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Uncertainty principles and weighted norm inequalities

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This paper is dedicated to the memory of Björn Jawerth

Abstract. The focus of this paper is weighted uncertainty principle inequalities in harmonic analysis. We start by reviewing the classical uncertainty principle inequality, and then proceed to extensions and refinements by modifying two major results necessary to prove the classical case. These are integration by parts and the Plancherel theorem. The modifications are made by means of generalizations of Hardy's inequality and weighted Fourier transform norm inequalities, respectively. Finally, the traditional Hilbert space formulation is given in order to construct new examples.

1. Introduction

1.1. Background and theme. Uncertainty principle inequalities abound in harmonic analysis, e.g., see [**62**], [**25**], [**28**], [**30**], [**29**], [**18**], [**27**], [**66**], [**8**], [**26**], [**9**], [**42**], [**20**], [**32**], [**38**], [**56**]. Having been developed in the context of quantum mechanics, the classical Heisenberg uncertainty principle is deeply rooted in physics, see [**45**], [**72**], [**71**], [**34**]. The classical mathematical uncertainty principle inequality was first stated and proved in the setting of $L^2(\mathbb{R})$, the space of Lebesgue measurable square-integrable functions on the real line $\mathbb{R}$, in 1924 by Norbert Wiener at a Göttingen seminar [**3**], also see [**49**]. This is Theorem 1.1. The proof of the basic inequality, (1.1) below, invokes integration by parts, Hölder's inequality, and the Plancherel theorem, see (1.3). For more complete proofs, see, for example, [**72**], [**9**], [**32**], [**38**].

Theorem 1.1. *(The classical uncertainty principle inequality) If $f \in L^2(\mathbb{R})$ and $x_0, \gamma_0 \in \mathbb{R}$, then*

$$||f||_2^2 \leqslant 4\pi ||(x - x_0)f(x)||_2 \;\; ||(\gamma - \gamma_0)\hat{f}(\gamma)||_2, \tag{1.1}$$

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and there is equality if and only if

$$f(x) = Ce^{2\pi i x\gamma_0}e^{-s(x-x_0)^2}, \tag{1.2}$$

for $C \in \mathbb{C}$ and $s > 0$. ($||\cdot||_2$ designates the L^2 norm, and the Fourier transform $\widehat{f}$ of f is formally defined as

$$\widehat{f}(\gamma) = \int_{\mathbb{R}} f(x)e^{-2\pi i x\gamma}dx.)$$

The uncertainty principle inequality (1.1) is a consequence of the following calculation for the case $(x_0, \gamma_0) = (0, 0)$ and for $f \in \mathscr{S}(\mathbb{R})$, the Schwartz class of infinitely differentiable rapidly decreasing functions defined on $\mathbb{R}$.

$$\begin{aligned} ||f||_2^4 &= \left(\int_{\mathbb{R}} x|f(x)^2|'dx\right)^2 \leqslant \left(\int_{\mathbb{R}} |x||f(x)^2|'dx\right)^2 \\ &\leqslant 4\left(\int_{\mathbb{R}} |x\overline{f(x)}f'(x)|dx\right)^2 \\ &\leqslant 4||xf(x)||_2^2||f'(x)||_2^2 = 16\pi^2||xf(x)||_2^2||\gamma\widehat{f}(\gamma)||_2^2. \end{aligned} \tag{1.3}$$

Integration by parts gives the first equality and the Plancherel theorem gives the second equality. The third inequality is a consequence of Hölder's inequality.

There is a result analogous to Theorem 1.1 for the case $d > 1$. This is Theorem 2.3, that is given in Section 2. The main difficulty in the $d > 1$ case is that the square integrability of the distributional derivatives of f in the inequality, arising from the analogue on $\widehat{\mathbb{R}}$ of $\gamma\widehat{f}(\gamma)$, does not afford easy technical manipulation, e.g., being able to deduce absolute continuity, see [**33**], [**48**].

One way to remedy this is to introduce the notion of a bi-Sobolev space. In this context, the argument is reduced to proving the uncertainty principle for smooth compactly supported functions on $\mathbb{R}^d$, and extending to $L^2(\mathbb{R}^d)$ by means of a density argument. This was originally done in [**8**]. Using more abstract ideas, Folland and Sitarum also gave a proof of the result for $L^2(\mathbb{R}^d)$ as a special case, see [**32**], pages 210-213.

The approach in Section 2, following [**8**], has the advantage of using the same method of integration by parts, Hölder's inequality, and the Plancherel theorem, as in the one-dimensional case, in order to obtain versions of the classical uncertainty principle inequality on $L^2(\mathbb{R}^d)$. It thereby serves as a stepping stone to proving more difficult classical cases involving weighted spaces as well as extending its theoretical tentacles far beyond Theorems 1.1 and 2.3.

Remark 1.2 (The additive uncertainty principle). Cowling and Price [**23**] proved the following strong additive version of the classical uncertainty principle inequality on $\mathbb{R}$ for arbitrary $p, q \in [1, \infty]$ and $a, b > 0$, and for the class of tempered functions f, i.e., essentially polynomial growth, for which $\widehat{f}$ is a function. *There is $C > 0$ such that*

$$\forall f, \quad ||f||_2^2 \leqslant C\left(|||x|^a f(x)||_p + |||\gamma|^b\widehat{f}(\gamma)||_q\right) \tag{1.4}$$

if and only if

$$a > \frac{1}{2} - \frac{1}{p} \quad \textit{and} \quad b > \frac{1}{2} - \frac{1}{q}.$$

($||\cdot||_p$ and $||\cdot||_q$ designate the L^p and L^q-norms.)

REMARK 1.3. The relevance of Theorem 1.1 for quantum mechanics can be illustrated by considering a freely moving mass point with varying location $l \in \mathbb{R}$. The term $||xf(x)||_2^2$ represents the average distance of l from its expected value $x_0 = 0$. In fact, the position l is interpreted as a random variable depending on the state function f; more precisely, the probability that x is in a given region $A \subseteq \mathbb{R}$ is defined as

$$\int_A |f(x)|^2 dx,$$

and $||xf(x)||_2^2$ is the variance of x.

Our *theme* is as follows. We shall extend and refine Theorems 1.1 and 2.3 in several ways. The main ingredients of our proofs, however, will remain the same: integration by parts will give way to conceptually similar ideas such as generalizations of Hardy's inequality, and the Plancherel theorem will be generalized to weighted Fourier transform norm inequalities.

1.2. Outline. In Section 2, we give a detailed proof of the classical uncertainty principle on $\mathbb{R}^d$.

Because of our theme for generalizing the classical uncertainty principle inequality, Sections 3 and 4 are devoted to Hardy's inequality and weighted Fourier transform norm inequalities, respectively. Then, in Section 5, the results in Sections 3 and 4 are used to obtain a variety of uncertainty principle inequalities.

In Section 6 we provide a proof of the traditional uncertainty principle inequality for general Hilbert spaces in order to exhibit several elementary and some new examples. We conclude with a brief Epilogue.

REMARK 1.4. Most of these topics have a long history with contributions by some of the most profound harmonic analysts. Our presentation has to be viewed in that context, notwithstanding the considerable number of references to the first named author. It was his intention to put together various uncertainty principle inequalities in which he was involved and that had a common point of view.

1.3. Notation. Generally, our notation is standard from modern analysis texts, e.g., [**68**], [**63**], [**31**], [**24**], [**10**], [**11**].

The *Fourier transform* $\widehat{f}$ of a complex-valued Lebesgue measurable function $f : \mathbb{R}^d \to \mathbb{C}$ on Euclidean space $\mathbb{R}^d$ is formally defined as

$$\widehat{f}(\gamma) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \gamma} dx,$$

where $\gamma \in \widehat{\mathbb{R}}^d = \mathbb{R}^d$ and $\widehat{\mathbb{R}}$ denotes a frequency or spectral domain.

In particular, we use the d-dimensional multi-index notation, where if α is a d-tuple of natural numbers, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$, then $\alpha \leqslant \beta$ means $\alpha_i \leqslant \beta_i$ for each $i \in \{1, \ldots, d\}$. Also, we write

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_d^{\alpha_d} \quad \text{and} \quad \partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \ldots \partial_d^{\alpha_d},$$

where $\partial_i^{\alpha_i} = \partial^{\alpha_i}/\partial x_i^{\alpha_i}$, and $|\alpha| = \alpha_1 + \cdots + \alpha_d$.

Further, we use the following notation. If $p \geqslant 1$, then p' is defined by $\frac{1}{p} + \frac{1}{p'} = 1$.

Let $\mathbb{R}^+ = [0, \infty)$. If $x \in \mathbb{R}^d$ and $r > 0$, then $B(x, r) \subseteq \mathbb{R}^d$ is the open ball of radius r centered at x. Let $v \geqslant 0$ on $\mathbb{R}^d$ and let $p \geqslant 1$. $L_v^p(\mathbb{R}^d)$ is the weighted space of Borel measurable functions $f : \mathbb{R}^d \to \mathbb{C}$ for which

$$||f||_{p,v} = \left(\int_{\mathbb{R}^d} |f(x)|^p v(x) dx \right)^{\frac{1}{p}} < \infty.$$

We shall usually omit the domain of integration in integrals when the setting is clear.

Finally, if X and Y are topological vector spaces, then $\mathscr{L}(X, Y)$ is the space of continuous linear mappings $X \to Y$.

2. The classical uncertainty principle inequality for $L^2(\mathbb{R}^d)$

In this section we describe the classical uncertainty principle inequality on $\mathbb{R}^d$.

DEFINITION 2.1. Given integers $m, n \geqslant 0$, and let $1 \leqslant p \leqslant \infty$. The *Sobolev space* $W^{m,p}(\mathbb{R}^d)$ is the Banach space of functions $f \in L^p(\mathbb{R}^d)$ with norm,

$$||f||_{m,p} = \sum_{|\alpha| \leqslant m} ||\partial^\alpha f||_p < \infty.$$

The *weighted space* $L^p_{0,n}(\mathbb{R}^d)$ is the Banach space of functions $f \in L^p(\mathbb{R}^d)$ with norm,

$$||f||_{p,n} = \sum_{|\beta| \leqslant n} ||t^\beta f(t)||_p < \infty.$$

The *bi-Sobolev* space $L^p_{m,n}(\mathbb{R}^d)$ is the Banach space of functions $f \in W^{m,p}(\mathbb{R}^d) \cap L^p_{0,n}(\mathbb{R}^d)$ with norm,

$$||f||_{m,n,p} = ||f||_{m,p} + ||f||_{p,n} < \infty.$$

The following result is a variant of a theorem of Meyers-Serrin [**55**] (1964). It is to be expected by a natural approximate identity strategy combined with truncations on larger and larger domains. We provide full details to show that the strategy works and because of the expository nature of a chapter such as this.

THEOREM 2.2. *Given integers $m, n \geqslant 0$. $C_c^\infty(\mathbb{R}^d)$ is dense in the Hilbert space $\left(L^2_{m,n}(\mathbb{R}^d), ||...||_{m,2,n}\right)$, with inner product,*

$$\langle\langle f, g \rangle\rangle = \sum_{|\alpha| \leqslant m} \langle \partial^\alpha f, \partial^\alpha g \rangle + \sum_{|\beta| \leqslant n} \langle\, t^\beta f, t^\beta g \rangle,$$

where $\langle \cdot\ ,\ \cdot \rangle$ is the usual inner product on $L^2(\mathbb{R}^d)$.

PROOF. *i.* Although it is well-known, we first show that $C_c^\infty(\mathbb{R}^d)$ is dense in $W^{m,p}(\mathbb{R}^d)$, $1 \leqslant p < \infty$. Let $f \in W^{m,p}(\mathbb{R}^d)$, and let $\{h_j\} \subseteq C_c^\infty(\mathbb{R}^d)$ be an L^1-approximate identity [**10**], Section 1.6. Assume without loss of generality that each $\operatorname{supp}(h_j) \subseteq B(0, 1)$. Choose $u \in C_c^\infty(\mathbb{R}^d)$ such that $0 \leqslant u \leqslant 1$ and $u = 1$ on $B(0, 1)$, and define $u_j(t) = u(t/j)$.

Now fix α with the property $|\alpha| \leqslant m$. Not only does $\partial^\alpha(f * h_j) = f * \partial^\alpha h_j$, but, by integration by parts,

$$\partial^\alpha(f * h_j) = \partial^\alpha f * h_j.$$

Thus, by Young's inequality,

$$||\partial^\alpha(f * h_j)||_p \leqslant ||\partial^\alpha f||_p ||h_j||_1,$$

and so each $f * h_j$ is an element of $W^{m,p}(\mathbb{R}^d)$, as is each $u_j(f * h_j)$.

The desired density will follow from the triangle inequality once we prove that

$$\forall \alpha \text{ satisfying } |\alpha| \leqslant m, \quad \lim_{j\to\infty} ||\partial^\alpha [(f * h_j)(u_j - 1)]||_p = 0. \tag{2.1}$$

To this end, note that Leibniz' formula gives

$$\begin{aligned} ||\partial^\alpha [(f * h_j)(u_j - 1)]||_p &\leqslant ||(u_j - 1)\partial^\alpha (f * h_j)||_p \\ &\quad + \sum_{\beta \leqslant \alpha, |\beta| \geqslant 1} |C_{\alpha\beta}| j^{-|\beta|} ||\partial^{\alpha-\beta}(f * h_j)(t)\partial^\beta u(t/j)||_p. \end{aligned} \tag{2.2}$$

The dominated convergence theorem and Young's inequality allow us to show that the first term on the right side of (2.2) tends to 0 as $j \to \infty$. Again, Young's inequality and the fact that

$$\lim_{j\to\infty} j^{-|\beta|} ||\partial^\beta u||_\infty = 0$$

for $|\beta| \geqslant 1$ show that the remaining terms on the right side of (2.2) tend to 0 as $j \to \infty$.

Thus, (2.1) is proved. This density in $W^{m,p}(\mathbb{R}^d)$ can also be proved by an equicontinuity argument, much like the one we now give in part *ii.*

ii. It is sufficient to prove that

$$\forall f \in L^2_{0,m}(\mathbb{R}^d), \qquad \lim_{j\to\infty} ||F_{\beta j}(f)||_2 = 0 \tag{2.3}$$

for each β for which $|\beta| \leqslant n$, where $F_{\beta j}(f) = F_j(f) = t^\beta(u_j(f * h_j) - f)$. To this end we first show that

$$\sup_j ||F_j(f)||_2 = C(f) < \infty. \tag{2.4}$$

This is accomplished by the estimate,

$$\begin{aligned} ||F_j(f)||_2 - ||t^\beta f(t)||_2 &\leqslant C(\beta) ||\partial^\beta (\widehat{f}\widehat{h}_j)||_2 \\ \leqslant \sum_{\gamma\leqslant\beta} |C_{\beta\gamma}| ||\partial^{\beta-\gamma}\widehat{f}\partial^\gamma \widehat{h}_j||_2 &\leqslant \sum_{\gamma\leqslant\beta} |C_{\beta\gamma}| ||\partial^{\beta-\gamma}\widehat{f}||_2 \sup_{\gamma\leqslant\beta} ||\partial^\gamma \widehat{h}_j||_\infty, \end{aligned}$$

and the fact, in the case h_j is the dilation $j^d h(jt)$, that

$$|\partial^\gamma \widehat{h}_j(\lambda)| = \left| C(\gamma) \int h(u) u^\gamma j^{-|\gamma|} e^{-2\pi i (u/j)\cdot\lambda} du \right| \leqslant K(\gamma) j^{-|\gamma|}$$

since $\mathrm{supp}(h)$ is compact. This last estimate used the Plancherel theorem so we note the fact that the distribution $\partial^\beta(\widehat{f}\widehat{h}_j)$ is an element of $L^2(\mathbb{R}^d)$.

It is straightforward to check that the elements of $L^2_{0,n}(\mathbb{R}^d)$ having compact support are dense in $L^2_{0,n}(\mathbb{R}^d)$ and that

$$\forall \beta \text{ satisfying } |\beta| \leqslant n, \quad \{F_{\beta j}\} \subseteq \mathcal{L}(L^2_{0,n}(\mathbb{R}^d), L^2(\mathbb{R}^d)).$$

Because of (2.4) we can invoke the uniform boundedness principle and obtain $\sup ||F_j||_2 = C < \infty$. Thus, $\{F_j\}$ is equicontinuous. On the other hand, it is routine to check that $\lim_{j\to\infty} ||F_j(f)||_2 = 0$ for compactly supported functions $f \in L^2_{0,n}(\mathbb{R}^d)$. This convergence on a dense subset of $L^2_{0,n}(\mathbb{R}^d)$ combined with the equicontinuity yield convergence on $L^2_{0,n}(\mathbb{R}^d)$, and the resulting limit $F(f)$ for $f \in L^2_{0,n}(\mathbb{R}^d)$ determines an element $F \in \mathcal{L}(L^2_{0,n}(\mathbb{R}^d), L^2(\mathbb{R}^d))$. Thus, (2.3) is obtained. □

Because of the caveat mentioned after Remark 1.2, Theorem 2.2 or a similar result is needed to prove the following $\mathbb{R}^d$ uncertainty principle inequalities in Theorem 2.3. The remainder of the proof of Theorem 2.3 is an adaptation of the basic calculation (1.3) on $\mathbb{R}$ given after the statement of Theorem 1.1.

THEOREM 2.3 (The classical uncertainty principle inequality). *If $f \in L^2(\mathbb{R}^d)$ and $(x_0, \gamma_0) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, then*

$$\forall j = 1, \ldots, d, \quad ||f||_2^2 \leqslant 4\pi ||(x_j - x_{0,j})f(x)||_2 \;\; ||(\gamma_j - \gamma_{0,j})\hat{f}(\gamma)||_2 \tag{2.5}$$

and

$$||f||_2^2 \leqslant \frac{4\pi}{d} |||x - x_0|f(x)||_2 \;\; |||\gamma - \gamma_0|\hat{f}(\gamma)||_2, \tag{2.6}$$

where, for example, $x_0 = (x_{0,1}, \ldots, x_{0,d})$ and

$$|x - x_0| = \left(\sum_{j=1}^{d} (x_j - x_{0,j})^2 \right)^{1/2}.$$

The constant $4\pi/d$ is optimal since equality is obtained in (2.6) for $f(x) = \exp(-\pi|x|^2)$, $x_0 = \gamma_0 = 0$.

3. Hardy type inequalities

3.1. Hardy's classical inequality. In this subsection we state Hardy's inequality, Theorem 3.2. This is background for Section 3.2, where we shall discuss a Hardy type inequality on $\mathbb{R}^{+d}$ due to Hernandez [**46**]. These inequalities can be viewed in a certain sense as generalizations of integration by parts.

DEFINITION 3.1. The *Hardy operator* is the positive linear operator P_d defined as

$$P_d(f)(x) = \int_0^{x_d} \cdots \int_0^{x_1} f(t_1, \ldots, t_d) dt_1 \ldots dt_d = \int_{\langle 0,x \rangle} f(t) dt$$

for Borel measurable functions f on $\mathbb{R}^{+d}$. The region $\langle 0, x \rangle \subseteq \mathbb{R}^d$ is $\{t = (t_1, \ldots, t_d) : x_j > 0 \text{ and } 0 < t_j < x_j \text{ for each } j = 1, \ldots, d\}$. The *dual Hardy operator* P'_d is defined as

$$P'_d(f)(x) = \int_{x_d}^{\infty} \cdots \int_{x_1}^{\infty} f(t_1, \ldots t_d) dt_1 \ldots dt_d = \int_{\langle x,\infty \rangle} f(t) dt.$$

The unbounded region $\langle x, \infty \rangle \subseteq \mathbb{R}^d$ is defined analagously to $\langle 0, x \rangle$.

THEOREM 3.2 (Hardy's inequality (1920) [**40**]). *Let $f \geqslant 0$ ($f \neq 0$) be Borel measurable and $p > 1$. Then,*

$$\int_0^{\infty} P_1(f)(t)^p t^{-p} dt < \left(\frac{p}{p-1} \right)^p \int_0^{\infty} f(t)^p dt. \tag{3.1}$$

G.H. Hardy, along with E. Landau, G. Pólya, I. Schur, M. Riesz, proved this inequality as well as the following discrete version between 1920 and 1925 [**39**].

THEOREM 3.3 (Hardy's discrete inequality). *Let $p > 1$ and let $\{a_k\}_{k=1}^{\infty}$ be a sequence of non-negative real numbers. Then,*

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} a_k \right)^p \leqslant \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n^p. \tag{3.2}$$

Since the constant $\left(\frac{p}{p-1}\right)^p$ is sharp, Theorems 3.2 and 3.3 not only express the fact that the Hardy operators are bounded mappings from L^p into L^p and l^p into l^p, respectively, for $p > 1$, but that each has norm $p' = \frac{p}{p-1}$.

REMARK 3.4. *a.* It is not difficult to see that restricting to step functions in the integral inequality (3.1) gives the discrete version. However, historically, a weaker form of the integral version was proved first, followed by the discrete version (as stated), and then finally the integral version (as stated) was proved.

b. Hardy's original motivation in studying these types of inequalities was to find a simpler proof for Hilbert's inequality [**47**] for double series. In fact, it can be shown that Hilbert's inequality follows from the discrete version. See [**50**] for a history of Hardy's inequality.

c. Hardy's inequality is striking in that it is an L^p inequality with an explicit optimal constant and that the only function for which equality is satisfied is the zero function.

REMARK 3.5. *a.* For the sake of context, we mention here that Hardy's inequality is a fundamental inequality in analysis that demonstrates two very useful principles. Using notation from the fractional calculus, the first principle is that an inverse power weight such as $1/|x|^\alpha$ may be dominated in an L^p sense, by the corresponding derivative $|\nabla|^\alpha$. Certain higher dimensional generalizations of Hardy's inequality on $\mathbb{R}^d$ take the form,

$$||f(x)/|x|^\alpha||_p \leqslant C_{\alpha,p,d}|||\nabla|^\alpha f||_p,$$

where α, p, d, and f satisfy certain conditions. These inequalities are fundamental in the study of partial differential equations that involve singular potentials or weights such as $1/|x|^\alpha$, e.g., [**43**], [**61**], [**54**], also see Section 4.4.

b. The second principle exemplified by Hardy's inequality is that a maximal average of a function is in many cases dominated in an L^p sense by the function itself. This can be seen by a different type of generalization, namely, the Hardy-Littlewood maximal function inequality, and its variants, in terms of the Hardy-Littlewood maximal function M defined in the following way. Given a locally integrable function $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, $Mf : \mathbb{R}^d \to \mathbb{R}$ is a function that at each point $x \in \mathbb{R}^d$ gives the maximum average value that f can take on balls centered at that point. More precisely, letting $B(x,r) \subseteq \mathbb{R}^d$ denote the open ball of radius r centered at x and letting $|B(x,r)|$ be its d-dimensional Lebesgue measure, $Mf(x)$ is defined as

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy.$$

THEOREM 3.6. *(Hardy-Littlewood maximal function inequality) Let $d \geqslant 1$. There is a constant $C_d > 0$ such that*

$$\forall f \in L^1(\mathbb{R}^d) \text{ and } \forall \lambda > 0 \quad |\{Mf > \lambda\}| < \frac{C_d}{\lambda}||f||_1,$$

where $|\{Mf > \lambda\}|$ is the Lebesgue measure of the set $\{x \in \mathbb{R}^d : Mf(x) > \lambda\}$.

This inequality is fundamental in harmonic analysis, ranging from the study of singular integral operators, for example the Hilbert transform, to the convergence of Fourier series, e.g., see [**58**], [**57**], [**68**], [**67**]. Both the discrete and continuous Hardy inequalities have been generalized and applied to problems in analysis and differential equations, e.g., see [**41**], [**59**], [**51**], [**50**].

3.2. Hernandez' weighted Hardy inequality. Hernandez [**46**] proved a far-reaching classical extension of Hardy's inequality that we now state in Theorem 3.8. The following result is needed for its proof. The proof of Lemma 3.7 is based on Hölder's inequality for positive linear operators. It should be compared with the Schur test for positive integral operators, see [**37**], Appendix A.

LEMMA 3.7 ([**46**], Theorem 3.1). *Given $1 < p \leqslant q < \infty$ and non-negative Borel measurable functions u and v on $X \subseteq \mathbb{R}^d$. Suppose $P : L^p_v(X) \to L^q_u(X)$ is a positive linear operator with canonical dual operator $P' : L^{q'}_{u^{-q'/q}}(X) \to L^{q'}_{v^{-p'/p}}(X)$, defined by the duality*

$$\int_X P(f)(x)g(x)dx = \int_X f(x)P'(g)(x)dx.$$

Assume there exist $K_1, K_2 > 0$ such that

$$\forall g \in L^{(q/p)'}(X),\ \text{for which } g \geqslant 0 \text{ and } ||g||_{(q/p)'} \leqslant 1,$$

there are non-negative functions,

$$f_1 \in L^p_v(X), \quad h_1 \in L^p_{u^{p/q}g}(X), \quad f_2 \in L^{p'}_{u^{-p'/q}g}(X), \quad h_2 \in L^{p'}_{v^{-p'/p}}(X),$$

with the properties,

$$P(f_1) \leqslant K_1 h_1 \text{ and } P'(f_2 g) \leqslant K_2 h_2 \tag{3.3}$$

and

$$v = f_1^{-p/p'} h_2 \text{ and } u = h_1^{-q/p'} f_2^{q/p}.$$

Then $P \in \mathcal{L}\left(L^p_v(X), L^q_u(X)\right)$, $P' \in \mathcal{L}\left(L^{q'}_{u^{-q'/q}}(X), L^{p'}_{v^{-p'/p}}(X)\right)$, and

$$||P||,\ ||P'|| \leqslant K_1^{1/p'} K_2^{1/p}.$$

Notice that if we set

$$f_1 = v^{-p'/p} P_d(v^{-p'/p})^{-1/p} \text{ and } h_1 = P_d(v^{-p'/p})^{-1/p'},$$
$$f_2 = u^{p/q} P'_d(u)^{-p/(qp')} \text{ and } h_2 = P_d(v^{-p'/p})^{-1/p'},$$

then (3.3) is valid for any non-negative $g \in L^{(q/p)'}(\mathbb{R}^{+d})$, for which $||g||_{(q/p)'} \leqslant 1$, as long as (3.4), (3.5), and (3.6) are assumed. As a result, Hernandez obtained the following version of Hardy's inequality on $\mathbb{R}^{+d}$.

THEOREM 3.8 ([**46**], Section 4.2). *Given $1 < p \leqslant q < \infty$ and non-negative Borel measurable functions u and v on $\mathbb{R}^{+d}$. Assume there exist positive K, $C_1(p)$, and $C_2(p)$ such that*

$$\sup_{s>0} \left(\int_{\langle s,\infty\rangle} u(x)dx \right)^{1/q} \left(\int_{\langle 0,s\rangle} v(x)^{-p'/p}dx \right)^{1/p'} = K, \tag{3.4}$$

$$\forall x \in \mathbb{R}^{+d}, \quad P_d\left(v^{-p'/p}(P_d(v^{-p'/p})^{-1/p}\right)(x) \leqslant C_1(p) P_d(v^{-p'/p})^{-1/p'}, \tag{3.5}$$

and

$$\forall x \in \mathbb{R}^{+d}, \quad P'_d\left(u(P'_d(u))^{-1/p'}\right)(x) \leqslant C_2(p)^{q/p}(P'_d u)^{1/p}. \tag{3.6}$$

Then, $P_d \in \mathcal{L}\left(L^p_v(\mathbb{R}^{+d}), L^q_u(\mathbb{R}^{+d})\right)$, $P'_d \in \mathcal{L}\left(L^{q'}_{u^{-q'/q}}(\mathbb{R}^{+d}), L^{p'}_{v^{-p'/p}}(\mathbb{R}^{+d})\right)$, and $||P_d||$, $||P'_d|| \leqslant K C_1(p)^{1/p'} C_2(p)^{1/p}$.

REMARK 3.9. Condition (3.4) is necessary and sufficient for weighted Hardy inequalities on $\mathbb{R}$ and necessary on $\mathbb{R}^d$, $d > 1$. Conditions (3.5) and (3.6) are automatically satisfied on $\mathbb{R}$. Conditions (3.4), (3.5), and (3.6), are sufficient but not necessary on $\mathbb{R}^d$, $d > 1$.

A different but important direction for establishing Hardy inequalities is found in [**13**].

3.3. The regrouping lemma. The following lemma will allow us to use the results from Subsection 3.2 to derive an uncertainty principle inequality in Subsection 5.3. Let Ω be the subgroup of the orthogonal group whose corresponding matrices with respect to the standard basis are diagonal with ± 1 entries. Each element $\omega \in \Omega$ can be identified with an element $(\omega_1, \dots, \omega_d) \in \{-1, 1\}^d$, and $\omega\gamma = (\omega_1\gamma_1, \dots \omega_d\gamma_d)$. Thus, formally,

$$\int_{\widehat{\mathbb{R}}^d} F(\gamma)d\gamma = \sum_{\omega\in\Omega} \int_{\widehat{\mathbb{R}}^{+d}} F(\omega\gamma)d\gamma.$$

Since

$$\sum_{\omega\in\Omega} a_\omega^{1/r} b_\omega^{1/r'} \leqslant \left(\sum_{\omega\in\Omega} a_\omega\right)^{1/r} \left(\sum_{\omega\in\Omega} b_\omega\right)^{1/r'},$$

for $1 < r < \infty$ and a_ω, $b_\omega \geqslant 0$, we have the following regrouping inequality.

PROPOSITION 3.10. *Given $1 < r < \infty$ and suppose $F \in L^r(\widehat{\mathbb{R}}^d)$, $G \in L^{r'}(\widehat{\mathbb{R}}^d)$. Then,*

$$\sum_{\omega\in\Omega} \left(\int_{\widehat{\mathbb{R}}^{+d}} |F(\omega\gamma)|^r d\gamma\right)^{1/r} \left(\int_{\widehat{\mathbb{R}}^{+d}} |G(\omega\gamma)|^{r'} d\gamma\right)^{1/r'} \leqslant ||F||_r ||G||_{r'}.$$

4. Weighted Fourier transform norm inequalities

4.1. Generalizations of Plancherel's theorem. The uncertainty principle inequalities on $L^2(\mathbb{R}^d)$, stated in Theorem 2.3, were statements about minimizing variance. However, in many applications, such as signal and image processing, as well as quantum mechanics itself, there are other optimization criteria that are of interest. Weighted uncertainty principle inequalities are one way of addressing this issue. For example, in linear system theory weights correspond to various filters in energy concentration problems, and in prediction theory weighted L^p- spaces arise for weights corresponding to power spectra of stationary stochastic processes [**9**].

Once a weighted uncertainty principle inequality is obtained, the goal is to determine a minimizer for this inequality, just as the Gaussian is a minimizer for the classical uncertainty principle inequality of Theorem 1.1.

Plancherel's theorem can be viewed as a specific example of a weighted norm inequality for the Fourier transform for the case of energy equivalence between space and spectral domains. Thus, an inequality of the form

$$||\widehat{f}||_{q,u} \leqslant C||f||_{p,v}, \tag{4.1}$$

where $u, v \geqslant 0$ are Borel measurable functions on $\mathbb{R}^d$, can be viewed as a generalization of the Plancherel theorem with an eye towards applications, where the value of p is a relevant parameter and the weights u and v are relevant "filters" or impulse responses.

The main problems concerning (4.1) are characterizing the relationship between the weights u and v to ensure its validity, and in this case finding the smallest possible constant C so that (4.1) is true for all $f \in L^p_v(\mathbb{R}^d)$.

THEOREM 4.1 (Hausdorff-Young inequality). *For all* $f \in S(\mathbb{R}^d)$,

$$||\hat{f}||_{p'} \leqslant B_d(p)||f||_p, \tag{4.2}$$

where $1 < p \leqslant 2$ *and*

$$B_d(p) = \left(p^{1/p}(p')^{-1/p'}\right)^{d/2}. \tag{4.3}$$

REMARK 4.2. *a.* Theorem 4.1 can be extended to $L^p(\mathbb{R}^d)$ since $S(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$. In particular, the Fourier transform is *well-defined* for each $f \in L^p(\mathbb{R}^d)$, $1 < p \leqslant 2$.

b. The optimal constants, $B_d(p)$, are due to Babenko [**2**](1961) and Beckner [**5**](1975), and represent an analytical tour de force. The extension of the Hausdorff-Young inequality for Fourier series to the case of Fourier transforms is due to Titchmarsh [**70**] in 1924.

4.2. A_p - weights.

DEFINITION 4.3 (A_p - weights). Let $1 < p < \infty$, and let $w \geqslant 0$ be a Borel measurable function on $\mathbb{R}^d$. We call w an A_p-weight, written $w \in A_p$, if

$$\sup_Q \left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-p'/p}dx\right)^{p/p'} = K < \infty,$$

where Q is a compact cube with sides parallel to the axes and having non-empty interior, see [**35**] for a definitive treatise.

A_p stands for Muckenhoupt weight classes. They are essential in characterizing the continuity of maximal functions and singular integral operators defined on weighted Lebesgue spaces, e.g., see [**35**], pages 411 ff., as well as the special case, Theorem 4.8, ahead, for the Riesz transform.

More surprising is the role of A_p in establishing the continuity of the Fourier transform considered as an operator defined on weighted Lebesgue spaces. The basic relationship between the Fourier transform and A_p is found in [**15**], cf. [**16**]. The authors began their theory with the following result.

THEOREM 4.4. [**15**] *Let* $w \geqslant 0$ *be an even Borel measurable function on* $\mathbb{R}$, *that is non-increasing on* $(0, \infty)$; *and let* $1 < p \leqslant 2$. *Then, there exists* $C > 0$ *such that*

$$\forall f \in C_c^\infty(\mathbb{R}), \quad \int_{\hat{\mathbb{R}}} |\hat{f}(\gamma)|^p |\gamma|^{p-2} w(1/\gamma) d\gamma \leqslant C||f||^p_{p,w}$$

if and only if $w \in A_p$.

Such inequalities naturally lead to subtle problems dealing with the proper definition of the Fourier transform on weighted Lebesgue spaces, see [**17**].

The extension of Theorem 4.4 to $\mathbb{R}^d$ is due to Heinig and Smith [**44**].

THEOREM 4.5 ([**44**]). *Let* $1 < p \leqslant q \leqslant p' < \infty$, *and let* $w \geqslant 0$ *be a Borel measurable function on* $\mathbb{R}^d$. *Assume* $w(|t|)$ *is increasing on* $(0, \infty)$. *Then, there* *exists $C > 0$ such that*

$$\forall f \in C_c^\infty(\mathbb{R}^d), \quad \left(\int_{\widehat{\mathbb{R}}^d} |\widehat{f}(\gamma)|^q |\gamma|^{d(q/p'-1)} w\left(\frac{1}{|\gamma|}\right)^{q/p} d\gamma \right)^{1/q} \leqslant C||f||_{p,w} \tag{4.4}$$

if and only if $w \in A_p$.

Example 4.6. Theorems 4.4 and 4.5 are generalizations of classical results. Consider the case over $\mathbb{R}$, where $p = q$ and $w = 1$. Then (4.4) is the Hardy-Littlewood-Paley theorem (1931):

$$\left(\int |\widehat{f}(\gamma)|^p |\gamma|^{p-2} d\gamma \right)^{1/p} \leqslant C||f||_p.$$

On the other hand, when $q = p'$ and $w = 1$, (4.4) is the Hausdorff-Young theorem; and if $w(t) = |t|^\alpha$, $0 \leqslant \alpha < p - 1$, then (4.4) becomes Pitt's theorem (1937):

$$\left(\int |\widehat{f}(\gamma)|^q |\gamma|^{-\beta} d\gamma \right)^{1/q} \leqslant C \left(\int |f(t)|^p |t|^\alpha dt \right)^{1/p},$$

where $\beta = \frac{q}{p}(\alpha + 1) + 1 - q$, cf. [**4**].

Definition 4.7. (Riesz transform) The d-dimensional Riesz transforms are the d singular integral operators $R_1, \ldots, R_d$ defined by the odd kernels $k_j(x) = \Omega_j(x)/|x|^d, j = 1, \ldots, d$, where $\Omega_j(x) = c_d x_j/|x_j|$ and $c_d = \Gamma\left((d+1)/2\right)/\pi^{(d+1)/2}$. In fact,

$$(R_j f)(x) = \lim_{T^{-1}, \epsilon \to 0} \int_{\epsilon \leqslant |t| \leqslant T} f(x - t) k_j(t) dt$$

exists a.e. for each $f \in L^p(\mathbb{R}^d)$, $1 < p < \infty$, and there is $C = C(p)$ such that

$$\forall f \in L^p(\mathbb{R}^d), \quad ||R_j f||_p \leqslant C||f||_p,$$

$j = 1, \ldots, d$. $C = C(p)$ does not depend on d. Also, we compute

$$\widehat{k}_j(\gamma) = -i\frac{\gamma_j}{|\gamma|}, \quad j = 1, \ldots, d.$$

Theorem 4.8 (Hunt, Muckenhoupt, and Wheeden, 1973). *Let $1 < p < \infty$, and suppose $w \in A_p$. Then,*

$$R_j : L^p_w(\mathbb{R}^d) \longrightarrow L^p_w(\mathbb{R}^d)$$

is a continuous linear mapping for $j = 1, \ldots, d$.

4.3. Weighted Fourier transform norm inequalities. It is convenient to begin with the following definition.

Definition 4.9. If $1 < p, q < \infty$ and if there is a constant $K > 0$ such that

$$\sup_{s>0} \left(\int_0^{1/s} u(\gamma) d\gamma \right)^{1/q} \left(\int_0^s v(t)^{-p'/p} dt \right)^{1/p'} = K, \tag{4.5}$$

then we write $(u, v) \in F(p, q)$.

The following theorem, proved in 1982, is a weighted Hausdorff-Young-Titchmarsh inequality.

THEOREM 4.10. [**12**] *Let $1 < p \leqslant q < \infty$, and let $u, v \geqslant 0$ be even Borel measurable functions defined on $\widehat{\mathbb{R}}$, $\mathbb{R}$, respectively, for which $(u, v) \in F(p, q)$ with constant K. Assume $1/u$ and v are increasing on $(0, \infty)$. Then, there is a constant $C(K)$ such that*

$$\forall f \in S(\mathbb{R}) \cap L^p_v(\mathbb{R}), \quad ||\widehat{f}||_{L^q_u} \leqslant C(K)||f||_{L^p_v}. \tag{4.6}$$

REMARK 4.11. If $p = 1$ and $q > 1$ then Theorem 4.10 is true for any positive Borel measurable function u. In this case the proof is routine and the constant $C(K)$ is explicit [**12**], pages 272-273. If $p > 1$, then the constant $C(K)$ is less explicit, but it can be estimated by examining the proof of Calderón's rearrangement inequality [**21**], that the authors also used in their proof of Theorem 4.10.

The authors of [**12**] continued this program of understanding weighted Fourier transform norm inequalities in a series of papers through to [**14**] in 2003. We state two of more of their results.

For the first inequality, $u^* : [0, \infty) \to [0, \infty)$ designates the decreasing rearrangement of any measurable function defined on a measure space.

THEOREM 4.12 ([**14**]). *Let $u, v \geqslant 0$ be Borel measurable functions on $\mathbb{R}^d$, and suppose $1 < p, q < \infty$. There is a constant $C > 0$ such that for all $f \in L^p_v(\mathbb{R}^d)$, the inequality,*

$$||\widehat{f}||_{L^q_u} \leqslant C||f||_{L^p_v}, \tag{4.7}$$

holds in the following ranges and with the following hypotheses on u and v:
(i) $1 < p \leqslant q < \infty$ and

$$\sup_{s>0} \left(\int_0^{1/s} u^*(t)dt \right)^{1/q} \left(\int_0^s (1/v)^*(t)^{p'-1} dt \right)^{1/p'} = B_1 < \infty;$$

(ii) for $1 < q < p < \infty$,

$$\left(\int_0^\infty \left(\int_0^{1/s} u^* \right)^{r/q} \left(\int_0^s (1/v)^{*(p'-1)} \right)^{r/q'} (1/v)^*(s)^{p'-1} ds \right)^{1/r} = B_2 < \infty,$$

where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$.

Moreover, the best constant C in (4.7) satisfies

$$C \leqslant \begin{cases} B_1(q')^{1/p'} q^{1/q}, & \text{if } 1 < p \leqslant q, \quad q \geqslant 2, \\ B_1(p')^{1/p'} p^{1/q}, & \text{if } 1 < p \leqslant q < 2, \\ B_2(p')^{1/q'} q^{1/q}, & \text{if } 1 < p < q < \infty. \end{cases}$$

Take $d > 1$. $SO(d)$ is the non-commutative special orthogonal group of proper rotations. $S \in SO(d)$ is a real $d \times d$ matrix whose transpose S^t is also its inverse S^{-1} and whose determinant $\det(S)$ is 1. A function ϕ on $\widehat{\mathbb{R}}^d$ is a *radial* if $\phi(S\gamma) = \phi(\gamma)$ for all $S \in SO(d)$.

Radial measures are defined in the following way.

DEFINITION 4.13. $\mu \in M(\widehat{\mathbb{R}}^d)$, $d > 1$, is *radial* if $S\mu = \mu$ for all $S \in SO(d)$, where $S\mu$ is defined as

$$\forall \phi \in C_c(\widehat{\mathbb{R}}^d), \quad \langle S\mu, \phi \rangle = \langle \mu(\gamma), \phi(S\gamma) \rangle.$$

If $d\mu(\gamma) = u(\gamma)d\gamma$, i.e., μ is identified with $u \in L^1_{\text{loc}}(\widehat{\mathbb{R}}^d)$, then $(Su)(\gamma) = u(S^{-1}\gamma)$ for $S \in SO(d)$; in fact,

$$\int (Su)(\gamma)\phi(\gamma)d\gamma = \int u(\gamma)\phi(S\gamma)d\gamma = \int u(S^{-1}\gamma)\phi(\gamma)d\gamma,$$

where the second equality follows since the Jacobian of any rotation is 1.

PROPOSITION 4.14. [**13**] *Given $\mu \in M(\widehat{\mathbb{R}}^d)$ and assume $\mu(\{0\}) = 0$. If μ is radial, then there is a unique measure $\nu \in M(0,\infty)$ such that for all radial functions $\phi \in C_c(\widehat{\mathbb{R}}^d)$,*

$$\langle \mu, \phi \rangle = \omega_{d-1} \int_{(0,\infty)} \rho^{d-1}\phi(\rho)d\nu(\rho), \tag{4.8}$$

where $\omega_{d-1} = 2\pi^{d/2}/\Gamma(d/2)$ is the surface area of the unit sphere Σ_{d-1} of $\widehat{\mathbb{R}}^d$.

Formula (4.8) extends to the radial elements of $L^1_\mu(\widehat{\mathbb{R}}^d)$ by Lebesgue's theorem.

Define

$$M_0(d) = \{f \in L^1(\mathbb{R}^d) : \text{supp} f \text{ is compact and } \widehat{f}(0) = 0\},$$

see Section 5 for more on moment spaces.

THEOREM 4.15. [**13**] *Given radial $\nu \in L^1_{loc}(\mathbb{R}^d)$, $\nu > 0$ a.e., and radial $\mu \in M_+(\widehat{\mathbb{R}}^d)$, $\mu(\{0\}) = 0$. Let $v \in M_+((0,\infty))$ denote the measure on $(0,\infty)$ corresponding to μ (as in Proposition 4.14). Assume $1 < p \leqslant q < \infty$ and $\nu^{1-p'} \in L^1_{loc}(\mathbb{R}^d \backslash B(0,y))$ for each $y > 0$. If*

$$B_1 = \sup_{y>0} \left(\int_{(0,y)} \rho^{d-1+q} d\nu\left(\frac{\rho}{\pi}\right) \right)^{1/q} \left(\int_0^{1/y} r^{d-1+p'} v(r)^{1-p'} dr \right)^{1/p'} < \infty \tag{4.9}$$

and

$$B_2 = \sup_{y>0} \left(\int_{(y,\infty)} \rho^{d-1} d\nu\left(\frac{\rho}{\pi}\right) \right)^{1/q} \left(\int_{1/y}^{\infty} r^{d-1} v(r)^{1-p'} dr \right)^{1/p'} < \infty, \tag{4.10}$$

then there is $C > 0$ such that, for all $f \in M_0(d) \cap L^p_\nu(\mathbb{R}^d)$,

$$||\widehat{f}||_{q,\mu} \leqslant C||f||_{p,\nu}. \tag{4.11}$$

Furthermore, C can be chosen as

$$C = 2\omega_{d-1}^{1/q+1/p'}\pi^{-(d-1)/q}(p)^{1/q}(p')^{1/p'}(B_1 + B_2).$$

The notation $d\nu(\rho/\pi)$ signifies $(1/\pi)\eta(\rho/\pi)d\rho$ in the case $d\nu(\rho) = \eta(\rho)d\rho$.

4.4. Weighted gradient inequalities.

THEOREM 4.16 ([**65**], Theorem 4.1). *Let $1 < q < \infty$, and let $u, v \geqslant 0$ be Borel measurable functions on $\mathbb{R}^d$.*

(a) Then,

$$\exists\, C > 0, \quad \forall g \in C_c^\infty(\mathbb{R}^d), \quad ||g||_{q,u} \leqslant C||t\nabla g(t)||_{q,v} \tag{4.12}$$

if and only if

$$\sup_{s \in \mathbb{R}^d} \left(\int_0^1 u(xs)x^{d-1}dx \right)^{1/q} \left(\int_1^\infty (v(xs)x^d)^{-q'/q} x^{-1} dx \right)^{1/q'} = K < \infty.$$

The constants C and K satisfy the inequalities, $K \leqslant C \leqslant K q^{1/q}(q')^{1/q'}$.
(b) Furthermore,

$$\exists\, C > 0, \quad \forall g \in C_c^{\infty}(\mathbb{R}^d) \text{ for which } g(0) = 0, \quad ||g||_{q,u} \leqslant C||t\nabla g(t)||_{q,v}$$

if and only if

$$\sup_{s \in \mathbb{R}^d} \left(\int_1^{\infty} u(xs) x^{d-1} dx \right)^{1/q} \left(\int_0^1 (v(xs)x^d)^{-q'/q} x^{-1} dx \right)^{1/q'} = K < \infty.$$

5. Uncertainty principle inequalities

5.1. Moment spaces. In this section we provide extensions and refinements of the classical uncertainty principle inequality by using the inequalities obtained in Sections 3 and 4.

We introduced the moment space $M_0(d)$ before Theorem 4.15 in Section 4. For all practical purposes, $M_0(d)$ can be replaced by the following subspaces of the Schwartz space $\mathscr{S}(\mathbb{R}^d)$:

$$\mathscr{S}_0(\mathbb{R}^d) = \{f \in \mathscr{S}(\mathbb{R}^d) : \hat{f}(0) = 0\}$$

and

$$\mathscr{S}_{0,a}(\mathbb{R}^d) = \{f \in \mathscr{S}(\mathbb{R}^d) : \hat{f}(\gamma_1, \ldots, \gamma_d) = 0 \text{ if some } \gamma_j = 0\}.$$

In particular, $f \in \mathscr{S}(\mathbb{R}^d)$ is an element of $\mathscr{S}_{0,a}(\mathbb{R}^d)$ if $\hat{f} = 0$ on the coordinate axes.

THEOREM 5.1. [**13**] *Let $v \in L^r_{\mathrm{loc}}(\mathbb{R}^d)$ for some $r > 1$, where $v > 0$ a.e., and choose $p \in (0, \infty)$. a. If $h \in L^p_v(\mathbb{R}^d)'$ annihilates $\mathscr{S}_0(\mathbb{R}^d) \bigcap L^p_v(\mathbb{R}^d)$, then h is a constant function.*

b. $\overline{\mathscr{S}_0(\mathbb{R}^d) \bigcap L^p_v(\mathbb{R}^d)} = L^p_v(\mathbb{R}^d)$ or $L^p_v(\mathbb{R}^d) \subseteq L^1(\mathbb{R}^d)$.

c. If $v^{1-p'} \notin L^1(\mathbb{R}^d)$, then $\overline{\mathscr{S}_0(\mathbb{R}^d) \bigcap L^p_v(\mathbb{R}^d)} = L^p_v(\mathbb{R}^d)$.

REMARK 5.2. *a.* The condition $p > 1$ is necessary in Theorem 5.1. In fact, if $p = 1$ and $v = 1$, then by a standard spectral synthesis result [**6**], the L^1-closure of $\mathscr{S}_0(\mathbb{R}^d)$ is the closed maximal ideal $\{f \in L^1(\mathbb{R}^d) : \hat{f}(0) = 0\}$.

b. Subsequent work dealing with $\mathscr{S}_0(\mathbb{R}^d)$ and weighted Lebesgue spaces is due to Carton-LeBrun [**19**].

Now consider $\mathscr{S}_0(\mathbb{R}^d)$ as a subspace of $L^2_{1,1}(\mathbb{R}^d)$. The following is not difficult to verify.

PROPOSITION 5.3. *a. $\mathscr{S}_0(\mathbb{R}^d)^{\perp}$ as a subspace of $L^2_{1,1}(\mathbb{R}^d)'$ is the set of constant functions on $\mathbb{R}^d$.*

b. The closure of $\mathscr{S}_0(\mathbb{R}^d)$ in $L^2_{1,1}(\mathbb{R}^d)$ is $\{f \in L^2_{1,1}(\mathbb{R}^d) : \hat{f}(0) = 0\}$.

Proposition 5.3*a* and the inclusion $L^2_{1,1}(\mathbb{R}^d) \subseteq L^1(\mathbb{R}^d)$ give Proposition 5.3*b*.

5.2. Weighted uncertainty principle inequalities on $\mathbb{R}$. We begin with the following.

THEOREM 5.4 ([**8**], Proposition 2.1.2). *Given $1 < p \leqslant 2$. Then,*

$$\forall f \in \mathscr{S}(\mathbb{R}), \quad ||f||_2^2 \leqslant 4\pi B_1(p)||xf(x)||_p||\gamma\hat{f}(\gamma)||_p, \tag{5.1}$$

where $B_1(p)$ was defined in (4.3).

The proof is similar to the proof of Theorem 1.1: the $L^p(\mathbb{R})$ - version of Hölder's inequality is used instead of the $L^2(\mathbb{R})$ - version, and the Hausdorff - Young inequality replaces the Plancherel theorem.

Heinig and Smith strengthened Theorem 5.4:

THEOREM 5.5 ([**44**], Theorem 1.1). *Given $1 < p \leqslant 2$. Then,*

$$\forall f \in \mathscr{S}_0(\mathbb{R}), \quad ||f||_2^2 \leqslant 2\pi p B_1(p)||xf(x)||_p||\gamma\widehat{f}(\gamma)||_p. \tag{5.2}$$

The constant in (5.2) is sharper than that in (5.1) for $1 < p < 2$. The proof of (5.2) is also similar to the proof of Theorem 1.1 but depends on Hardy's inequality in the following way:

$$\begin{aligned}\int_0^\infty \left|\widehat{f}(\gamma)\right|^2 d\gamma &\leqslant \left(\int_0^\infty \left|\gamma\widehat{f}(\gamma)\right|^p d\gamma\right)^{1/p}\left(\int_0^\infty \left|\frac{1}{\gamma}\widehat{f}(\gamma)\right|^{p'} d\gamma\right)^{1/p'}\\ &= \left(\int_0^\infty \left|\gamma\widehat{f}(\gamma)\right|^p d\gamma\right)^{1/p}\left(\int_0^\infty \left|\frac{1}{\gamma}P_1((\widehat{f})')(\gamma)\right|^{p'} d\gamma\right)^{1/p'}\\ &\leqslant p\left(\int_0^\infty \left|\gamma\widehat{f}(\gamma)\right|^p d\gamma\right)^{1/p}\left(\int_0^\infty \left|(\widehat{f})'(\gamma)\right|^{p'} d\gamma\right)^{1/p'}.\end{aligned}$$

We can then prove the following weighted uncertainty principle inequality, see [**7**], page 408.

THEOREM 5.6. [**7**] *Given $1 < p \leqslant q < \infty$ and even Borel measurable functions $v,\ w \geqslant 0$, that are increasing on $(0, \infty)$. Assume $(1/w, v) \in F(p,q)$ with constant K (as in (4.5)). Then, there is a $C = C(K) > 0$ such that*

$$\forall f \in \mathscr{S}(\mathbb{R}), \quad ||f||_2^2 \leqslant 4\pi C(K)||xf(x)||_{p,v}||\gamma\widehat{f}(\gamma)||_{q',w^{q'/q}}. \tag{5.3}$$

PROOF. The proof is a consequence of the estimate,

$$\begin{aligned}||f||_2^2 = ||\widehat{f}||_2^2 &\leqslant 2\int\left|\gamma\overline{\widehat{f}(\gamma)}(\widehat{f})'(\gamma)\right| d\gamma\\ &= 2\int\left|\gamma\overline{\widehat{f}(\gamma)}w(\gamma)^{1/q}\right|\left|(\widehat{f})'(\gamma)w(\gamma)^{-1/q}\right| d\gamma\\ &\leqslant 2\left(\int\left|\gamma\overline{\widehat{f}(\gamma)}\right|^{q'} w(\gamma)^{q'/q}d\gamma\right)^{1/q'}\left(\int\left|(\widehat{f})'(\gamma)\right|^q w(\gamma)^{-1}d\gamma\right)^{1/q}\\ &\leqslant 2C\left|\left|\left((\widehat{f})'\right)^\vee(x)\right|\right|_{p,v}\left|\left|\gamma\widehat{f}(\gamma)\right|\right|_{q',w^{q'/q}},\end{aligned}$$

and the fact that $((\widehat{f})')^\vee(x) = 2\pi i x f(x)$. $\square$

5.3. Weighted uncertainty principle inequalities on $\mathbb{R}^d$. Combining Theorem 3.8 and the regrouping lemma (Proposition 3.10), we obtain the following.

THEOREM 5.7. *Given $1 < r < \infty$ and Borel measurable functions $v,\ w \geqslant 0$. Suppose $u = w^{-r'/r}$. Assume that for all $\omega \in \Omega$, the weights $u(\omega\gamma)$ and $v(\omega\gamma)$ satisfy conditions (3.4), (3.5), (3.6) on $\widehat{\mathbb{R}}^{+d}$ for $p = q = r'$ and constants $K(\omega)$, $C_1(p,\omega)$, $C_2(p,\omega)$. If $C = \sup_{\omega\in\Omega} K(\omega)C_1(p,\omega)^{1/p'}C_2(p,\omega)^{1/p}$, then*

$$\forall f \in \mathscr{S}_{0,a}(\mathbb{R}^d), \quad ||f||_2^2 \leqslant C||\widehat{f}||_{r,w}||\nabla\widehat{f}||_{r',v}. \tag{5.4}$$

REMARK 5.8. At this point, generalizations of Theorems 5.5 and 5.6 can be obtained by applying $d-$dimensional versions of Theorem 4.10 to the factor $||\partial_1, \ldots \partial_d \widehat{f}||_{r',v}$ on the right side of (5.4). We shall not look at all forms of rearrangements, but confine ourselves to the following.

If $v = 1$ and $p = q = r'$, then the supremum in Theorem 3.8 has the form,

$$\sup_{s>0} \quad (s_1 \ldots s_d)^{1/r} \left(\int_{\langle s, \infty \rangle} u(y) dy \right)^{1/r'} = K \tag{5.5}$$

and (3.5) is satisfied for $C_1(r') = r^d$. For this setting we have the following uncertainty principle inequality.

THEOREM 5.9. *Given $1 < r \leqslant 2$ and let the non-negative Borel measurable weight w be invariant under the action of Ω. Assume $K < \infty$ (in (5.5)) for $u = w^{-r'/r}$ and that*

$$P'_d(w^{-r'/r}(P'_d(w^{-r'/r})^{-1/r}) \leqslant C_2(r')(P'_d(w^{-r'/r})^{1/r'}. \tag{5.6}$$

Then, for all $f \in \mathscr{S}_{0,a}(\mathbb{R}^d)$,

$$\begin{aligned} ||f||_2^2 \leqslant & (2\pi)^d r^{d/r} K C_2(r')^{1/r'} B_d(r) ||t_1 \ldots t_d f(t)||_r ||\widehat{f}||_{r,w} \\ \leqslant & (2\pi)^d r^{d/r} d^{-d/2} K C_2(r')^{1/r'} B_d(r) |||t|^d f(t)||_r ||\widehat{f}||_{r,w}. \end{aligned}$$

The weight $w(\gamma) = |\gamma_1 \ldots \gamma_d|^r$, $1 < r \leqslant 2$, is Ω-invariant, $K = (r'-1)^{-d/r'}$ in (5.5), and (5.6) is satisfied for $C_2(r') = (r(r'-1))^d$. Thus, we obtain the following d-dimensional generalization of Theorem 5.5.

THEOREM 5.10 ([**8**], Section 2.1.11). *Given $1 < r \leqslant 2$. Then, for all $f \in \mathscr{S}_{0,a}(\mathbb{R}^d)$,*

$$||f||_2^2 \leqslant (2\pi r)^d B_d(r) ||t_1 \ldots t_d f(t)||_r ||\gamma_1 \ldots \gamma_d \widehat{f}(\gamma)||_r.$$

Using Theorem 4.16, Theorem 4.8, and Minkowski's inequality we obtain the following inequality.

THEOREM 5.11 ([**8**], [**9**], Theorem 7.62). *Given $1 < r \leqslant 2$ and a nonnegative radial weight $w \in A_r$ on $\mathbb{R}^d$ for which $w(|t|)$ is increasing on $(0, \infty)$. Assume*

$$\sup_{s \in \mathbb{R}^d} \left(\int_0^1 \frac{w(xs)^{-r'/r}}{|xs|^{r'}} x^{d-1} dx \right)^{1/r'} \left(\int_1^\infty w\left(\frac{1}{|xs|}\right)^{-1} |xs|^r x^{-1-(dr/r')} dx \right)^{1/r} = K < \infty. \tag{5.7}$$

Then, there is a constant $C = C(K) > 0$ such that

$$\forall f \in C_c^\infty(\mathbb{R}^d), \quad ||f||_2^2 \leqslant C |||t| f(t)||_{r,w} |||\gamma| \widehat{f}(\gamma)||_{r,w}.$$

PROOF. For $1 < r < \infty$ we have

$$||f||_2^2 \leqslant ||\ |t| f(t)||_{r,w} ||f||_{r',u}, \tag{5.8}$$

where

$$u(t) = |t|^{-r'} w(t)^{-r'/r}.$$

The second factor on the right side of (5.8) is estimated by means of Theorem 4.16a, where q and v in (4.12) are $q = r'$ and

$$v(t) = |t|^{-r'} w\left(\frac{1}{|t|}\right)^{r'/r},$$

respectively. Thus,

$$||f||_{r',u} \leqslant C_1||t\nabla f(t)||_{r',v} \tag{5.9}$$

if and only if (5.7) holds.

By Minkowski's inequality the right side of (5.9) is bounded by

$$C_1 \sum_{j=1}^{d} \left(\int |(R_j G_j)^{\vee}(t)|^{r'} w\left(\frac{1}{|t|}\right)^{r'/r} dt \right)^{1/r'}, \tag{5.10}$$

where $G_j^{\vee}(t) = \partial_j f(t)$. Combining (5.9) and (5.10), and applying Theorem 4.5 for the case $p = r$ and $q = p'$ (so that $1 < r \leqslant 2$), we obtain

$$||f||_{r',u} \leqslant C_1 C_2 \sum_{j=1}^{d} ||R_j G_j||_{r,w}. \tag{5.11}$$

Finally, combining (5.8) and (5.11) and applying Theorem 4.8 to the right side of (5.11), we have the estimate

$$\begin{aligned} ||f||_2^2 &\leqslant C_1C_2C_3|||t|f(t)||_{r,w} \sum_{j=1}^{d} ||(\partial_j f)^{\vee}(\gamma)||_{r,w} \\ &\leqslant 2\pi d^{1/r'} C_1C_2C_3|||t|f(t)||_{r,w} \left(\int |\widehat{f}(\gamma)|^r \left(\sum_{j=1}^{d} |\gamma_j|^r \right) w(\gamma) d\gamma \right)^{1/r} \\ &\leqslant 2\pi d^{1/2} C_1C_2C_3|||t|f(t)||_{r,w}|||\gamma|\widehat{f}(\gamma)||_{r,w}. \end{aligned}$$

□

Corollary 5.12. *Given $1 < r \leqslant 2$ and $d > r'$, there is $C > 0$ such that*

$$\forall f \in S(\mathbb{R}^d), \quad ||f||_2^2 \leqslant C|||t|f(t)||_r|||\gamma|\widehat{f}(\gamma)||_r. \tag{5.12}$$

Remark 5.13. *a.* The constant C in Corollary 5.12 is of the form

$$C = 2\pi d^{1/2} C_1(r, d) B_d(r) C_3(r).$$

Since it is of interest to measure the growth of C as d increases, we note that $C_1(r, d)$ can be estimated in terms of K in (5.7) for any w.

b. Theorem 4.16b gives rise to an analogue of Theorem 5.11 which, for $w = 1$, yields (5.12) for $d < r'$.

See [**14**] for a summary of these and further results.

6. An uncertainty principle inequality for Hilbert spaces

6.1. An uncertainty principle inequality. We shall prove a well-known uncertainty principle inequality for Hilbert spaces [**71**], [**8**], [**9**], [**32**]. This result is also referred to as the Robertson uncertainty relation [**62**].

Definition 6.1. Let A, B be self-adjoint operators on a complex Hilbert space H. (A and B need not be continuous.) Define the *commutator* $[A, B] = AB - BA$, the *expectation* or *expected value* $E_x(A) = \langle Ax, x\rangle$ of A at $x \in D(A)$, where $D(A)$ denotes the domain of A, and the *variance* $\Delta_x^2(A) = E_x(A^2) - \{E_x(A)\}^2$ of A at $x \in D(A^2)$.

THEOREM 6.2 ([**9**], Theorem 7.2). *Let A, B be self-adjoint operators on a complex Hilbert space H (A and B need not be continuous). If*

$$x \in D(A^2) \cap D(B^2) \cap D(i[A,B])$$

and $||x|| \leqslant 1$, then

$$\{E_x(i[A,B])\}^2 \leqslant 4\Delta_x^2(A)\Delta_x^2(B). \tag{6.1}$$

PROOF. By self-adjointness, we first compute

$$\begin{aligned} E_x(i[A,B]) =& i\left(\langle Bx, Ax\rangle - \langle Ax, Bx\rangle\right) \\ =& 2\mathrm{Im}\langle Ax, Bx\rangle. \end{aligned} \tag{6.2}$$

Also note that $D(A^2) \subseteq D(A)$.

Since $||x|| \leqslant 1$ and $\langle Ax, x\rangle$, $\langle Bx, x\rangle \in \mathbb{R}$ by self-adjointness, we have

$$||(B+iA)x||^2 - |\langle (B+iA)x, x\rangle|^2 \geqslant 0 \tag{6.3}$$

and

$$|\langle (B+iA)x, x\rangle|^2 = \langle Bx, x\rangle^2 + \langle Ax, x\rangle^2. \tag{6.4}$$

By the definition of $||\cdot||$, we compute

$$||(B+iA)x||^2 = ||Bx||^2 + ||Ax||^2 - 2\mathrm{Im}\,\langle Ax, Bx\rangle. \tag{6.5}$$

Substituting (6.4) and (6.5) into (6.3) yields the inequality,

$$\begin{aligned} ||Ax||^2 - \langle Ax, x\rangle^2 + ||Bx||^2 - \langle Bx, x\rangle^2 \\ \geqslant 2\mathrm{Im}\,\langle Ax, Bx\rangle. \end{aligned} \tag{6.6}$$

Letting $r, s \in \mathbb{R}$, so that rA and sB are also self-adjoint, (6.6) becomes

$$\begin{aligned} r^2\left(||Ax||^2 - \langle Ax, x\rangle^2\right) + s^2\left(||Bx||^2 - \langle Bx, x\rangle^2\right) \\ \geqslant 2rs\mathrm{Im}\,\langle Ax, Bx\rangle. \end{aligned} \tag{6.7}$$

Setting $r^2 = ||Bx||^2 - \langle Bx, x\rangle^2$ and $s^2 = ||Ax||^2 - \langle Ax, x\rangle^2$, substituting into (6.7), squaring both sides and dividing, we obtain

$$\left(||Ax||^2 - \langle Ax, x\rangle^2\right)\left(||Bx||^2 - \langle Bx, x\rangle^2\right) \geqslant \left(\mathrm{Im}\langle Ax, Bx\rangle\right)^2.$$

From this inequality and (6.2) the uncertainty principle inequality (6.1) follows. □

6.2. Examples.

EXAMPLE 6.3. (The classical uncertainty inequality) The classical uncertainty principle inequality, Theorem 1.1, is a corollary of Theorem 6.2 for the case $H = L^2(\mathbb{R})$, where the operators A and B are defined as

$$A(f)(t) = (t - t_0)f(t)$$

and

$$B(f)(t) = i\left(2\pi i(\gamma - \gamma_0)\widehat{f}(\gamma)\right)^{\vee}(t).$$

Straightforward calculations show that A and B are self-adjoint, and that

$$E_f(A) = \int (t - t_0)|f(t)|^2 dt,$$

$$E_f(B) = -2\pi \int (\gamma - \gamma_0)|\widehat{f}(\gamma)|^2 d\gamma,$$

$$\Delta_f^2(A) = \int |t-t_0|^2|f(t)|^2dt - \left(\int (t-t_0)|f(t)|^2dt\right)^2,$$

$$\Delta_f^2(B) = 4\pi^2\left(\int |\gamma-\gamma_0|^2|\widehat{f}(\gamma)|^2d\gamma - \left(\int (\gamma-\gamma_0)|\widehat{f}(\gamma)|^2d\gamma\right)^2\right),$$

$$\{E_f(i[A,B])\}^2 = ||f||_2^4.$$

Example 6.4 (Pauli and generalized Gell-Mann matrices). The following matrices, referred to as the Pauli matrices, are important in quantum mechanics, where they occur in the Pauli equation which models the interaction of a particle's spin with external electromagnetic fields [**64**]:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Note that each Pauli matrix is Hermitian, and together with the identity matrix, the Pauli matrices span the vector space of 2×2 Hermitian matrices. From a quantum mechanical point of view, Hermitian matrices are observables, and thus the Pauli matrices span the space of observables of the 2-dimensional complex Hilbert space.

The Pauli matrices may be generalized to the Gell-Mann matrices in dimension 3, and then these to the so-called generalized Gell-Mann matrices in any dimension d [**36**]. In dimension d, this is the following family of matrices. Let $E_{j,k}$ denote the $d \times d$ matrix with 1 in the $jk-$th entry. Define the following matrices:

$$A_{k,j}^d = \begin{cases} E_{k,j} + E_{j,k}, & \text{for } k < j, \\ -i(E_{j,k} - E_{k,j}), & \text{for } k > j, \end{cases}$$

and

$$h_k^d = \begin{cases} I_d, & \text{for } k = 1, \\ h_k^{d-1} \oplus 0, & \text{for } 1 < k > d, \\ \sqrt{\frac{2}{d(d-1)}}\left(h_1^{d-1} \oplus (1-d)\right), & \text{for } k = d. \end{cases}$$

Thus, for any dimension d, Theorem 6.2 can be applied to any pair of generalized Gell-Mann matrices, to obtain inequalities regarding the the components of vectors in the unit disc in $\mathbb{C}^d$. For example, when $d = 2$ and $H = \mathbb{C}^2$, we apply Theorem 6.2 to the operators A and B (as defined above). Straightforward calculations give the following inequality.

Corollary 6.5. *Let $z = (z_1, z_2) \in \mathbb{C}^2$, $|z| \leqslant 1$. Then,*

$$\left(|z_1|^2 - |z_2|^2\right)^2 \leqslant \left(|z|^2 + (z_1\bar{z}_2 - \bar{z}_1z_2)^2\right)\left(|z|^2 - (\bar{z}_1z_2 - z_1\bar{z}_2)^2\right).$$

Similarly, using the pair B and C we obtain the following inequality.

Corollary 6.6. *Let $z = (z_1, z_2) \in \mathbb{C}^2$, $|z| \leqslant 1$. Then,*

$$(\bar{z}_1z_2 + z_1\bar{z}_2)^2 \leqslant \left(|z|^2 + (z_1\bar{z}_2 - \bar{z}_1z_2)^2\right)\left(|z|^2 - (|z_1|^2 - |z_2|^2)^2\right).$$

Example 6.7 (Ornstein-Uhlenbeck operator). Let $L^2_\mu(\mathbb{R}^d)$ denote $L^2(\mathbb{R}^d)$ with respect to the Gaussian measure $d\mu$, where

$$d\mu(x) = \frac{1}{\pi^{d/2}}e^{|x|^2}dx.$$

The Ornstein-Uhlenbeck operator, L, is the self-adjoint second-order differential operator defined as

$$Lf(x) = \Delta f(x) - x\nabla f(x).$$

Let Q be the position operator

$$Qf(x) = xf(x).$$

Taking $d = 1$ and working on $L^2_\mu(\mathbb{R}^d)$, we compute

$$[L, Q]f(x) = 2f'(x) - xf(x),$$

$$E_f([L,Q]) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \left(2f'(x)f(x) - xf(x)^2\right) e^{-x^2}\, dx,$$

$$E_f(L) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} (f''(x) - xf'(x))f(x)e^{-x^2}\, dx, \quad E_f(Q) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} xf(x)^2 e^{-x^2}\, dx,$$

$$E_f(L^2) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} f(x) \left(f''''(x) - 2xf'''(x) + xf''(x) - 2f''(x) + xf'(x)\right) e^{-x^2}\, dx,$$

and

$$E_f(Q^2) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} x^2 f(x)^2 e^{-x^2}\, dx.$$

Then, applying Theorem 6.2 and assuming sufficient differentiability, we have the following inequality.

Corollary 6.8.

$$\left| \int_{\mathbb{R}} \left(2f'f - xf^2\right) d\mu \right|$$

$$\leqslant 2 \left(\frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} (f'' - xf')f d\mu \right)^{1/2} \left(\int_{\mathbb{R}} f \left(f'''' - 2xf''' + xf'' - 2f'' + xf'\right) d\mu \right)^{1/2}.$$

Remark 6.9. *a.* The theory that developed around the Ornstein-Uhlenbeck operator can be viewed as a model of harmonic analysis in which Lebesgue measure is replaced by a Gaussian measure. This theory has applications to quantum physics and probability. In an infinite dimensional setting, the theory leads to the Malliavin calculus [**1**], [**53**].

b. The Hermite polynomials form an orthogonal system with respect to the Gaussian measure in Euclidean space, and they are the eigenfunctions of the Ornstein-Uhlenbeck operator.

7. Epilogue

In 1982, when the first named author began to travel (literally, driving from Toronto to Ottawa) with Hans Heinig on the path of weighted Fourier transform norm inequalities and uncertainty principle inequalities, he also had the great good fortune to begin a correspondence with John F. Price. This, combined with Fritz Carlson's inequality (1934) and the Bell Labs inequalities of Henry J. Landau, David Slepian, and Henry Pollack [**52**], led to the exposition [**8**] in 1989 featuring local uncertainty principle inequalities, spearheaded by Faris [**28**], Cowling and Price [**22**], [**23**], and Price [**60**], and in the context of more classical work inspired by Carlson's work.

Subsequently, others have exposited the local theory, but there is an argument to update the current state of affairs, especially in light of the uncertainty principle inequalities of Donoho and Stark [**27**] and Tao [**69**], and the advent of thinking in terms of sparsity, compressive sensing, and dimension reduction, as well as quantum inequalities emanating from the role of Gårding's inequality, see, e.g., [**30**].

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