to make good choices for the permutations p_0 and $\{p_k\}$, at least for a small number N of vectors. In fact, Algorithm YW gives a way to find good permutations p for large N, by expressing p as a union of smaller permutations.

We now formulate *Algorithm 1* that serves the purpose of finding good permutations for FUNTFs for $\mathbb{C}^d$. We are going to use Algorithm 1 for step (1) of the Algorithm YW in order to find a good starting permutation p, and also as a supplement to step (2) of Algorithm YW in order to find $\{p_k\}$ in smaller subcubes. Here is Algorithm 1.

Let $(x_k)_{k=1}^N$ be a FUNTF for $\mathbb{C}^d$.

(1) Let $k_1 = 1$, $y_1 = x_{k_1}$, $J_1 = \{k_1\}$,

(2) Let $k_n = argmax_{k\notin J_{n-1}}|Re\langle y_{n-1}, x_k\rangle|$, $J_n = J_{n-1}\cup\{k_n\}$,

(3) Let $y_n = sign(Re\langle y_{n-1}, x_{k_n}\rangle)x_{k_n}$.

$(y_k)_{k=1}^N$ is unitarily equivalent to $(x_k)_{k=1}^N$, because it results from multiplying the permuted frame elements by ± 1.

Basically, Algorithm 1 starts with a frame element, and chooses another one from the remaining set of frame elements that maximizes the absolute value of the inner product. It repeats this at every step until all of the frame elements are used.

Note that, applying Algorithm 1 to $\{e_n\}$ or to $\{w_n = (Re(e_n), Im(e_n))\}$ results in the same permutation since

$$Re\langle e_n, e_m\rangle = Re(e_n)Re(e_m) + Im(e_n)Im(e_m) = \langle w_n, w_m\rangle.$$

Intuitively, Algorithm 1 is greedy, and so it should reduce the frame variation. In fact, since Algorithm 1 maximizes the absolute value of the real part of the successive inner products $|Re\langle y_{n-1}, x_k\rangle|$ among the remaining set of frame elements $\{x_k : k \notin J_{n-1}\}$ at each step, and since

$$\forall n, \quad \|y_n - y_{n-1}\|^2 = \|y_{n-1}\|^2 + \|y_n\|^2 - 2Re\langle y_{n-1}, y_n\rangle = 2 - 2|Re\langle y_{n-1}, x_{k_n}\rangle|,$$

then it must minimize the successive differences $\{\|y_n - y_{n-1}\|\}$ between the new frame elements at each step. In particular, if we let $p(n) = k_n$, and let p_{id} be the identity permutation of $\{1, \ldots, N\}$, then we have

$$\sigma((y_k)_{k=1}^N, p_{id}) = 2 - 2\sum_{n=1}^{N-1} |Re(\langle x_{p(n-1)}, x_{p(n)}\rangle)| \le \sigma((x_k)_{k=1}^N, p).$$

Algorithm 1 multiplies some of the frame elements by -1, but this is not a detriment, and, in fact, it is really a benefit to us. To see this, suppose that, for a given x, the $\Sigma\Delta$ system (2.1) gives rise to

$$\widetilde{x} = \frac{d}{N}\sum_{k=1}^N q_k y_k.$$

We also have the following intrinsic property of $\mathcal{A}_K^\delta$:

$$a \in \mathcal{A}_K^\delta \Leftrightarrow \pm a \in \mathcal{A}_K^\delta.$$

Therefore, we can find $\{\widetilde{q_k}\}$, where either $\widetilde{q_k} = q_k$ or $\widetilde{q_k} = -q_k$, and where

$$\widetilde{x} = \frac{d}{N}\sum_{k=1}^N q_k y_k = \frac{d}{N}\sum_{k=1}^N \widetilde{q_k} x_k.$$

The second equality depends on step (3) of Algorithm 1. Thus, we can obtain a representation of $\tilde{x}$ in terms of the original frame $\{x_n\}$. On the other hand, in Algorithm 1, we search for a frame with small frame variation, not only among the permutations of $\{x_n\}$, but also among the permutations of a class of unitary equivalents of it. This increases the chance of finding a frame with a much smaller frame variation.

Remark 5.4. This section proides an algorithm to reduce frame variation for generic FUNTFs. Frames with more structure, such as frames with uniformly bounded frame variation, may have better error bounds. See [**5, 4**].

6. Examples

In the following examples, we use a combination of Algorithm YW and Algorithm 1 for FUNTFs for $\mathbb{R}^d$. For each example, we calculate the frame variation of the original frame in the given order, the frame variation of the frame produced by the algorithms, and the upper bound from Theorem 5.1.

The gray scale figures depict the real parts of the Grammian matrices for the frames with which we begin, as well as the frames produced by the algorithms. In particular, Figure 4 depicts the Grammian of the 25-element frame in Example 6.2; and Figure 5 is the corresponding Grammian produced by the algorithms. A white square represents a 1 in the corresponding entry of a matrix, while a black square represents a -1. Lighter squares in the second diagonal entries of the figures for the permuted frames show that the real part of the inner products of the successive frame elements are close to 1. (By second diagonal, we mean below the main diagonal.) Darker areas off of the second diagonal of the real part of the Grammian of the new frame confirm the success Algorithm 1.

Example 6.1. The rows of the matrix in Table 1 form a FUNTF for $\mathbb{R}^2$ with $N = 21$ elements. We calculated this FUNTF by solving (6.1).

The frame variation of the original frame in the given order is 24.4779, and the frame variation of the frame that the algorithms produce is 2.56091. Also, the upper bound given by Corollary 5.2 is

$$4\sqrt{d+3}N^{1-\frac{1}{d}} - 4\sqrt{d+3} = 32.0436.$$

It was shown in [**1**] that the minimizers of the frame potential

$$TFP(\{x_n\}) = \sum_{m=1}^{N}\sum_{n=1}^{N} |\langle x_m, x_n\rangle|^2$$

subject to the constraints $\|x_n\| = 1$, $n = 1, \dots, N$, are FUNTFs.

The MATLAB code below uses sequential quadratic programming (SQP) to solve the constrained minimization problem

$$\begin{aligned} &\textit{minimize} && \sum_{m=1}^{N}\sum_{n=1}^{N} |\langle x_m, x_n\rangle|^2 \\ &\textit{subject to} && \|x_n\|^2 = 1, \quad \forall n = 1, \dots, N. \end{aligned} \tag{6.1}$$

```
function L=randfuntf(N,d,x)

 if nargin<3; x=2*rand(N,d)-1; end
 lambda = N^2/d; p=2;

 optns=optimset('Display','on','Largescale','off',...
     'MaxIter',1e+3,'TolX',1e-6,'TolFun',1e-6,'TolCon',1e-8,...
     'MaxFunEvals',1e+4,'DiffMinChange',1e-6,...
     'Gradobj','off','Hessian','off','GradConstr','off');

 x=fmincon(@(x)TFP(x),x,[],[],[],[],[],[],@FUNTFconst,optns);
 L=x;
```

$$\begin{pmatrix}
-0.7228 & 0.69106 \\
0.37772 & 0.92592 \\
-0.75132 & 0.65994 \\
0.81978 & 0.57267 \\
-0.65436 & 0.75618 \\
-0.99649 & -0.083726 \\
0.99664 & 0.081908 \\
-0.70853 & -0.70568 \\
0.093379 & 0.99563 \\
0.05196 & 0.99865 \\
-0.91746 & -0.39784 \\
0.84169 & 0.53996 \\
0.52699 & -0.84987 \\
0.78632 & -0.61782 \\
0.99884 & 0.048212 \\
0.7079 & -0.70631 \\
-0.39328 & -0.91942 \\
0.62502 & -0.78061 \\
-0.49371 & -0.86962 \\
0.1665 & -0.98604 \\
0.95312 & 0.30258
\end{pmatrix}$$

TABLE 1

```
%%%%% Frame Potential Function %%%%%%
function F = TFP(L)
 F = sum(sum((L*L').^2));

%%%%% Constraints %%%%%%
function [c, ceq] = FUNTFconst(L)
 [N d]= size(L);
 un_const= sum((L.^2)') - ones(1,N);
 c = []; % inequality constraint c>=0
 ceq = [un_const]; % equality constraint ceq=0
```

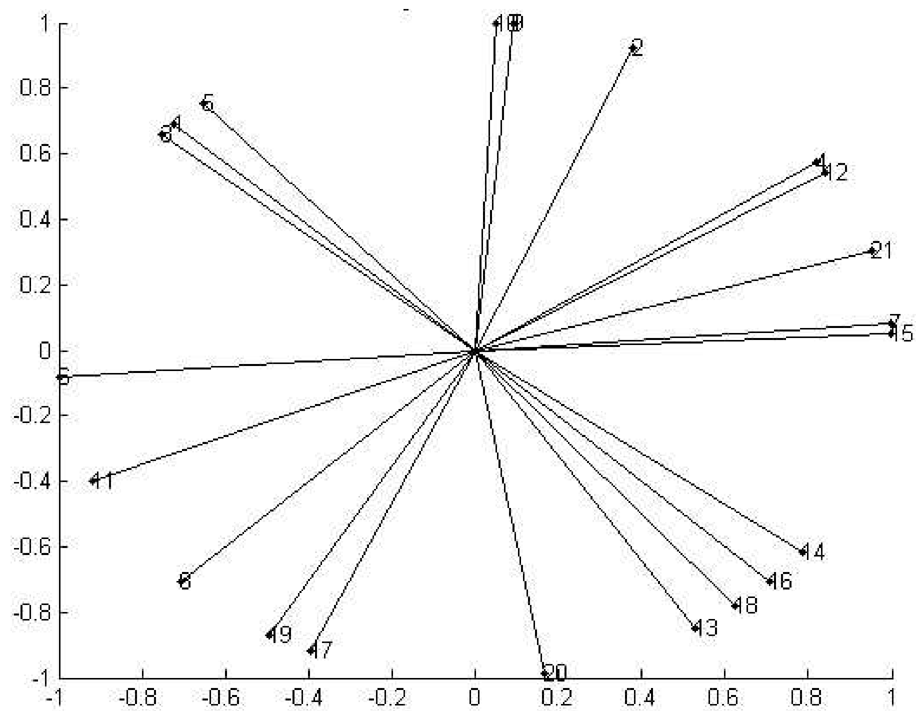

FIGURE 2. 2D plot of the FUNTF in Example 6.1

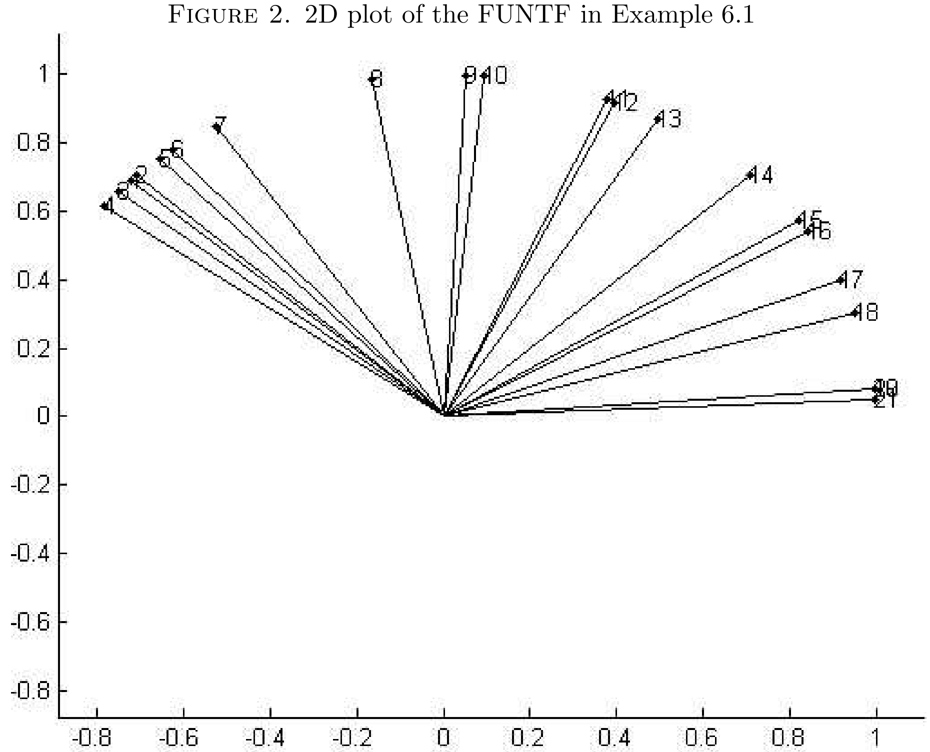

FIGURE 3. 2D plot of the frame produced by the algorithms in Example 6.1

$$\begin{pmatrix}
-0.37504 & -0.38496 & -0.070224 & 0.71156 & -0.4471 \\
0.054793 & -0.56558 & 0.40751 & 0.69414 & -0.17092 \\
-0.10414 & -0.1458 & 0.037379 & 0.21973 & 0.95824 \\
-0.044991 & -0.83328 & 0.36519 & -0.41215 & -0.019822 \\
0.204 & 0.011008 & -0.16624 & 0.90786 & 0.3262 \\
-0.29678 & -0.35433 & 0.6884 & -0.54997 & 0.10004 \\
-0.26537 & 0.33617 & -0.54834 & -0.21413 & -0.68559 \\
-0.23439 & -0.84495 & -0.45732 & -0.032102 & 0.14476 \\
0.088218 & -0.46106 & -0.36962 & 0.70666 & -0.37903 \\
0.16061 & -0.29044 & -0.78254 & -0.52594 & 0.029268 \\
0.55915 & -0.40843 & -0.70111 & -0.12308 & -0.11764 \\
-0.35653 & -0.43616 & -0.018993 & -0.77622 & -0.28244 \\
-0.6365 & -0.17318 & 0.28307 & 0.12267 & -0.68535 \\
0.47483 & 0.22986 & 0.20764 & 0.21582 & -0.79499 \\
-0.80924 & -0.31991 & -0.17755 & 0.3776 & -0.26207 \\
-0.88533 & 0.35056 & 0.043721 & -0.22079 & 0.20649 \\
0.1202 & -0.17113 & -0.89432 & -0.011497 & -0.39538 \\
0.37785 & -0.68901 & 0.61307 & -0.055072 & -0.060083 \\
-0.42291 & 0.56805 & -0.68048 & 0.17251 & -0.07518 \\
-0.78414 & 0.47486 & 0.21235 & -0.17423 & 0.29015 \\
-0.16818 & -0.2941 & -0.59159 & -0.01329 & 0.73148 \\
-0.34009 & -0.090196 & 0.2614 & -0.066493 & -0.89636 \\
0.90878 & 0.17222 & 0.23475 & -0.27235 & -0.12317 \\
-0.25709 & -0.27966 & 0.22858 & 0.7985 & 0.40723 \\
-0.24237 & -0.80442 & -0.27565 & -0.44036 & 0.15573
\end{pmatrix}$$

TABLE 2

EXAMPLE 6.2. The rows of the matrix in Table 2 form a FUNTF for $\mathbb{R}^5$ with $N = 25$ elements. We calculated this FUNTF by solving (6.1).

The frame variation of the original frame in the given order is 33.5730, and the frame variation of the frame that the algorithms produce is 14.7342. Also, the upper bound given by Corollary 5.2 is

$$4\sqrt{d+3}N^{1-\frac{1}{d}} - 4\sqrt{d+3} = 137.2652.$$

EXAMPLE 6.3. The rows of the matrix in Table 3 form a FUNTF for $\mathbb{R}^3$ with $N = 20$ elements. We calculated this FUNTF by solving (6.1).

The frame variation of the original frame in the given order is 30.0417, and the frame variation of the frame that the algorithms produce is 6.25892. Also, the upper bound given by Corollary 5.2 is

$$4\sqrt{d+3}N^{1-\frac{1}{d}} - 4\sqrt{d+3} = 62.3940.$$

EXAMPLE 6.4. 41st roots of unity frame for $\mathbb{R}^2$. Nth roots of unity frames for $\mathbb{R}^2$ are defined by

$$\{e_n^N\}_{n=1}^N, \quad e_n^N = (\cos(2\pi n/N), \sin(2\pi n/N)).$$

The algorithms produce the *discretized semicircle frame path* described in [**5**]. The frame variation of the original frame in the given order is 6.1239, and the

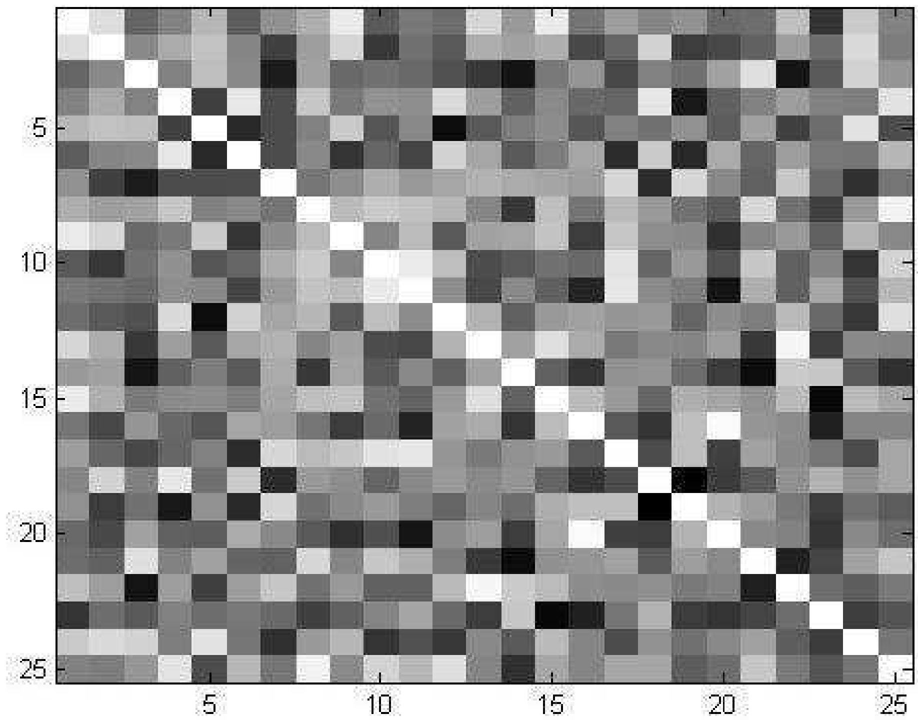

FIGURE 4. Grammian of the frame given in Example 6.2, with frame variation = 33.5730

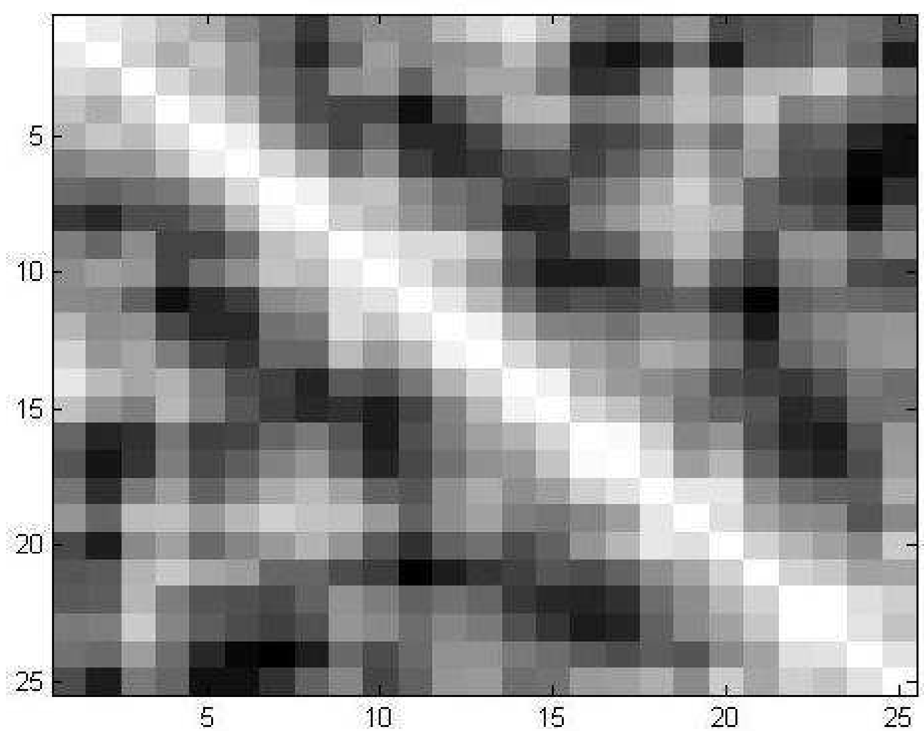

FIGURE 5. Grammian of the frame produced by the algorithms in Example 6.2, with frame variation = 14.7342

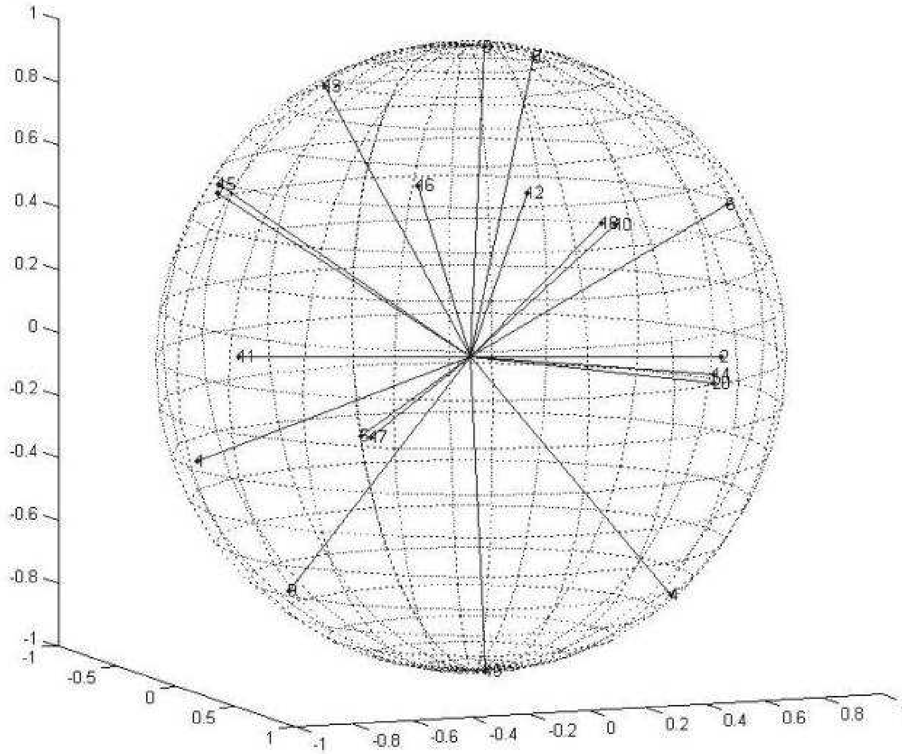

FIGURE 6. 3D plot of the FUNTF in Example 6.3

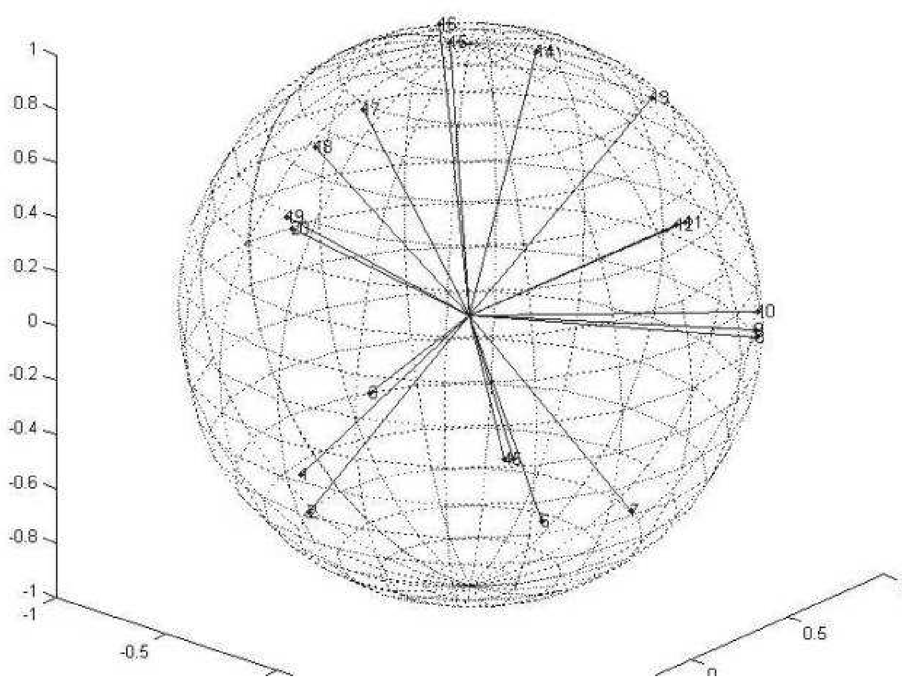

FIGURE 7. 3D plot of the frame produced by the algorithms in Example 6.3

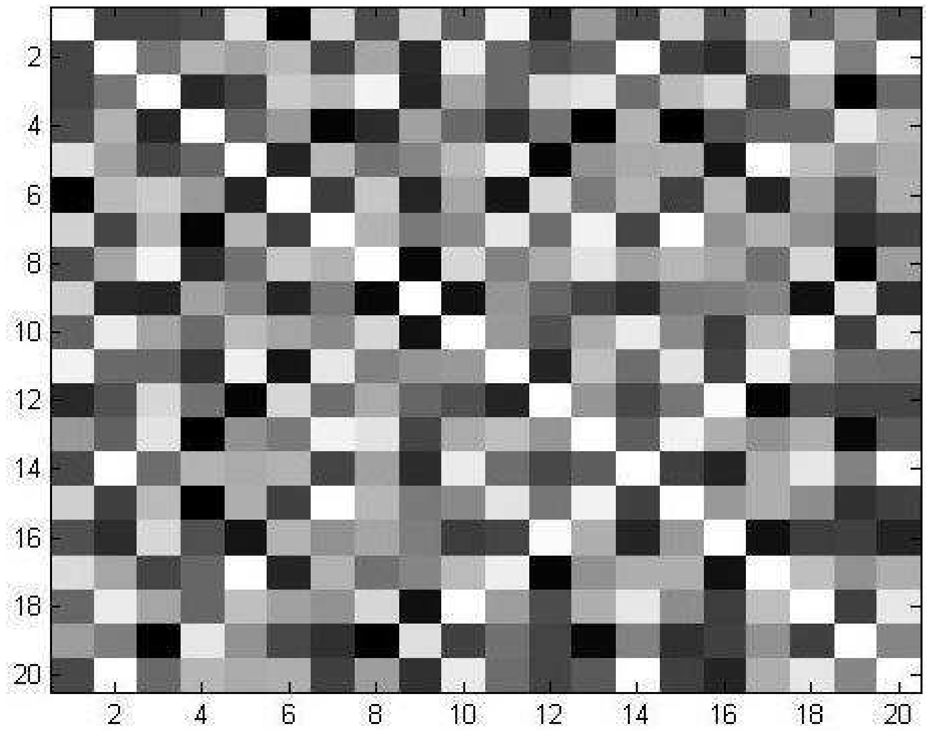

FIGURE 8. Grammian of the frame given in Example 6.3, with frame variation = 30.0417

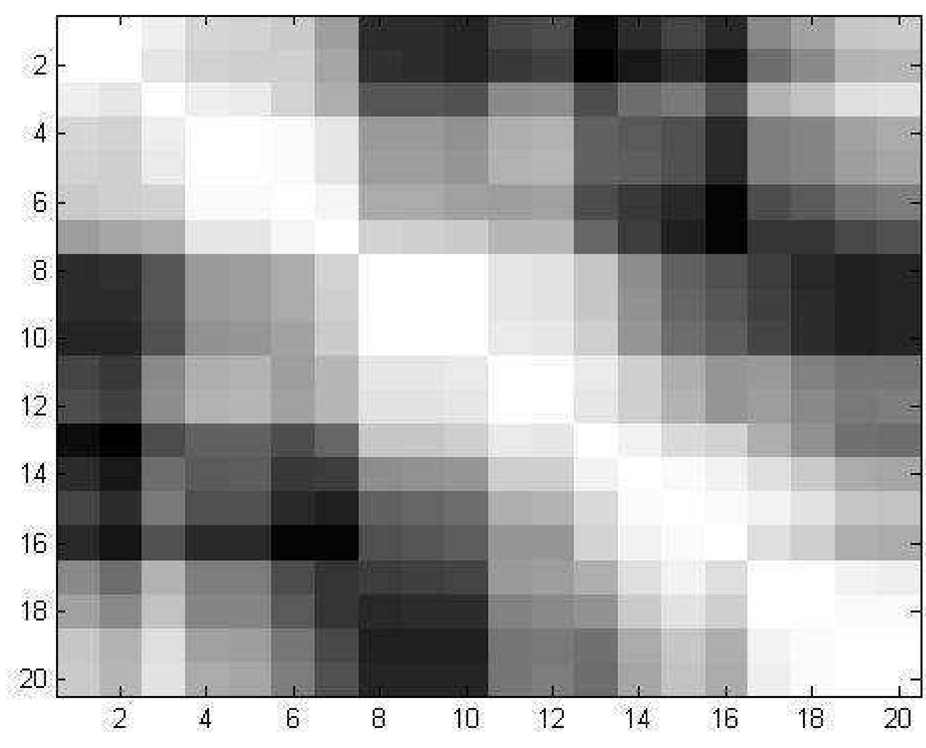

FIGURE 9. Grammian of the frame produced by the algorithms in Example 6.3, with frame variation = 6.25892

$$\begin{pmatrix} 0.070033 & -0.95985 & -0.27165 \\ 0.85466 & 0.51198 & 0.086172 \\ -0.27521 & 0.16122 & 0.94777 \\ 0.16184 & 0.61873 & -0.76876 \\ 0.73411 & -0.66763 & -0.12388 \\ -0.058269 & 0.89998 & 0.43202 \\ -0.083444 & -0.8277 & 0.55493 \\ 0.17208 & 0.14815 & 0.97388 \\ -0.44222 & -0.43689 & -0.78331 \\ 0.83844 & 0.15561 & 0.52231 \\ 0.35466 & -0.92996 & 0.096869 \\ -0.76667 & 0.50535 & 0.39603 \\ -0.07697 & -0.46577 & 0.88155 \\ 0.87966 & 0.47423 & 0.036022 \\ -0.1239 & -0.80569 & 0.57924 \\ -0.8871 & 0.18068 & 0.42474 \\ 0.75447 & -0.64445 & -0.12434 \\ 0.84174 & 0.10608 & 0.52936 \\ 0.063063 & 0.02824 & -0.99761 \\ 0.8801 & 0.47474 & 0.0072052 \end{pmatrix}$$

TABLE 3

frame variation of the frame that the algorithms produce is 3.06422. Also, the upper bound given by Corollary 5.2 is

$$4\sqrt{d+3}N^{1-\frac{1}{d}} - 4\sqrt{d+3} = 48.3270.$$

EXAMPLE 6.5. The rows of the matrix in Table 4 form a finite Heisenberg frame $\mathbb{H}(\phi)$ for $\mathbb{C}^4$, generated by the fiducial vector

$$\phi = [0.37203-0.61017i, 0.54061-0.26808i, 0.11708+0.19828i, 0.086747+0.25419i],$$

and consisting of 16 elements.

Let $\phi \in \mathbb{C}^d$ be unit norm. The system $\mathbb{H}(\phi) \subseteq \mathbb{C}^d$ is defined as

$$\mathbb{H}(\phi) = \{M^bT^a\phi : a, b = 0, \dots d-1\},$$

where the matrices M and T are defined by

$$\begin{aligned} (M\phi)[n] &= e^{2\pi i n/d}\phi[n], \quad n = 1, \dots, d, \\ (T\phi)[n] &= \phi[n+1], \quad n = 1, \dots, d-1, \quad \text{and} \quad (T\phi)[d] = \phi[0]. \end{aligned}$$

If ϕ is not an eigenvector of M or T, the elements of $\mathbb{H}(\phi)$ are all distinct. As a result, $N = d^2$. Otherwise, $\mathbb{H}(\phi)$ is a union of d ONBs for $\mathbb{C}^d$, each of which is a constant multiple by a dth root of unity of another [**13, 16**].

Finite Heisenberg frames are always tight frames [**13, 6**].

The r-th row of the matrix given in Table 4 is $M^bT^a\phi$, where $r = bd + a$. Numerical evidence shows that this natural ordering of finite Heisenberg frames results in "good" frame variation. However, the algorithms presented in this paper generally found permutations resulting in lower frame variation than that given by this natural ordering.

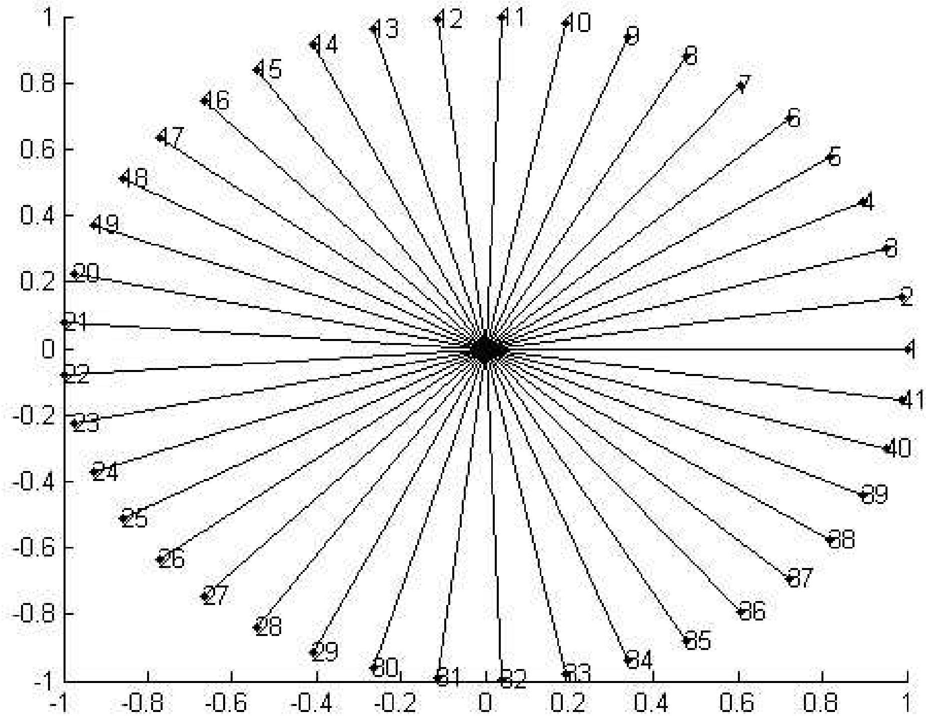

Figure 10. 2D plot of 41st roots of unity frame in Example 6.4. This ordering is going in the counter-clockwise direction starting at $(1, 0)$

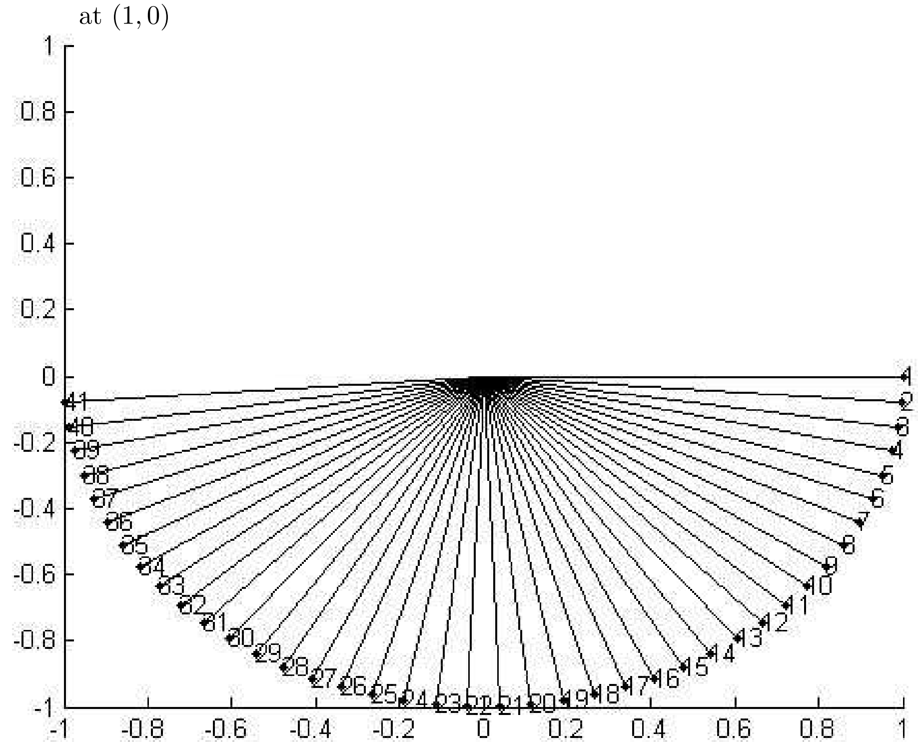

Figure 11. 2D plot of the frame produced by the algorithms in Example 6.4. This ordering is going in the clockwise direction starting at $(1, 0)$

$$\begin{pmatrix}
0.37203 + 0.61017i & 0.54061 + 0.26808i & 0.11708 - 0.19828i & 0.086747 - 0.25419i \\
0.54061 + 0.26808i & 0.11708 - 0.19828i & 0.086747 - 0.25419i & 0.37203 + 0.61017i \\
0.11708 - 0.19828i & 0.086747 - 0.25419i & 0.37203 + 0.61017i & 0.54061 + 0.26808i \\
0.086747 - 0.25419i & 0.37203 + 0.61017i & 0.54061 + 0.26808i & 0.11708 - 0.19828i \\
0.37203 + 0.61017i & 0.26808 - 0.54061i & -0.11708 + 0.19828i & 0.25419 + 0.086747i \\
0.54061 + 0.26808i & -0.19828 - 0.11708i & -0.086747 + 0.25419i & -0.61017 + 0.37203i \\
0.11708 - 0.19828i & -0.25419 - 0.086747i & -0.37203 - 0.61017i & -0.26808 + 0.54061i \\
0.086747 - 0.25419i & 0.61017 - 0.37203i & -0.54061 - 0.26808i & 0.19828 + 0.11708i \\
0.37203 + 0.61017i & -0.54061 - 0.26808i & 0.11708 - 0.19828i & -0.086747 + 0.25419i \\
0.54061 + 0.26808i & -0.11708 + 0.19828i & 0.086747 - 0.25419i & -0.37203 - 0.61017i \\
0.11708 - 0.19828i & -0.086747 + 0.25419i & 0.37203 + 0.61017i & -0.54061 - 0.26808i \\
0.086747 - 0.25419i & -0.37203 - 0.61017i & 0.54061 + 0.26808i & -0.11708 + 0.19828i \\
0.37203 + 0.61017i & -0.26808 + 0.54061i & -0.11708 + 0.19828i & -0.25419 - 0.086747i \\
0.54061 + 0.26808i & 0.19828 + 0.11708i & -0.086747 + 0.25419i & 0.61017 - 0.37203i \\
0.11708 - 0.19828i & 0.25419 + 0.086747i & -0.37203 - 0.61017i & 0.26808 - 0.54061i \\
0.086747 - 0.25419i & -0.61017 + 0.37203i & -0.54061 - 0.26808i & -0.19828 - 0.11708i
\end{pmatrix}$$

TABLE 4

In this particular example, the frame variation of the original frame in the given order is 19.001, and the frame variation of the frame that the algorithms produce is 13.9715. Also, the upper bound given by Corollary 5.2 is

$$4\sqrt{2d+3}N^{1-\frac{1}{2d}} - 4\sqrt{2d+3} = 136.8268.$$

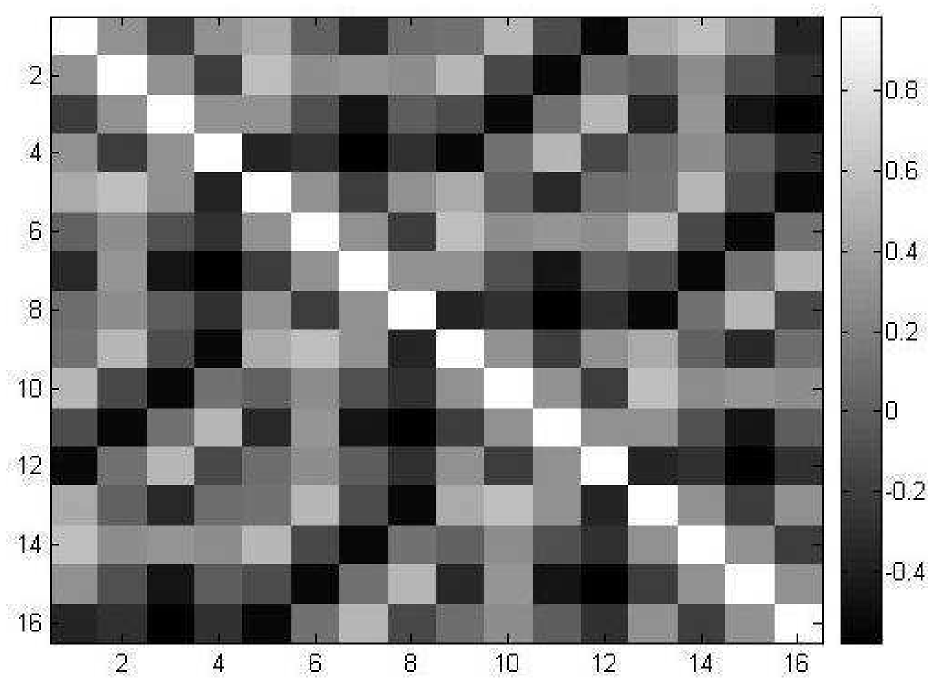

FIGURE 12. Real part of the Grammian of the frame given in Example 6.5, with frame variation = 19.001

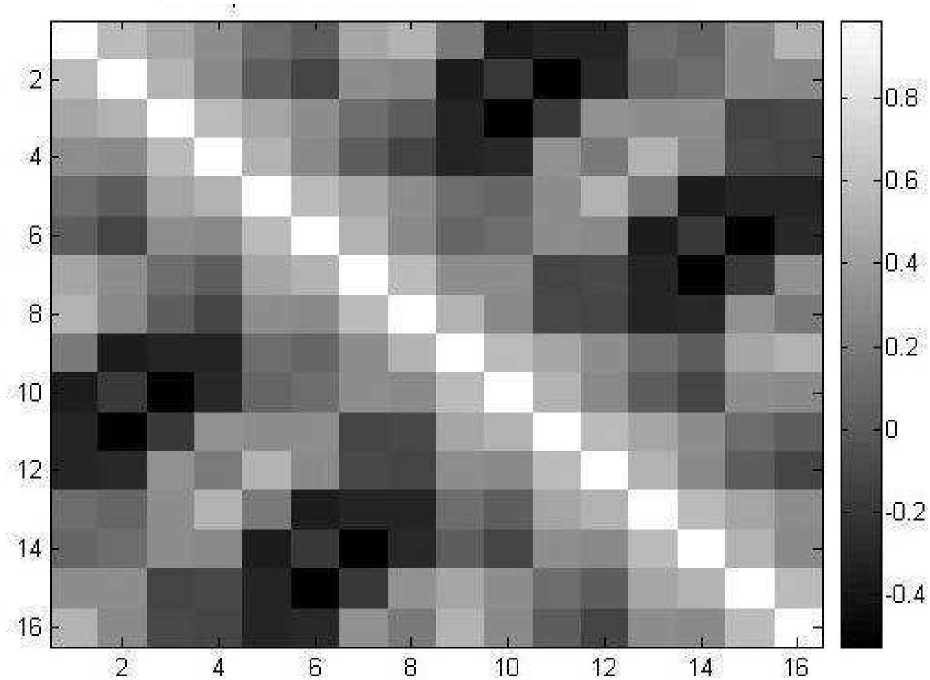

FIGURE 13. Real part of the Grammian of the frame produced by the algorithms in Example 6.5, with frame variation = 13.9715

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