Smooth functions associated with wavelet sets on $\mathbb{R}^d$, $d \geq 1$, and frame bound gaps

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Abstract

The theme is to smooth characteristic functions of Parseval frame wavelet sets by convolution in order to obtain implementable, computationally viable, smooth wavelet frames. We introduce the following: a new method to improve frame bound estimation; a shrinking technique to construct frames; and a nascent theory concerning frame bound gaps. The phenomenon of a *frame bound gap* occurs when certain sequences of functions, converging in L^2 to a Parseval frame wavelet, generate systems with frame bounds that are uniformly bounded away from 1. We prove that smoothing a Parseval frame wavelet set wavelet on the frequency domain by convolution with elements of an approximate identity produces a frame bound gap. Furthermore, the frame bound gap for such frame wavelets in $L^2(\mathbb{R}^d)$ increases and converges as d increases.

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1 Introduction

1.1 Problem

Wavelet theory for $\mathbb{R}^d$, $d > 1$, was historically associated with multiresolution analysis (MRA), *e.g.*, [Mey90]. In particular, for dyadic wavelets, it is well-known that $2^d - 1$ wavelets are required to provide a wavelet orthonormal basis (ONB) with an MRA for $L^2(\mathbb{R}^d)$, cf., [Mad92], [Aus95], and [Str93]. In fact, until the mid-1990s, it was assumed that it would be impossible to construct a single dyadic wavelet ψ generating an ONB for $L^2(\mathbb{R}^d)$. This changed with the groundbreaking work of Dai and Larson [DL98] and Dai, Larson, and Speegle [DLS97], [DLS98]. The earliest known examples of such single dyadic wavelets for $d > 1$ had complicated spectral properties, see [BMM99], [BL99], [BL01], [BL98], [DL98], [DLS97], [DLS98], [HWW96], [HWW97], [SW98], [Zak96]. Further, such wavelets have discontinuous Fourier transforms. As such it is a natural problem to construct single wavelets with better temporal decay. Further, even on $\mathbb{R}$, in order improve the temporal decay, one must consider systems of frames rather than orthonormal bases [BJMP06], [CH97], [Han94], [Han97] or wavelets which have an MRA structure [HWW96], [HWW97]. We shall address the problem of smoothing $\widehat{\psi}$ by convolution, where ψ is derived by the so-called neighborhood mapping method, see Section 1.3. This method has the advantage of being general and constructive. Although there are other smoothing techniques that have been introduced in the area of wavelet theory, e.g., [Han94] and [Han97], we choose to smooth by convolution because of its theoretical simplicity and computational effectiveness. However, as will be shown later in the paper, convolutional smoothing on the frequency domain yields counterintuitive results.

1.2 Preliminaries

Definition 1. Let $\psi \in L^2\left(\mathbb{R}^d\right)$ and define the *wavelet system*,

$$\mathcal{W}(\psi) = \{D_n T_k \psi(x) : n \in \mathbb{Z}, k \in \mathbb{Z}^d\} = \{2^{nd/2}\psi\left(2^n x - k\right) : n \in \mathbb{Z}, k \in \mathbb{Z}^d\}.$$

If $\mathcal{W}(\psi)$ is an orthonormal basis for $L^2\left(\mathbb{R}^d\right)$, then ψ is an *orthonormal dyadic wavelet* or simply a *wavelet* for $L^2(\mathbb{R}^d)$.

The *Haar wavelet* is the function $\psi = \mathbb{1}_{[0,1/2)} - \mathbb{1}_{[1/2,1)}$, where $\mathbb{1}_S$ is the characteristic function of S. The Haar wavelet is well localized in the time domain but not in the frequency domain. There are wavelets which are characteristic functions in the frequency domain and thus are not localized in the time domain. We shall define these shortly.

Notationally, we write $\int f(x)dx = \int_{\mathbb{R}^d} f(x)dx$. Also, for $f \in L^1(\mathbb{R}^d)$, we shall use the following definition of the *Fourier transform of* f:

$$\mathcal{F}(f)(\gamma) = \hat{f}(\gamma) = \int f(x)e^{-2\pi i x\cdot\gamma}dx,$$

$\gamma \in \widehat{\mathbb{R}}^d$, where $\widehat{\mathbb{R}} = \mathbb{R}$ considered as a spectral domain. $\mathcal{F}$ can be defined on $L^2(\mathbb{R}^d)$ by the Plancherel theorem. Formally, the inverse Fourier transform of f is written as $\mathcal{F}^{-1}(f) = \check{f}$.

A classical example of a wavelet which is the inverse Fourier transform of a characteristic function is the *Shannon* or *Littlewood-Paley wavelet*, $\check{\mathbb{1}}_{[-1,-1/2)\cup[1/2,1)}$. Another example is the Journé wavelet,

$$\check{\mathbb{1}}_{[-\frac{16}{7},-2)\cup[-\frac{1}{2},-\frac{2}{7})\cup[\frac{2}{7},\frac{1}{2})\cup[2,\frac{16}{7})}.$$

At an AMS special session in 1992, Dai and Larson introduced the term *wavelet set*, which generalizes this phenomenon. Their original publications concerning wavelet sets are [DL98] and also [DLS97] and [DLS98], which were written with Speegle. Hernàndez, Wang, and Weiss developed a similar theory in [HWW96] and [HWW97], using the terminology *minimally supported frequency (MSF) wavelets.*

Definition 2. If K is a measurable subset of $\widehat{\mathbb{R}}^d$ and $\check{\mathbb{1}}_K$ is a wavelet for $L^2(\mathbb{R}^d)$, then K is a *wavelet set.*

We now introduce the notion of frames. The theory is due to Duffin and Schaeffer [DS52].

Definition 3. A sequence $\{e_j\}_{j\in J}$ in a Hilbert space $\mathcal{H}$ is a *frame* for $\mathcal{H}$ if there exist constants $0 < A \leq B < \infty$ such that

$$\forall f \in \mathcal{H}, \qquad A\|f\|^2 \leq \sum_{j\in J} |\langle f, e_j\rangle|^2 \leq B\|f\|^2. \tag{1}$$

The maximal such A and minimal such B are the *optimal frame bounds.* In this paper, the phrase *frame bound* will always mean the optimal frame bound, where A is the *lower frame bound* and B is the *upper frame bound.* A frame is *tight* if $A = B$, and it is *Parseval* if $A = B = 1$. If a frame $\{e_j\}_{j\in J}$ for $\mathcal{H}$ has the property that for all $k \in J$, $\{e_j\}_{j\neq k}$ is not a frame for $\mathcal{H}$, then $\{e_j\}_{j\in J}$ is a *Riesz basis* for $\mathcal{H}$. If the second inequality of (1) is true, but possibly not the first, then $\{e_j\}_{j\in J}$ is a *Bessel sequence.* In this case, we shall still refer to B as the upper frame bound.

In Definition 1, we deal with wavelet systems that are orthonormal bases. However, there is no reason that we should not consider systems $\mathcal{W}(\psi)$ which form frames (respectively, Bessel sequences) for $L^2(\mathbb{R}^d)$. In this case, ψ is a *frame wavelet* (respectively, *Bessel wavelet*).

Definition 4. If L is a measurable subset of $\widehat{\mathbb{R}}^d$ and $\mathcal{W}(\check{\mathbb{1}}_L)$ is a frame (respectively, tight frame or Parseval frame) for $L^2(\mathbb{R}^d)$, then L is a *frame* (respectively, *tight frame* or *Parseval frame*) *wavelet set.*

We need the following definition in order to characterize wavelet sets and Parseval frame wavelet sets.

Definition 5. Let K and L be two measurable subsets of $\widehat{\mathbb{R}}^d$. A *partition* of K is a collection $\{K_l : l \in \mathbb{Z}\}$ of subsets of K such that $\bigcup_l K_l$ and K differ by a set of measure 0 and, for all $l \neq j$, $K_l \cap K_j$ is a set of measure 0. If there exist a partition $\{K_l : l \in \mathbb{Z}\}$ of K and a sequence $\{k_l : l \in \mathbb{Z}\} \subseteq \mathbb{Z}^d$ such that $\{K_l + k_l : l \in \mathbb{Z}\}$ is a partition of L, then K and L are $\mathbb{Z}^d$*-translation congruent*. Similarly, if there exist a partition $\{K_l : l \in \mathbb{Z}\}$ of K and a sequence $\{n_l : l \in \mathbb{Z}\} \subseteq \mathbb{Z}$, where $\{2^{n_l} K_l : l \in \mathbb{Z}\}$ is a partition of L, then K and L are *dyadic-dilation congruent.*

Proposition 6. *Let $K \subseteq \widehat{\mathbb{R}}^d$ be measurable. The following are equivalent:*

- *K is a (Parseval frame) wavelet set.*
- *K is $\mathbb{Z}^d$-translation congruent to (a subset of) $[0,1)^d$, and K is dyadic-dilation congruent to $[-1,1)^d \backslash [-\frac{1}{2}, \frac{1}{2})^d$.*
- *$\left\{K + k : k \in \mathbb{Z}^d\right\}$ is a partition of (a subset of) $\widehat{\mathbb{R}}^d$ and $\{2^n K : n \in \mathbb{Z}\}$ is a partition of $\widehat{\mathbb{R}}^d$.*

1.3 Neighborhood mapping construction

An infinite iterative construction of wavelet sets, called the *neighborhood mapping construction*, is given by Leon, Sumetkijakan, and one of the authors in [BS06], [BL99], and [BL01]. See also [Zak96], [BMM99], and [SW98]. In dimensions $d \geq 2$, the example wavelet sets K formed by this process are fractal-like but not fractals. Following a question by E. Weber, the authors proved that the sets $(K_m \backslash A_m)$ they defined, formed after a finite number of steps of the neighborhood mapping construction, are actually Parseval frame wavelet sets.

These frame wavelet sets are finite unions of convex sets. The delicate, complicated shape of an orthonormal wavelet set K constructed in [BS06] makes it difficult to use natural methods with which to smooth it. It is for this reason that we shall deal with frame wavelets and with the smoothing of $\check{\mathbb{1}}_L$, where L is a $K_m \backslash A_m$. We shall use the following collection of sets in Section 2.

Example 7. One may use the neighborhood mapping construction to find the Journé wavelet set. A generalization of this construction to higher dimensions begins with

$$K_0 \backslash A_0 = \left[-\frac{1}{2}, \frac{1}{2}\right)^d \backslash \left[-\frac{1}{4}, \frac{1}{4}\right)^d.$$

It should be mentioned that Merrill [Mer08] has recently found examples of orthonormal wavelet sets for $d = 2$ which may be represented as finite unions of 5 or more convex sets. She uses the generalized scaling set technique from [BMM99]. It is unknown if the construction can be used for $d > 2$. Moreover, the question of existence of orthonormal wavelet sets in $\widehat{\mathbb{R}}^d$ for $d > 2$, which are the finite union of convex sets, is still an open problem. Furthermore, in [BS06], it is shown that a wavelet set in $\widehat{\mathbb{R}}^d$ can not be decomposed into a union of d or fewer convex sets. It is possible that this bound is not sharp for $d = 2$; that is, it is still not known if there exists a wavelet set in $\widehat{\mathbb{R}}^2$ which may be written as the union of 3 or 4 convex sets.

1.4 Outline and results

We shall smooth Parseval wavelet sets L by convolving $\mathbb{1}_L$ with auxiliary functions to obtain $\hat{\psi}$ and consider the properties of $\mathcal{W}(\psi)$. In many cases, the resulting $\mathcal{W}(\psi)$ is a frame. In Section 2, we develop methods to estimate the resulting frame bounds. We apply those methods to a canonical example in Section 3. However, we see in Section 4 that there exists a Parseval wavelet set L such that $\mathcal{W}((\mathbb{1}_L * \frac{m}{2}\mathbb{1}_{[-\frac{1}{m},\frac{1}{m}]})^\vee)$ is not a frame for any $m > 0$. Later in Section 4, we introduce the *shrinking method*, with which we modify the preceding example to obtain a frame. This method may be used to modify Parseval frame wavelets sets in such a way that they may be smoothed using our techniques or other methods, like those in [Han97]. Section 5 contains Theorems 35 and

39, which show that *frame bound gaps* occur with many wavelet sets. In fact, for certain Parseval frame wavelet sets L and approximate identities $\{k_\lambda\}$, the system $\mathcal{W}((\mathbb{1}_L * k_\lambda)^\vee)$ does not have frame bounds that converge to 1 as $\lambda \to \infty$, even though, for all $1 \leq p < \infty$,

$$\lim_{\lambda\to\infty} \|\mathbb{1}_L * k_\lambda - \mathbb{1}_L\|_{L^p(\widehat{\mathbb{R}}^d)} = 0.$$

Furthermore, when we smooth a specific class of Parseval frame wavelet sets $L_d \subseteq \widehat{\mathbb{R}}^d$ with certain approximate identities $k_{\lambda,d} = \otimes_{i=1}^d k_\lambda$, the corresponding upper frame bounds increase and converge to 2 as $d \to \infty$. We conclude the paper with Section 6 which contains a review of previously known methods to smooth frame wavelet set wavelets.

2 Frame bounds and approximate identities

2.1 Approximating frame bounds

In this section we give several methods, mostly well-known, to evaluate frame bounds. Our goal is to manipulate Parseval frame wavelet set wavelets on the frequency domain in order to construct frames with faster temporal decay than the original Parseval frames.

Definition 8. For $k \in \mathbb{Z}^d$, define the modulation operator M_k on $L^2(\mathbb{R}^d)$ by $M_k f(x) = e^{2\pi i k\cdot x} f(x)$. As operators, $\mathcal{F}T_k = M_{-k}\mathcal{F}$.

Remark 9. The following calculation and ones similar to it are commonly used to prove facts about frame wavelet bounds. Define $Q_n = [0, 2^{-n}]^d$ and $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Using the Parseval-Plancherel theorem on both $\mathbb{R}^d$ and $\mathbb{T}^d$ as well as a standard L^1 periodization technique, we let $\psi \in L^2(\mathbb{R}^d)$ and have the following formal calculation:

$$\forall f \in L^2(\mathbb{R}^d), \quad \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} |\langle f, D_n T_k \psi\rangle|^2 =$$

$$= \sum_n 2^{dn} \sum_k \left| \int_{Q_n} \sum_{l \in \mathbb{Z}^d} \hat{f}(\gamma + 2^{-n}l) e^{2\pi i k \cdot 2^n(\gamma + 2^{-n}l)} \overline{\hat{\psi}(2^n\gamma + l)} d\gamma \right|^2$$

$$= \sum_n \int_{Q_n} \left| \sum_l \hat{f}(\gamma + 2^{-n}l) \overline{\hat{\psi}(2^n\gamma + l)} \right|^2 d\gamma$$

$$= \sum_n \int \sum_k \hat{f}(\gamma) \overline{\hat{f}(\gamma + 2^{-n}k) \hat{\psi}(2^n\gamma)} \hat{\psi}(2^n\gamma + k) d\gamma \tag{2}$$

$$= \int \left| \hat{f}(\gamma) \right|^2 \sum_n \left| \hat{\psi}(2^n\gamma) \right|^2 d\gamma + \int \sum_n \sum_{k \neq 0} \hat{f}(\gamma) \overline{\hat{f}(\gamma + 2^{-n}k) \hat{\psi}(2^n\gamma)} \hat{\psi}(2^n\gamma + k) d\gamma. \tag{3}$$

Here, (2) and (3) are formally computed, but are valid for a large class of functions $\psi \in L^2(\mathbb{R}^d)$.

To simplify notation, we define

$$F(f) = \int \left| \hat{f}(\gamma) \right|^2 \sum_n \left| \hat{\psi}(2^n\gamma) \right|^2 d\gamma + \int \sum_n \sum_{k \neq 0} \hat{f}(\gamma) \overline{\hat{f}(\gamma + 2^{-n}k) \hat{\psi}(2^n\gamma)} \hat{\psi}(2^n\gamma + k) d\gamma. \tag{4}$$

We would like to find explicit upper and lower bounds of $F(f)$ in terms of $\|f\|^2$. Clearly, these bounds correspond to frame bounds for the system $\mathcal{W}(\psi)$. Specifically, if $\mathcal{W}(\psi)$ has frame bounds A, B, then

$$A = \inf_{\|f\|_2 = 1} F(f) \quad \text{and} \quad B = \sup_{\|f\|_2 = 1} F(f).$$

Consequently, if $f \in L^2\left(\mathbb{R}^d\right)$ has unit norm, then $A \leq F(f) \leq B$.

Calculations such as this play a basic role in proving the following well-known theorem ([Dau92], [Chr02]) and its variants.

Theorem 10. *Let $\psi \in L^2(\widehat{\mathbb{R}}^d)$, and let $a > 0$ be arbitrary. Define*

$$\begin{aligned} \mu_\psi(\gamma) &= \sum_{k \in \mathbb{Z}^d} \sum_{n \in \mathbb{Z}} \left| \hat{\psi}\left(2^n\gamma\right) \hat{\psi}\left(2^n\gamma + k\right) \right| \quad \textit{and} \\ M_\psi &= \operatorname{esssup}_{\gamma \in \widehat{\mathbb{R}}^d} \mu_\psi(\gamma) = \operatorname{esssup}_{a \leq \|\gamma\| \leq 2a} \mu_\psi(\gamma). \end{aligned}$$

If $M_\psi < \infty$, then $\mathcal{W}\left(\psi\right)$ is a Bessel sequence with upper frame bound B, and $M_\psi \geq B$. Similarly, *define*

$$\begin{aligned}\nu_\psi(\gamma) &= \left[\sum_{n\in\mathbb{Z}}\left|\hat{\psi}\left(2^n\gamma\right)\right|^2 - \sum_{k\neq 0}\sum_{n\in\mathbb{Z}}\left|\hat{\psi}\left(2^n\gamma\right)\hat{\psi}\left(2^n\gamma+k\right)\right|\right] \text{ and} \\ N_\psi &= \operatorname{essinf}_{\gamma\in\widehat{\mathbb{R}}^d}\nu_\psi(\gamma) = \operatorname{essinf}_{a\leq\|\gamma\|\leq 2a}\nu_\psi(\gamma).\end{aligned}$$

If $N_\psi > 0$, then $\mathcal{W}(\psi)$ is a frame with lower frame bound $A \geq N_\psi$.

We refer to M_ψ and N_ψ as the *Daubechies-Christensen bounds.* Christensen proved Theorem 10 for functions $\psi \in L^2(\mathbb{R})$, but his proof extends to $L^2(\mathbb{R}^d)$ with only minor modifications.

Chui and Shi proved necessary conditions for a wavelet system in $L^2(\mathbb{R})$ to have certain frame bounds, [CS93b]. Jing extended this result to $L^2(\mathbb{R}^d)$ for $d \geq 1$, [Jin99].

Proposition 11. *Define $\kappa_\psi(\gamma) = \sum_{n\in\mathbb{Z}}\left|\hat{\psi}\left(2^n\gamma\right)\right|^2$. If $\mathcal{W}(\psi)$ is a wavelet frame for $L^2(\mathbb{R}^d)$ with bounds A and B, then, for almost all $\gamma \in \widehat{\mathbb{R}}^d$,*

$$A \leq \kappa_\psi(\gamma) \leq B.$$

We may combine the previous two results to obtain the following corollary.

Corollary 12. *Let $\psi \in L^2(\mathbb{R}^d)$. Let $a > 0$ be arbitrary. If $M_\psi < \infty$, then $\mathcal{W}(\psi)$ is a Bessel sequence with bound B satisfying $\operatorname{esssup}_{a\leq\|\gamma\|\leq 2a}\kappa_\psi(\gamma) \leq B \leq M_\psi$. If, further, $N_\psi > 0$, then $\mathcal{W}(\psi)$ is a frame with lower frame bound A satisfying $N_\psi \leq A \leq \operatorname{essinf}_{a\leq\|\gamma\|\leq 2a}\kappa_\psi(\gamma)$.*

Many of the ψ that we mention in this paper are continuous. In these cases, we shall simply calculate the supremum and infimum of κ_ψ, rather than the essential supremum and essential infimum.

2.2 Approximate Identities

Definition 13. An *approximate identity* is a family $\{k_{(\lambda)} : \lambda > 0\} \subseteq L^1(\mathbb{R}^d)$ of functions with the following properties:

i. $\forall \lambda > 0$, $\int k_{(\lambda)}(x)dx = 1$;

ii. $\exists K$ such that $\forall \lambda > 0$, $\|k_{(\lambda)}\|_{L^1(\mathbb{R}^d)} \leq K$;

iii. $\forall \eta > 0$, $\lim_{\lambda\to\infty} \int_{\|x\|\geq\eta} |k_{(\lambda)}(x)|dx = 0$.

The following result is well-known, e.g., [Ben97], [Fol99], [SW71].

Proposition 14. *Suppose $k \in L^1(\mathbb{R}^d)$ satisfies $\int k(x)dx = 1$. Define the family,*

$$\{k_\lambda : k_\lambda(x) = \lambda^d k(\lambda x), \lambda > 0\},$$

of dilations. Then, the following assertions hold.

a. $\{k_\lambda\}$ *is an* approximate identity*;*

b. If $f \in L^p(\mathbb{R}^d)$ for some $1 \leq p < \infty$, then $\lim_{\lambda\to\infty} \|f * k_\lambda - f\|_{L^p(\mathbb{R}^d)} = 0$;

c. If k is an even function, there exists a subsequence $\{\lambda_m\}$ of $\{\lambda\}$ such that

$$\lim_{m\to\infty} \int f(u) T_x k_{\lambda_m}(u)du = f(x) \ \text{a.e. } x \in \mathbb{R}^d.$$

We shall use approximate identities on $\widehat{\mathbb{R}}^d$. The following notation will streamline our arguments.

Definition 15. Fix a non-negative, compactly supported, bounded, even function $k : \widehat{\mathbb{R}}^d \to \mathbb{C}$ with the property that $\int k(\gamma)d\gamma = 1$. Then, $k \in L^1 \cap L^2(\widehat{\mathbb{R}}^d)$ and the results of Proposition 14 hold. For $\omega \in \widehat{\mathbb{R}}^d$ and $\alpha > 0$, define $g_{\lambda,\alpha,\omega} \in L^2(\widehat{\mathbb{R}}^d)$ by $g_{\lambda,\alpha,\omega} = \sqrt{\alpha T_\omega k_\lambda}$. If $\alpha = 1$, we write $g_{\lambda,\omega} = g_{\lambda,1,\omega}$. Note that $\|g_{\lambda,\omega}\|_2 = 1$ for all λ, ω.

3 A canonical example

For this section, let $L = \left[-\frac{1}{2}, -\frac{1}{4}\right) \cup \left[\frac{1}{4}, \frac{1}{2}\right)$, which is $K_0 \backslash A_0$ from the 1-d Journé construction, see Example 7.

Example 16. We shall compute some Bessel bounds.

a. $\mathcal{W}(\mathbb{1}_L^\vee)$ is a Parseval frame. Smooth $\mathbb{1}_L$ by defining $\hat{\psi} = \mathbb{1}_L * 8\mathbb{1}_{[-\frac{1}{16}, \frac{1}{16}]}$. We would like to determine if $\mathcal{W}(\psi)$ is a Bessel sequence and, if so, to determine its upper frame bound. We compute $\sup_\gamma \kappa_\psi(\gamma) = \frac{17}{16}$. Within the dyadic interval $\left[\frac{9}{32}, \frac{9}{16}\right)$ this supremum occurs at $\frac{7}{16}$. Also, $M_\psi = \frac{17}{16}$, where the supremum occurs at the same point. Thus, by Corollary 12, the upper frame bound of $\mathcal{W}(\psi)$ is $\frac{17}{16}$.

b. Similarly, if $\hat{\psi} = \mathbb{1}_{[-\frac{1}{2}, \frac{1}{2})^2 \backslash [-\frac{1}{4}, \frac{1}{4})^2} * 64\mathbb{1}_{[-\frac{1}{16}, \frac{1}{16}]^2}$, then the upper frame bound of $\mathcal{W}(\psi)$ is $\frac{305}{256}$.

Example 17. Once again, let $\hat{\psi} = \mathbb{1}_L * 8\mathbb{1}_{[-\frac{1}{16}, \frac{1}{16}]}$.

a. We have that $\inf_\gamma \kappa_\psi(\gamma) = \frac{9}{20}$ and $N_\psi = \frac{2}{9}$. It now follows from Corollary 12 that $\mathcal{W}$ is a frame with lower frame bound A, satisfying $\frac{2}{9} \leq A \leq \frac{9}{20}$. We would like to tighten these bounds around A. This is a delicate operation. For this estimate, we shall use functions consisting of multiple spikes, scaled by positive and negative numbers. We have that $\inf_\gamma \kappa_\psi(\gamma)$ occurs at $\frac{21}{40}$ within the dyadic interval $\left[\frac{9}{32}, \frac{9}{16}\right)$. By symmetry, this infimum is also achieved at $-\frac{21}{40}$. Further,

$$\sup_\gamma \sum_n \sum_{l \neq 0} \hat{\psi}(2^n\gamma)\overline{\hat{\psi}(2^n\gamma + l)} = \frac{1}{4}.$$

This supremum occurs at $\pm\frac{1}{2}$. In order to compute the lower frame bound, we need to minimize $F(f)$, defined in (4), over all $f \in L^2(\mathbb{R}^d)$. We shall refer to the summands,

$$\hat{f}(\gamma)\overline{\hat{f}(\gamma + 2^{-n}k)\hat{\psi}(2^n\gamma)}\hat{\psi}(2^n\gamma + k),$$

in $F(f)$ as *cross terms.* We would like to find an $\hat{f} \in L^2(\widehat{\mathbb{R}}^d)$ that allows us to use the cross terms to mitigate the other terms as much as possible. Since $\pm\frac{21}{40}$ is close to $\pm\frac{1}{2}$, one possibility is to set $\hat{f}_\lambda = g_{\lambda,\frac{1}{2},\frac{1}{2}} - g_{\lambda,\frac{1}{2},-\frac{1}{2}}$. The centers of the bumps are chosen to be a distance 1 apart from each other so that the cross terms do not disappear as λ gets larger, while the negative coefficient is chosen so that the cross terms cancel out some of the other terms. For large enough λ, $\operatorname{supp}(g_{\lambda,\frac{1}{2},\frac{1}{2}}) \cap \operatorname{supp}(g_{\lambda,\frac{1}{2},-\frac{1}{2}}) = \emptyset$. We may always rescale the k which generates the $g_{\lambda,\frac{1}{2},\pm\frac{1}{2}}$ so that these supports are disjoint for all λ. Thus, without loss of generality, assume that the supports are disjoint for all λ. We have $\left|\hat{f}_\lambda\right|^2 = \frac{1}{2}T_{\frac{1}{2}}k_\lambda + \frac{1}{2}T_{-\frac{1}{2}}k_\lambda$. Also,

$$
\begin{aligned}
\hat{f}_\lambda(\gamma)\hat{f}_\lambda(\gamma+1) &= -\frac{1}{2}T_{\frac{1}{2}}k_\lambda(\gamma) \\
\text{and} \quad \hat{f}_\lambda(\gamma)\hat{f}_\lambda(\gamma-1) &= -\frac{1}{2}T_{-\frac{1}{2}}k_\lambda(\gamma).
\end{aligned}
$$

These equalities rely on the evenness of the k_λ. For an appropriate subsequence λ_ℓ, it is true that

$$
\begin{aligned}
F(f_{\lambda_\ell}) \quad \rightarrow \quad & \frac{1}{2}\Bigg\{ \sum_n \left[\left|\hat{\psi}(2^n\frac{1}{2})\right|^2 + \left|\hat{\psi}(2^n\left(-\frac{1}{2}\right))\right|^2 \right] \\
- & \sum_n \sum_{l\neq 0} \left[\hat{\psi}(2^n\frac{1}{2})\hat{\psi}(2^n\frac{1}{2}+l) + \hat{\psi}(2^n\left(-\frac{1}{2}\right))\hat{\psi}(2^n\left(-\frac{1}{2}\right)+l) \right] \Bigg\} = \frac{1}{4}
\end{aligned}
$$

as $\ell \to \infty$. Thus, the lower frame bound A of ψ is bounded above by $\frac{1}{4}$.

b. Can we use similar methods to tighten this lower frame bound estimate? For example, although the maximum of the cross terms occurs at $\frac{1}{2}$, the minimum of the remaining terms occurs at $\frac{21}{40}$. Perhaps it would be better to consider $\hat{f}_\lambda = g_{\lambda,\frac{1}{2},\frac{21}{40}} - g_{\lambda,\frac{1}{2},-\frac{19}{40}}$. Further, values of α different from $\frac{1}{2}$ might yield better results. Actually, neither of these options changes the results. If we choose $0 < \alpha < 1$ and $\omega \in [\frac{7}{16}, \frac{9}{16})$ and set $\hat{f}_\lambda = g_{\lambda,\alpha,\omega} - g_{\lambda,1-\alpha,1-\omega}$, then the minimum bound obtained for A using the same method as in part a is $\frac{1}{4}$. We note that ω must be chosen from the interval $\left[\frac{7}{16}, \frac{9}{16}\right)$ (or the reflection of the interval to the negative $\mathbb{R}$ axis) because that is the only region in the support of $\hat{\psi}$ where, for γ lying in that region, $\hat{\psi}(\gamma)\hat{\psi}(\gamma + l)$ is non-zero for any $l \in \mathbb{Z}\backslash\{0\}$.

c. Recalling that the Daubechies-Christensen bound is $\frac{2}{9}$, we conclude that the lower frame bound satisfies $\frac{2}{9} \leq A \leq \frac{1}{4}$.

This method of fine tuning lower frame bounds is difficult to generalize.

A natural idea that arises when attempting to obtain Parseval frames with frequency smoothness is to use elements of an approximate identity to convolve with $\mathbb{1}_L$ in order to obtain $\mathcal{W}(\psi)$ with frame bounds A and B which are arbitrarily close to 1, specifically using an approximate identity, $\{\phi_m\}$, that consists of the dilations of a non-negative function ϕ with L^1-norm 1. We know that $\mathbb{1}_L * \phi_m$ converges to $\mathbb{1}_L$ in L^p, $1 \leq p < \infty$. Thus, there is a subsequence which converges almost everywhere to $\mathbb{1}_L$. However, what is *prima facie* hopeful, but, in fact, not valid, is that the corresponding frame bounds converge to 1.

Proposition 18. *Consider the approximate identity* $\{\phi_m = \frac{m}{2}\mathbb{1}_{[-\frac{1}{m},\frac{1}{m}]} : m > 12\}$. *Although* $\mathbb{1}_L * \phi_m \to \mathbb{1}_L$ *in* L^p, $1 \leq p < \infty$, *the upper frame bounds of* $\mathcal{W}((\mathbb{1}_L * \phi_m)^\vee)$ *are all* $\frac{17}{16}$, *and the lower frame bounds are bounded between* $\frac{2}{9}$ *and* $\frac{1}{4}$.

The proof of this fact uses essentially the same calculations as the ones found in Example 16.

One may hope to improve the frame bounds of the smooth frame wavelets, e.g., by bringing both of the bounds closer to 1, by convolving with a linear spline. The following proposition shows that, in this case, the resulting upper frame bound is closer to 1, than for the case of Proposition 18, but that it also constant for large enough m. Further, in the limit, there is a positive gap between upper and lower frame bounds.

Proposition 19. *Consider the approximate identity* $\{\phi_m : m > 12\}$, *where* $\phi_m(\gamma) = \max(m(1 - m|\gamma|), 0)$, $\gamma \in \widehat{\mathbb{R}}$. *Although* $\mathbb{1}_L * \phi_m \to \mathbb{1}_L$ *pointwise a.e. and in* L^p, $1 \le p < \infty$, *the upper frame bounds of* $\mathcal{W}((\mathbb{1}_L * \phi_m)^\vee)$ *are all* $\frac{65}{64}$, *and the lower frame bounds are bounded between* $\frac{2}{9}$ *and* $\frac{1}{4}$.

Proof. Let $\hat{\psi}_m = \mathbb{1}_L * \phi_m$. By utilizing basic methods of optimization from calculus, we evaluate

$$\sup_\gamma \kappa_{\psi_m}(\gamma) = M_{\psi_m} = \frac{65}{64}.$$

It follows from Corollary 12 that the upper frame bound of $\mathcal{W}(\psi_m)$ is equal to $\frac{65}{64}$, independent of which $m > 12$ is used.

As in Example 17, set $\hat{f}_\lambda = g_{\lambda,\frac{1}{2},\frac{1}{2}} - g_{\lambda,\frac{1}{2},-\frac{1}{2}}$. Then, we can verify that there exists a subsequence λ_ℓ such that

$$\begin{aligned} F(f_{\lambda_\ell}) \ \to \ & \frac{1}{2}\Big\{ \sum_n \left[\left|\hat{\psi}(2^n\left(\frac{1}{2}\right))\right|^2 + \left|\hat{\psi}(2^n\left(-\frac{1}{2}\right))\right|^2 \right] \\ - & \sum_n \sum_{l \neq 0} \left[\hat{\psi}(2^n\left(\frac{1}{2}\right))\hat{\psi}(2^n\left(\frac{1}{2}\right) - l) + \hat{\psi}(2^n\left(-\frac{1}{2}\right))\hat{\psi}(2^n\left(-\frac{1}{2}\right) - k) \right] \Big\} = \frac{1}{4}, \end{aligned}$$

as $\ell \to \infty$. Also, the lower Daubechies-Christensen bound is $\frac{2}{9}$, yielding the desired bounds on the lower frame bound. $\square$

We shall call the phenomenon which occurs in Propositions 18 and 19 a *frame bound gap.*

The results presented in this section prompt the following questions, which we address in Sections 4 and 5.

- Do we obtain a frame when we try to smooth $K_1 \backslash A_1$ from the 1-d Journé neighborhood mapping construction?

- Can we ever precisely determine the lower frame bound?

- What happens when we smooth $K_0 \backslash A_0$ from higher dimensional Journé constructions?

- Does a frame bound gap occur for other wavelet sets and other approximate identities?

4 A shrinking method to obtain frames

4.1 The shrinking method

When we try to smooth $\mathbb{1}_L$ for other sets L obtained using the neighborhood mapping constrution, we do not necessarily obtain a frame.

Example 20. Let

$$L = \left[-\frac{9}{4}, -2\right) \cup \left[-\frac{1}{2}, -\frac{9}{32}\right) \cup \left[\frac{9}{32}, \frac{1}{2}\right) \cup \left[2, \frac{9}{4}\right),$$

which is $K_1 \backslash A_1$ from the neighborhood mapping construction of the 1-d Journé set (Example 7). For $m \in \mathbb{N}$, define $\hat{\psi}_m = \mathbb{1}_L * \frac{m}{2}\mathbb{1}_{[-\frac{1}{m}, \frac{1}{m}]}$. Then $\mathcal{W}(\psi_m)$ is not a frame for any m. This can be shown by considering $F((g_{\lambda,\frac{1}{2},\frac{1}{2}} - g_{\lambda,\frac{1}{2},-\frac{1}{2}})^{\vee})$ for arbitrarily large λ, just as in Example 17. Specifically, a subsequence of $F((g_{\lambda,\frac{1}{2},\frac{1}{2}} - g_{\lambda,\frac{1}{2},-\frac{1}{2}})^{\vee})$ converges to 0, while each $g_{\lambda,\frac{1}{2},\frac{1}{2}} - g_{\lambda,\frac{1}{2},-\frac{1}{2}}$ has unit norm. However, for arbitrary m, $\mathcal{W}(\psi_m)$ is a Bessel sequence, and for any $m > 64$, the Bessel bound is bounded between $\frac{305}{256}$ and $\frac{11}{8}$. Again, we see that the upper frame bound does not converge to 1.

It seems reasonable to assume that smoothing $\mathbb{1}_L$ in Example 20 with a linear spline may yield a frame; however, the following example shows that this does not happen.

Example 21. Let

$$L = \left[-\frac{9}{4}, -2\right) \cup \left[-\frac{1}{2}, -\frac{9}{32}\right) \cup \left[\frac{9}{32}, \frac{1}{2}\right) \cup \left[2, \frac{9}{4}\right),$$

and for $m > 64$, let ϕ_m be the linear spline $\phi_m(\gamma) = \max(m(1 - m|\gamma|), 0)$. Set $\hat{\psi} = \mathbb{1}_L * \phi_m$. Then

$$\begin{aligned} M_\psi &= \frac{41}{32} \approx 1.28125 \\ \sup_{\gamma \in \widehat{\mathbb{R}}^d} \kappa_\psi(\gamma) &\approx 1.14833 \\ \inf_{\gamma \in \widehat{\mathbb{R}}^d} \kappa_\psi(\gamma) &\approx 0.38092 \\ N_\psi &= 0. \end{aligned}$$

In fact, for some subsequence $\{\lambda_\ell\}$,

$$F((g_{\lambda_\ell, \frac{1}{2}, \frac{1}{2}} - g_{\lambda_\ell, \frac{1}{2}, -\frac{1}{2}})^\vee) \to \nu_\psi(2) = 0$$

If $\mathcal{W}(\psi)$ formed a frame, then it would have a lower frame bound $0 = N_\psi \leq A \leq \nu_\psi(2) = 0$. Thus $\mathcal{W}(\psi)$ is not a frame, but it is a Bessel sequence with upper frame bound $1.14833 \leq B \leq 1.28125$.

We would not only like to construct frames, but also to determine the exact lower frame bound of such a frame rather than a range of possible values. The following definitions and theorem will help us do that.

Definition 22. For any measurable subset $L \subseteq \widehat{\mathbb{R}}^d$ define

$$\Delta(L) = \operatorname{dist}\big(L, \bigcup_{k \in \mathbb{Z}^d \setminus \{0\}} (L + k)\big).$$

Definition 23. If f is a function defined on $\widehat{\mathbb{R}}^d$ which takes only non-negative values, for $\epsilon \geq 0$, define

$$\operatorname{supp}_\epsilon f = \{\gamma \in \widehat{\mathbb{R}}^d : f(\gamma) > \epsilon\}.$$

Thus, for any function g on $\widehat{\mathbb{R}}^d$, $\operatorname{supp} g$ is the topological closure of $\operatorname{supp}_0 |g|$.

Theorem 24. *Let $\hat{\psi} \in L^\infty_c(\widehat{\mathbb{R}}^d)$ be a non-negative function. If there exists an $\epsilon > 0$ such that for $L = \operatorname{supp}_\epsilon \hat{\psi}$, $\bigcup_{n \in \mathbb{Z}} 2^n L = \widehat{\mathbb{R}}^d$ up to a set of measure 0, and for $\tilde{L} = \operatorname{supp}_0 \hat{\psi}$, $\Delta(\tilde{L}) > 0$,* *and* $\operatorname{dist}(0, \tilde{L}) > 0$. *Then,* $\mathcal{W}(\psi)$ *is a frame for* $L^2(\mathbb{R}^d)$. *The frame bounds are* $\operatorname{essinf}_\gamma \kappa_\psi(\gamma)$ *and* $\operatorname{esssup}_\gamma \kappa_\psi(\gamma)$.

Remark 25. If the $L \subseteq \widehat{\mathbb{R}}^d$ is a Parseval frame wavelet set and the closure $\overline{L} \subseteq (-\frac{1}{2}, \frac{1}{2})^d$, then $\hat{\psi} = \mathbb{1}_L$ and $0 < \epsilon < 1$ satisfy the hypotheses with $L = \tilde{L}$.

Proof. We first note that since $\hat{\psi}$ is compactly supported and bounded, it lies in $L^2(\widehat{\mathbb{R}}^d)$. Thus $\psi \in L^2(\mathbb{R}^d)$. We now prove that $\mathcal{W}(\psi)$ is a frame. Since $\Delta(\tilde{L}) > 0$,

$$\forall \gamma \in \widehat{\mathbb{R}}^d \qquad \sum_{n\in\mathbb{Z}} \sum_{k\neq 0} \left| \hat{\psi}(2^n\gamma)\hat{\psi}(2^n\gamma + k) \right| = 0. \tag{5}$$

So

$$M_\psi = \operatorname{esssup}_{\gamma\in\widehat{\mathbb{R}}^d} \kappa_\psi(\gamma).$$

By assumption, $\hat{\psi}$ is bounded. Furthermore, since $\operatorname{dist}(0, \tilde{L}) > 0$ and $\tilde{L}$ in bounded, for any $\gamma \in \widehat{\mathbb{R}}^d$, $\hat{\psi}(2^n\gamma)$ is non-zero for only finitely many $n \in \mathbb{Z}$. Putting these two facts together, we conclude that

$$\operatorname{esssup}_{\gamma\in\widehat{\mathbb{R}}^d} \kappa_\psi(\gamma) < \infty.$$

Similarly,

$$N_\psi = \operatorname{essinf}_{\gamma\in\widehat{\mathbb{R}}^d} \kappa_\psi(\gamma).$$

Since dyadic dilations of L cover $\widehat{\mathbb{R}}^d$, for almost every $\gamma \in \widehat{\mathbb{R}}^d$, there exists $n \in \mathbb{Z}$ such that $2^n\gamma \in L$, which implies that $\hat{\psi}(2^n\gamma) > \epsilon$. Thus $\operatorname{essinf}_{\gamma\in\widehat{\mathbb{R}}^d} \kappa_\psi(\gamma) > 0$. Thus, by Corollary 12, $\mathcal{W}(\psi)$ is a frame with bounds A and B which satisfy

$$\begin{aligned} A &= \operatorname{essinf}_{\gamma\in\widehat{\mathbb{R}}^d} \kappa_\psi(\gamma) \\ \text{and} \quad B &= \operatorname{esssup}_{\gamma\in\widehat{\mathbb{R}}^d} \kappa_\psi(\gamma). \end{aligned}$$

$\square$

A statement very similar to the preceding theorem appears camouflaged as Theorem 8 in [CS93a].

Remark 26. Let $\psi \in L^2(\mathbb{R})$ satisfy the the hypotheses of Theorem 24. Then for $C = \max\{A^{-1}, B\}$ and almost all $\gamma \in \widehat{\mathbb{R}}$

$$0 < C^{-1} \leq \kappa_\psi(\gamma) \leq C < \infty.$$

Furthermore, it follows from Line 5 that for almost all $\gamma \in \widehat{\mathbb{R}}$,

$$\overline{\hat{\psi}\,(2^n\gamma)}\hat{\psi}\,(2^n\gamma + 2^n k) = 0 \quad \forall k \in \mathbb{Z}\backslash 2\mathbb{Z}\ ,\ k \in \mathbb{N} \cup \{0\}.$$

Thus, by Proposition 2.2 of [DH02], if $S : L^2(\widehat{\mathbb{R}}) \to L^2(\widehat{\mathbb{R}})$ is the frame operator defined as

$$Sf = \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} \langle f, D_n T_k \psi\rangle D_n T_k \psi,$$

then S is translation invariant. That is, for all $x \in \mathbb{R}$, $ST_x = T_x S$ as operators.

Corollary 27. *Let L be a Parseval frame wavelet set from the neighborhood mapping construction. Let $\delta = \operatorname{dist}(0, L) > 0$. Let $\alpha > 0$ be such that the closure $\alpha\overline{L} \subseteq (-\frac{1}{2} + \epsilon, \frac{1}{2} - \epsilon)^d$, for some $0 < \epsilon < \frac{1}{2}$. Further let ϕ be an essentially bounded non-negative function such that $\operatorname{supp}_0 \phi \subseteq \min\{\frac{\alpha\delta}{2}, \epsilon\} \cdot (-1, 1)^d$ and $\operatorname{supp}_0 \phi$ contains a neighborhood about the origin. Then if $\hat{\psi} = \mathbb{1}_{\alpha L} * \phi$, $\mathcal{W}(\psi)$ is a frame for $L^2(\mathbb{R}^d)$.*

Proof. Define $\tilde{L} = \operatorname{supp}_0 \hat{\psi}$. Since $\operatorname{supp}_0 \phi$ contains a neighborhood about the origin, ϕ is non-negative, and $\hat{\psi}$ is continuous, there exists an $\epsilon > 0$ such that

$$\alpha L \subseteq \hat{\psi}^{-1}(\epsilon, \infty).$$

Thus, for this ϵ,

$$\widehat{\mathbb{R}}^d = \alpha\widehat{\mathbb{R}}^d = \alpha \bigcup_{n\in\mathbb{Z}} 2^n L \subseteq \bigcup_{n\in\mathbb{Z}} 2^n \operatorname{supp}_\epsilon \hat{\psi},$$

up to a set of measure zero. As the convolution of two essentially bounded functions with compact support, $\hat{\psi} \in L^\infty$ immediately. It follows from Theorem 24 that $\mathcal{W}(\psi)$ is a frame for $L^2(\mathbb{R}^d)$. $\square$

Example 28. Let

$$L = \left[-\frac{9}{32}, -\frac{1}{4}\right) \cup \left[-\frac{1}{16}, -\frac{9}{256}\right) \cup \left[\frac{9}{256}, \frac{1}{16}\right) \cup \left[\frac{1}{4}, \frac{9}{32}\right).$$

Then L is $K_1 \backslash A_1$ from the 1-d Journé construction, shrunk by a factor of 8. Further let $\hat{\psi}_m = \mathbb{1}_L * \frac{m}{2}\mathbb{1}_{[-\frac{1}{m},\frac{1}{m}]}$. Then for any $m \geq 384$, $\mathcal{W}(\psi_m)$ is frame with bounds $\frac{81}{260}$ and $\frac{305}{256}$. Note that $\mathcal{W}((\mathbb{1}_{8L} * \frac{m}{2}\mathbb{1}_{[-\frac{1}{m},\frac{1}{m}]})^\vee)$ is not a frame for any $m > 0$ (Example 20).

Example 29. Let $L_a = \left[-a, -\frac{a}{2}\right) \cup \left[\frac{a}{2}, a\right)$ for $0 < a < \frac{1}{2}$. Then L_a is $\left[-\frac{1}{2}, -\frac{1}{4}\right) \cup \left[\frac{1}{4}, \frac{1}{2}\right)$ from the 1-d Journé construction, dilated by a factor of $2a < 1$. Recall from Proposition 18 that

$$\mathcal{W}((\mathbb{1}_{[-\frac{1}{2},-\frac{1}{4})\cup[\frac{1}{4},\frac{1}{2})} * \frac{m}{2}\mathbb{1}_{[-\frac{1}{m},\frac{1}{m})})^\vee)$$

is a frame with upper frame bound $\frac{17}{16}$ and lower frame bound between $\frac{2}{9}$ and $\frac{1}{4}$. Define $\hat{\psi}_{m,a} = \mathbb{1}_{L_a} * \frac{m}{2}\mathbb{1}_{[-\frac{1}{m},\frac{1}{m}]}$. For $0 < a < \frac{1}{2}$ and $m \geq \max\{\frac{2}{1-2a}, \frac{6}{a}\}$, $\mathcal{W}(\psi_{m,a})$ is a frame with with frame bounds $\frac{9}{20}$ and $\frac{17}{16}$.

It follows from the calculations in Example 17 that the lower frame bound of $\mathcal{W}(\psi_{m,\frac{1}{2}})$ is bounded above by $\frac{1}{4}$, while the shrinking process brings the lower frame bound up to $\frac{9}{20}$, for $\mathcal{W}(\psi_{m,a})$, $0 < a < \frac{1}{2}$. Corollary 12, which is based on previously known results, only implies that the lower frame bound of $\mathcal{W}(\psi_{m,\frac{1}{2}})$ is bounded between $\frac{2}{9}$ and $\frac{9}{20}$. Thus without the methods introduced in Example 17, we would not know that the shrinking method actually improves the lower frame bound.

Further note that $\frac{17}{16} < \frac{305}{256}$ and $\frac{9}{20} > \frac{81}{260}$. Thus the frame bounds corresponding to shrinking $K_0 \backslash A_0$ from the 1-d Journé construction are closer to 1 than the bounds obtained by shrinking $K_1 \backslash A_1$ in the Example 28.

4.2 Oversampling

Corollary 27 yields an easy method to obtain wavelet frames with certain decay properties from Parseval frame wavelet sets. It almost seems counterintuitive to believe that simply shrinking the support of the frequency domain can change a function which is not a frame generator into a function that is one. Although we have proven that this does indeed happen, we now give a heuristic argument that this method should work for dyadic-shrinking. If the collection $\{D_nT_k\psi : n \in \mathbb{Z}, k \in \mathbb{Z}^d\} \subseteq L^2(\mathbb{R}^d)$ is a Bessel sequence, then it is not a frame if and only if there exists a sequence $\{f_m : \|f_m\|_2 = 1, m \in \mathbb{Z}\} \subseteq L^2(\mathbb{R}^d)$ such that $\lim_{m\to\infty} \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} |\langle f_m, D_nT_k\psi\rangle|^2 = 0$. If we add more elements to $\{D_nT_k\psi : n \in \mathbb{Z}, k \in \mathbb{Z}^d\}$, it is more likely that the system will be more complete. We would like to show that shrinking the support of $\hat{\psi}$ will add more elements to the system.

For $\alpha > 0$ and $\psi \in L^2(\mathbb{R}^d)$, let $\hat{\varphi}(\gamma) = \hat{\psi}(\alpha\gamma)$. Then $\mathcal{F}\varphi = \alpha^{-d/2} D_{\log_2 \alpha}\mathcal{F}\psi$,

$$\begin{aligned}
\Rightarrow \varphi &= \mathcal{F}^{-1}\mathcal{F}\varphi \\
&= \mathcal{F}^{-1}(\alpha^{-d/2} D_{\log_2 \alpha}\mathcal{F})\psi \\
&= \mathcal{F}^{-1}(\alpha^{-d/2} \mathcal{F} D_{-\log_2 \alpha})\psi \\
&= \alpha^{-d/2} D_{-\log_2 \alpha}\psi \\
\Rightarrow D_nT_k\varphi &= \alpha^{-d/2} D_nT_kD_{-\log_2 \alpha}\psi \\
&= \alpha^{-d/2} D_{n-\log_2 \alpha}T_{\frac{k}{\alpha}}\psi.
\end{aligned}$$

Hence if $\alpha = 2^N$, for $N \in \mathbb{N}$,

$$\operatorname{span}\{D_nT_k\varphi : n \in \mathbb{Z}, k \in \mathbb{Z}^d\} = \operatorname{span}\{D_nT_{\frac{k}{2^N}}\psi : n \in \mathbb{Z}, k \in \mathbb{Z}^d\}.$$

Thus dyadic shrinking on the Fourier domain has the effect of increasing the size of the system generated by dilations and translations by a power of 2. One may call this an *oversampling* of the continuous wavelet system $\{D_{\log_2 r} T_s \psi : r > 0, s \in \mathbb{R}\}$. If $L \subseteq \widehat{\mathbb{R}}^d$ is Parseval frame wavelet set and $\phi \in L_c^\infty(\widehat{\mathbb{R}}^d)$, $\mathcal{W}\left((\mathbb{1}_L * \phi)^\vee\right)$ is a Bessel sequence but perhaps not a frame. Hence dyadic shrinking increases the likelihood that $\mathcal{W}\left((\mathbb{1}_L * \phi)^\vee\right)$ is complete and thus has a positive lower frame bound. In general, shrinking by any $\alpha > 1$ has the effect of increasing the number of translations in the original wavelet system and shifts each of the dilation operators by the same amount. We compare and contrast our results with the following two oversampling theorems found in [CS93a].

Theorem 30. *Let $\mathcal{W}(\psi)$ be a frame for $L^2(\mathbb{R})$ with frame bounds A and B. Then for every odd positive integer N, the family*

$$\{D_n T_{\frac{k}{N}} \psi : n, k \in \mathbb{Z}\}$$

is a frame with bounds $\tilde{A}$ and $\tilde{B}$ which satisfy $\tilde{A} \geq NA$ and $\tilde{B} \leq NB$.

Theorem 31. *Let $\psi \in L^2(\mathbb{R})$ decay sufficiently fast and satisfy $\int \psi(x)dx = 0$. If $\mathcal{W}(\psi)$ forms a frame, then for any positive integer N,*

$$\{D_n T_{\frac{k}{N}} \psi : n, k \in \mathbb{Z}\}$$

is a frame also.

Remark 32. The specific decay conditions in the hypothesis of Theorem 31 are described in [CS93a], but are too lengthy to list here. The smoothed frame wavelets mentioned in this paper all satisfy the decay conditions.

Only dyadic shrinking corresponds to oversampling in the Chui and Shi sense. Oversampling may potentially create a frame system from a pre-existing frame system, but we see in Example 28 that oversampling may change a non-frame system to a frame system. Furthermore, in Example

29 we see that oversampling can bring frame bounds closer to 1, rather than just scaling them as in Theorem 30.

5 Frame bound gaps

Definition 33. Let $\psi \in L^2(\widehat{\mathbb{R}}^d)$ be a Parseval frame wavelet and $\{\psi_m\}_{m\in\mathbb{N}} \subseteq L^2(\widehat{\mathbb{R}}^d)$ be a sequence of frame wavelets (or Bessel wavelets) with lower frame bounds A_m and upper frame bounds B_m (or just upper frame bounds B_m) for which

$$\lim_{m\to\infty} \|\psi - \psi_m\|_{L^2(\widehat{\mathbb{R}}^d)} = 0.$$

If $\overline{\lim}_{m\to\infty} A_m < 1$ or $\underline{\lim}_{m\to\infty} B_m > 1$, then there is a *frame bound gap.* By Parseval's equality, $\|\psi - \psi_m\|_{L^2(\widehat{\mathbb{R}}^d)} = \|\hat{\psi} - \hat{\psi}_m\|_{L^2(\widehat{\mathbb{R}}^d)}$, so it suffices to check for convergence on the frequency domain.

Many examples of frame bound gaps occur in the previous sections. We shall now prove that this phenomenon occurs in more general situations. First we make a quick comment.

Remark 34. Let $L \subseteq \widehat{\mathbb{R}}^d$ be bounded and measurable and $g \in L^1_{loc}(\widehat{\mathbb{R}}^d)$. For $m > 1$ define

$$\begin{aligned} g_{(m)}(\gamma) &= mg(m\gamma), \text{ and} \\ \hat{\psi}_m &= \mathbb{1}_L * g_{(m)}. \end{aligned}$$

Then

$$\begin{aligned} \hat{\psi}_m(u) &= \int \mathbb{1}_L(u-\gamma) g_{(m)}(\gamma) d\gamma \\ &= \int \mathbb{1}_L(u - \frac{\gamma}{m}) g(\gamma) d\gamma \\ &= \int_{-mL+mu} g(\gamma) d\gamma. \end{aligned}$$

Theorem 35. *For $0 < a < 1/2$, let $L \subseteq \widehat{\mathbb{R}}^d$ be the Parseval frame wavelet set $[-a,a]^d \backslash [-\frac{a}{2}, \frac{a}{2}]^d$. Also let $g : \widehat{\mathbb{R}}^d \to \mathbb{R}$ satisfy the following conditions:*

i. $\operatorname{supp}_0 |g| \subseteq \prod_{i=1}^d [-b_i, c_i]$, *where for all i, $b_i, c_i > 0$ and $\operatorname{supp}_0 g$ contains a neighborhood of 0;*

ii. $\int g(\gamma) d\gamma = 1$; *and*

iii. $0 < \int_{\prod_{i=1}^d [\frac{c_i}{2}, c_i]} g(\gamma) d\gamma < 1$ *and* $0 < \int_{\prod_{i=1}^d [-\frac{b_i}{2}, c_i]} g(\gamma) d\gamma < 1$.

Define $\hat{\psi}_m = \mathbb{1}_L * g_{(m)}$. For any

$$m > \max_{1 \le i \le d} \{ \max \left\{ \frac{2(b_i + c_i)}{a}, \frac{b_i + c_i}{1 - 2a}, \frac{4b_i + c_i}{a}, \frac{4c_i + b_i}{a} \right\} \},$$

$\mathcal{W}(\psi_m)$ is a frame with frame bounds A_m and B_m, and there exist $\alpha < 1$ and $\beta > 1$, both independent of m, such that $A_m \le \alpha$ and $B_m \ge \beta$. In particular, there are frame bound gaps.

Remark 36. Any non-negative function $g : \widehat{\mathbb{R}}^d \to \mathbb{R}$ which integrates to 1 and has support $\prod_{i=1}^d [-b_i, c_i] \supseteq \operatorname{supp}_0 g \supseteq \prod_{i=1}^d (-b_i, c_i)$ satisfies the hypotheses.

Remark 37. This result holds true if $m \in \mathbb{N}$ or $m \in \mathbb{R}$.

Proof. Let $m > \max_{1 \le i \le d} \{ \max \{ \frac{2(b_i+c_i)}{a}, \frac{b_i+c_i}{1-2a}, \frac{4b_i+c_i}{a}, \frac{4c_i+b_i}{a} \} \}$.

Since $m > \frac{b_i+c_i}{1-2a}$, $\Delta(\operatorname{supp} \hat{\psi}_m) > 0$. Thus,

$$\mu_{\psi_m}(u) = \nu_{\psi_m}(u) = \kappa_{\psi_m}(u),$$

where κ_{ψ_m} is compactly supported. Further, for all $1 \le i \le d$,

$$m > \frac{2(b_i + c_i)}{a} > \max\{ \frac{2b_i}{a}, \frac{2c_i}{a} \},$$

so $\operatorname{dist}(0, \operatorname{supp} \hat{\psi}_m) > 0$. It follows from Theorem 24 and Corollary 12, that $\mathcal{W}(\psi_m)$ is a frame with bounds $A_m = \inf_u \kappa_{\psi_m}(u)$ and $B_m = \sup_u \kappa_{\psi_m}$.

As $m > \max_{1\le i\le d}\{\max\{\frac{4b_i+c_i}{a}, \frac{4c_i+b_i}{a}\}\}$, for $u \in \left(\prod_{i=1}^d[-a-\frac{b_i}{m}, a+\frac{c_i}{m}]\right) \setminus \left(\prod_{i=1}^d[-\frac{a}{2}-\frac{b_i}{2m}, \frac{a}{2}+\frac{c_i}{2m}]\right)$,

$$\begin{aligned}
\kappa_{\psi_m}(u) &= (\hat{\psi}_m(u))^2 + (\hat{\psi}_m(\frac{u}{2}))^2, \text{ where} \\
\hat{\psi}_m(u) &= \int_{-mL+mu} g(\gamma)d\gamma.
\end{aligned}$$

To bound B_m, we evaluate $\kappa_{\psi_m}(v)$ where $v = (a-\frac{b_1}{m}, a-\frac{b_2}{m}, \ldots, a-\frac{b_d}{m})$. We first compute $\hat{\psi}_m(v)$. Since $[\frac{a}{2}, a]^d \subseteq L$,

$$\prod_{i=1}^d [-b_i, \frac{ma}{2} - b_i] \subseteq -mL + mv.$$

As $m > \frac{2(b_i+c_i)}{a}$ for all $1 \le i \le d$, $\prod_{i=1}^d[-b_i, c_i] \subseteq \prod_{i=1}^d[-b_i, \frac{ma}{2}-b_i]$. Hence,

$$\hat{\psi}_m(v) = \int_{-mL+mv} g(\gamma)d\gamma = \int_{\prod_{i=1}^d[-b_i,c_i]} g(\gamma)d\gamma = 1.$$

We now compute $\hat{\psi}_m(\frac{v}{2})$. Since m is sufficiently large, for $1 \le i \le d$,

$$\begin{aligned}
[-b_i, c_i] \cap (-m[\frac{a}{2}, a] + \frac{m}{2}(a - \frac{b_i}{m})) &= [-b_i, c_i] \cap ([-\frac{ma}{2} - \frac{b_i}{2}, -\frac{b_i}{2}]) \\
&= [-b_i, -\frac{b_i}{2}],
\end{aligned}$$

$$\begin{aligned}
[-b_i, c_i] \cap (-m[-\frac{a}{2}, \frac{a}{2}] + \frac{m}{2}(a - \frac{b_i}{m})) &= [-b_i, c_i] \cap ([-\frac{b_i}{2}, ma - \frac{b_i}{2}]) \\
&= [-\frac{b_i}{2}, c_i], \text{ and}
\end{aligned}$$

$$\begin{aligned}
[-b_i, c_i] \cap (-m[\frac{a}{2}, a] + \frac{m}{2}(a - \frac{b_i}{m})) &= [-b_i, c_i] \cap ([ma - \frac{b_i}{2}, \frac{3ma}{2} - \frac{b_i}{2}]) \\
&= \emptyset.
\end{aligned}$$

It follows that

$$\left(\prod_{i=1}^d [-b_i, c_i]\right) \cap \left(-mL + m(\frac{v}{2})\right) = \left(\prod_{i=1}^d [-b_i, c_i]\right) \setminus \left(\prod_{i=1}^d [-\frac{b_i}{2}, c_i]\right),$$

and that

$$\hat{\psi}_m(\frac{v}{2}) = \int_{-mL+m(\frac{v}{2})} g(\gamma)d\gamma = 1 - \int_{\prod_{i=1}^d[-\frac{b_i}{2}, c_i]} g(\gamma)d\gamma.$$

Define $\beta = \kappa_{\psi_m}(v) = 1 + \left(1 - \int_{\prod_{i=1}^d [-\frac{b_i}{2}, c_i]} g(\gamma) d\gamma\right)^2$. Then, $B_m \geq \kappa_{\psi_m}(v) = \beta > 1$, and β is independent of m.

Let $\omega = (a + \frac{c_1}{m}, a + \frac{c_2}{m}, \ldots, a + \frac{c_d}{m})$. We shall show that $\kappa_{\psi_m}(\omega)$ is strictly less than 1. We compute,

$$\begin{aligned} -mL + m\omega &= \left(\prod_{i=1}^d [c_i, 2ma + c_i]\right) \setminus \left(\prod_{i=1}^d [\frac{ma}{2} + c_i, \frac{3ma}{2} + c_i]\right) \\ &\Rightarrow \left(\prod_{i=1}^d [-b_i, c_i]\right) \cap (-mL + m\omega) = \emptyset. \end{aligned}$$

Thus, $\hat{\psi}_m(\omega) = 0$. Furthermore,

$$-mL + m(\frac{\omega}{2}) = \left(\prod_{i=1}^d [-\frac{ma}{2} + \frac{c_i}{2}, \frac{3ma}{2} + \frac{c_i}{2}]\right) \setminus \left(\prod_{i=1}^d [\frac{c_i}{2}, ma + \frac{c_i}{2}]\right).$$

It follows from our choice of m that for all $1 \leq i \leq d$, $-\frac{ma}{2} + \frac{c_i}{2} < -b_i$ and $c_i < ma + \frac{c_i}{2} < \frac{3ma}{2} + \frac{c_i}{2}$. Hence,

$$\left(\prod_{i=1}^d [-b_i, c_i]\right) \cap (-mL + m(\frac{\omega}{2})) = \left(\prod_{i=1}^d [-b_i, c_i]\right) \setminus \left(\prod_{i=1}^d [\frac{c_i}{2}, c_i]\right), \text{ and}$$

$$\hat{\psi}_m(\frac{\omega}{2}) = 1 - \int_{\prod_{i=1}^d [\frac{c_i}{2}, c_i]} g(\gamma) d\gamma.$$

We define

$$\alpha = \kappa_{\psi_m}(\omega) = \left(1 - \int_{\prod_{i=1}^d [\frac{c_i}{2}, c_i]} g(\gamma) d\gamma\right)^2.$$

Consequently, $A_m \leq \alpha < 1$ for all sufficiently large m. $\square$

Corollary 38. *For $0 < a < \frac{1}{2}$, let $L_d \subseteq \widehat{\mathbb{R}}^d$ be the wavelet set $[-a, a]^d \setminus [-\frac{a}{2}, \frac{a}{2}]^d$. Also, let $g : \widehat{\mathbb{R}} \to \mathbb{R}$ satisfy the following conditions:*

i. $\operatorname{supp}_0 |g| \subseteq [-b, c]$ *for some $b, c > 0$ and* $\operatorname{supp}_0 g$ *contains a neighborhood of 0;*

ii. $\int g(\gamma) d\gamma = 1$*; and* *iii.* $0 < \int_{\frac{c}{2}}^{c} g(\gamma)d\gamma < 1$ *and* $0 < \int_{-\frac{b}{2}}^{c} g(\gamma)d\gamma < 1$.

Define $g_d = \bigotimes_{i=1}^{d} g : \widehat{\mathbb{R}}^d \to \mathbb{R}$. *Further define* $\hat{\psi}_{m,d} = \mathbb{1}_{L_d} * g_{d_{(m)}}$. *Then, for each*

$$m > \max\{\frac{2(b+c)}{a}, \frac{b+c}{1-2a}, \frac{4b+c}{a}, \frac{4c+b}{a}\},$$

and $d \geq 1$, $\mathcal{W}(\psi_{m,d})$ *is a frame with bounds* $A_{m,d}$ *and* $B_{m,d}$ *which satisfy*

$$\begin{aligned} A_{m,d} &\leq \left(1 - \left(\int_{\frac{c}{2}}^{c} g(\gamma)d\gamma\right)^d\right)^2 < 1, \textit{ and} \\ B_{m,d} &\geq \left(1 - \left(\int_{-\frac{b}{2}}^{c} g(\gamma)d\gamma\right)^d\right)^2 + 1 > 1. \end{aligned}$$

Also for such m, $\lim_{d\to\infty} B_{m,d} = 2$.

Proof. All of the hypotheses of Theorem 35 are satisfied, so

$$\begin{aligned} A_{m,d} &\leq \left(1 - \left(\int_{\frac{c}{2}}^{c} g(\gamma)d\gamma\right)^d\right)^2 < 1, \text{ and} \\ B_{m,d} &\geq \left(1 - \left(\int_{-\frac{b}{2}}^{c} g(\gamma)d\gamma\right)^d\right)^2 + 1 > 1, \end{aligned}$$

where

$$\lim_{d\to\infty} \left(1 - \left(\int_{-\frac{b}{2}}^{c} g(\gamma)d\gamma\right)^d\right)^2 + 1 = 2,$$

since $0 < \int_{-\frac{b}{2}}^{c} g(\gamma)d\gamma < 1$. Furthermore,

$$\begin{aligned} B_{m,d} &= \sup_{u \in [-a-\frac{b}{m}, a+\frac{c}{m}]^d \setminus [-\frac{a}{2}-\frac{b}{2m}, \frac{a}{2}+\frac{c}{2m}]^d} \kappa_{\psi_{m,d}}(u) \\ &= \sup_{u \in [-a-\frac{b}{m}, a+\frac{c}{m}]^d \setminus [-\frac{a}{2}-\frac{b}{2m}, \frac{a}{2}+\frac{c}{2m}]^d} (\hat{\psi}_{m,d}(u))^2 + (\hat{\psi}_{m,d}(\frac{u}{2}))^2 \\ &\leq 2. \end{aligned}$$

Thus, $\lim_{d\to\infty} B_{m,d} = 2$ for all large enough m. □

A similar result holds for a large class of wavelet sets in $\widehat{\mathbb{R}}$.

Theorem 39. *Let* $L = \bigcup_{j\in\mathcal{J}\subseteq\mathbb{Z}} [a_j, b_j]$, *with* $a_j < b_j$ *for all* $j \in \mathcal{J}$, *be a Parseval frame wavelet set. Let* $g : \widehat{\mathbb{R}} \to \mathbb{R}$ *be a non-negative function satisfying* $\int g(\gamma)d\gamma = 1$ *and with support* $\operatorname{supp}_0 g = [-c, d]$, *where* $c, d > 0$ *and which contains a neighborhood of zero. Define* $\hat{\psi}_m = \mathbb{1}_L * g_{(m)}$ *for* $m > \frac{c+d}{b_j - a_j}$ *for all* $j \in \mathcal{J}$. *Then if* $\mathcal{W}(\psi_m)$ *forms a Bessel sequence, the upper frame bound satisfies* $B_m \geq \beta > 1$, *where* β *is independent of* m. *In particular, there is a frame bound gap.*

Proof. Set $a_k = \min\{a_j > 0 : j \in \mathcal{J}\}$, and let $b_i \in \{b_j\}_{j\in\mathcal{J}}$ be the unique $b_j > 0$ such that there exists $N \in \mathbb{N} \cup \{0\}$ with $2^N a_k = b_i$. We wish to bound $\kappa_{\psi_m}(b_i - \frac{c}{m})$. Since $m > \frac{c+d}{b_i - a_i}$,

$$\begin{aligned}
-mL + m(b_i - \frac{c}{m}) &\supseteq m[-b_i, -a_i] + m(b_i - \frac{c}{m}) \\
&= [-c, m(b_i - a_i) - c] \\
&\supseteq [-c, d],
\end{aligned}$$

implying that

$$\hat{\psi}_m(b_i - \frac{c}{m}) = \int_{-mL+m(b_i-\frac{c}{m})} g(\gamma)d\gamma = 1.$$

Similarly,

$$\begin{aligned}
-mL + m(2^{-N}(b_i - \frac{c}{m})) &= -mL + ma_k - 2^{-N}c \\
&\supseteq m[-b_k, -a_k] + ma_k - 2^{-N}c \\
&= [m(a_k - b_k) - 2^{-N}c, -2^{-N}c] \\
&\supseteq [-c, -2^{-N}c].
\end{aligned}$$

So

$$\hat{\psi}_m(2^{-N}(b_i - \frac{c}{m})) \geq \int_{-c}^{-2^{-N}c} g(\gamma)d\gamma > 0.$$

Hence,

$$B_m \geq \kappa_{\psi_m}(b_i - \frac{c}{m}) \geq 1 + \left(\int_{-c}^{-2^{-N}c} g(\gamma)d\gamma\right)^2 > 1.$$

□

Corollary 40. *Let* $L = \bigcup_{j=1}^{J} [a_j, b_j] \subseteq (-\frac{1}{2}, \frac{1}{2})$*, with* $a_j < b_j$ *for all* $j \in \mathcal{J}$*, be a Parseval frame wavelet set. Let* $g : \widehat{\mathbb{R}} \to \mathbb{R}$ *be a non-negative function satisfying* $\int g(\gamma) d\gamma = 1$ *and with support* $\operatorname{supp}_0 g = [-c, d]$*, where* $c, d > 0$ *and which contains a neighborhood of zero. Define* $\hat{\psi}_m = \mathbb{1}_L * g_{(m)}$ *for all* m *large enough that*

$$m > \max\left\{\frac{c+d}{(\min_j a_j) - (\max_j b_j) + 1}, \max_j \{\frac{c+d}{b_j - a_j}\}, \frac{d}{\operatorname{dist}(0, L)}, \frac{c}{\operatorname{dist}(0, L)}\right\}.$$

Then $\mathcal{W}(\psi_m)$ *forms a frame with upper frame bound* $B_m \geq \beta > 1$*, where* β *is independent of* m*.*

Proof. Since $m > \frac{c+d}{(\min_j a_j) - (\max_j b_j) + 1}$, $\mu_{\psi_m} = \nu_{\psi_m} = \kappa_{\psi_m}$. Because $\operatorname{supp} \hat{\psi}_m \supsetneq L$, $\inf \kappa_{\psi_m} > 0$. Finally, since $m > \max\{\frac{d}{\operatorname{dist}(0,L)}, \frac{c}{\operatorname{dist}(0,L)}\}$ and $\operatorname{supp} \hat{\psi}_m$ is compact, $\sup \kappa_{\psi_m} < \infty$. Hence $\mathcal{W}(\psi_m)$ is a frame.

The remainder of the claim follows from Theorem 39. □

In this section Parseval frame wavelets are smoothed on the frequency domain by elements of successive elements of approximate identities. However, the corresponding frame bounds do not converge to 1 even though L is a Parseval frame wavelet set. We contrast these facts to the case of time domain smoothing. In [ABG01], the Haar wavelet is smoothed using convolution on the time domain with members of particular approximate identities $\{k_\lambda\}$. The smoothed functions generate Riesz basis wavelets which have frame bounds which approach 1 as $\lambda \to \infty$.

6 Other Methods

6.1 C^∞ Parseval frames

In his Master's thesis, [Han94], as well as the paper, [Han97], Bin Han used Parseval frame wavelet sets to construct C^∞ Parseval frames for a particular subspace of $L^2(\mathbb{R})$. This construction is easily extended to other subspaces of $L^2(\mathbb{R})$, but there complications arise when one tries to generalize the construction to $L^2(\mathbb{R}^d)$, $d > 1$, see [Kin08]. The results of this paper have implications for Han's work. The shrinking method introduced in Section 4 may be used to modify Parseval frame wavelet sets so that they satisfy the hypothesis of Han's construction. Furthermore, since the process of smoothing Parseval frame wavelets in [Han97] results in Parseval frame wavelets, it follows from Theorem 39 that these wavelets are not the result of convolution with a non-negative, continuous function.

6.2 MSF smoothing

As mentioned above, Hernàndez, Wang, and Weiss, [HWW96], created the theory of MSF wavelets. They characterize wavelets ψ for which $\hat{\psi}$ has support in $[-\frac{8}{3}\alpha, 2 - \frac{4}{3}\alpha]$, for $0 < \alpha \leq \frac{1}{2}$, and prove that these are all associated with a multiresolution analysis (MRA). The authors then smoothed these MSF wavelets [HWW97]. Their smoothing procedure was accomplished by deforming given low-pass filters to obtain new filters. This process sometimes results in non-bandlimited orthonormal wavelets. This process will not work to improve the frame wavelet set wavelets which were constructed in [BS06] because it relies heavily on the associated MRA structure.

6.3 Baggett, Jorgensen, Merrill, and Packer smoothing

A different smoothing idea is employed in [BJMP06]. The authors smooth the 1-d Journé wavelet using a Generalized Multiresolution Analysis. Frame wavelets for $L^2(\mathbb{R})$ are constructed which have the same dimension function,

$$\sum_{k\in\mathbb{Z}}\sum_{n=1}^{\infty}|\hat{\psi}\left(2^n\left(x+k\right)\right)|^2,$$

as the Journé wavelet set but are arbitrarily differentiable and have C^∞ Fourier transforms. As in [HWW97], they do not regularize the members of the frame directly but rather define auxillary functions which build wavelets sharing certain traits with the original wavelet. Since they construct Parseval frame wavelets, we know from Theorem 39 that their functions cannot result from convolutional smoothing on the frequency domain.

6.4 Operator interpolation

Let K and L be (orthonormal) wavelet sets. By Proposition 6, K and L are $\mathbb{Z}^d$-translation congruent and tile $\widehat{\mathbb{R}}^d$ by dyadic dilation. Dai and Larson use these facts to construct a unitary operator U on $L^2(\widehat{\mathbb{R}}^d)$ in [DL98]. If the group generated by U commutes with the Fourier transformed dilation and translation operators when applied to $\mathbb{1}_K$, then the wavelet sets admit *operator interpolation* and an *interpolated wavelet* is found using this U. If K and L satisfy futher conditions, then this interpolated wavelet is continuous in the frequency domain. In [Han03], this process is extended to Parseval frame wavelet sets. It is not known if any pairs of Parseval frame wavelet sets formed using the neighborhood mapping construction satisfy the hypotheses of operator interpolation. Although operator interpolation is a clever application of von Neumann algebra theory to regularization of (sub-frame) wavelet set wavelets, it is not helpful in our endeavor.

6.5 Stability results

Stability results give conditions under which perturbations of a frame or Riesz basis is again a frame or Riesz basis. Chistensen and Heil proved the following sufficient condition in [CH97]:

Theorem 41. *Let $\{e_n\}$ be a frame for a Hilbert space $\mathcal{H}$ with frame bounds A and B. Let $\{f_n\} \subseteq \mathcal{H}$. If $\{e_n - f_n\}$ is a Bessel sequence for $\mathcal{H}$ with bound $M < A$, then $\{g_n\}$ is a frame with frame bounds $\tilde{A}$ and $\tilde{B}$ satisfying $A\left(1-\sqrt{M/A}\right)^2 \geq \tilde{A}$ and $\tilde{B} \leq B\left(1+\sqrt{M/B}\right)^2$.*

The hypothesis for this result is much weaker than many pre-existing basis-type assumptions, and, thus, it is sometimes practical to use it. However, due to the use of the triangle inequality in the proof of Theorem 41, the theorem does not work well when comparing a smoothed Parseval wavelet set wavelet with the original Parseval frame wavelet.

There are other stability results in [FZ95] and [Jin99], which are also not applicable for our purposes.

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