Local Frames

Anthony Teolis * John J. Benedetto †

University of Maryland, College Park, MD 20742

ABSTRACT

In this paper we introduce the concept of a *local* Hilbert space frame and develop theory for the representation and reconstruction of signals using local frames. The theory of global frames is due to Duffin and Schaeffer. Local frames are defined with respect to a global frame and a particular element from a Hilbert space $\mathcal{H}$. For any signal $f_* \in \mathcal{H}$, $\mathcal{H}$ may be decomposed into two signal dependent subspaces: a finite dimensional one which essentially contains the signal f_* and one to which the signal is essentially orthogonal. The frame elements associated with the former subspace constitute the local frame around f_*.

1 INTRODUCTION

In many signal and image processing applications, a fundamental approach involves the decomposition of signals into underlying primitives. It seems natural that if we are interested in representing a class of signals, e.g., audio data, in a concise form that these primitives should adapt to the class of interest. This is one of the primary factors motivating the development of local frame theory.

Suppose $\mathcal{H}$ is a Hilbert space of interest. On the one hand the global theory of frames allows the reconstruction of *every* signal $f \in \mathcal{H}$ from its frame representation. On the other hand a typical processing goal is to reconstruct only a *particular* signal $f_* \in \mathcal{H}$ from its frame representation. Although the global frame representation is a viable discrete representation which meets our objective, its ability to recover every signal is far more than is required. Since it is not necessary to reconstruct every signal in the entire Hilbert space $\mathcal{H}$ it is natural to ask if there is some method in which the global frame representation may be *localized* about a particular signal. These ideas lead directly to the notion of a *local frame*.

Localization can be viewed in terms of a decomposition of the Hilbert space $\mathcal{H}$ into two signal dependent subspaces. If $f_* \in \mathcal{H}$ is the particular signal of interest, localization results in a decomposition of $\mathcal{H}$ as

$$\mathcal{H} = \mathcal{H}(f_*) \oplus \mathcal{H}(f_*)^{\perp}$$

where f_* is (almost) contained in a subspace $\mathcal{H}(f_*)$ which is finite dimensional.

In Section 2 we present the mathematical notation which is employed in this paper. Section 3 reviews global frame theory setting the stage for the concept of a local frame in Section 4. Local frames are developed in terms of truncations of global frame representations.

*Institute for Systems Research and Dept. of Electrical Engineering, Univ. of Maryland.
†Dept. of Mathematics and Institute for Systems Research, Univ. of Maryland.

0-8194-1283-X/93/$6.00

2 PRELIMINARIES

$\mathbf{R}$ is the set of real numbers and $\mathbf{R}^+$ is the set of positive real numbers. $L^2(\mathbf{R})$ is the space of complex-valued finite energy signals defined on the real line $\mathbf{R}$. The *norm* of an element $f \in L^2(\mathbf{R})$ is $\|f\| \equiv \left(\int |f(t)|^2 dt\right)^{\frac{1}{2}} < \infty$, where integration is over $\mathbf{R}$, and the *inner product* of $f, g \in L^2(\mathbf{R})$ is $\langle f, g\rangle = \int f(t)\overline{g}(t)dt$. $\ell^2(\mathbf{Z})$ is the space of complex-valued finite energy sequences defined on the integers $\mathbf{Z}$. The *norm* of an element $c \in \ell^2(\mathbf{Z})$ is $\|c\| \equiv \left(\sum |c_n|^2\right)^{\frac{1}{2}} < \infty$, where summation is over $\mathbf{Z}$, and the *inner product* of $c, d \in L^2(\mathbf{R})$ is $\langle c, d\rangle = \sum c_n \overline{d}_n$. $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is the space of bounded linear operators which map the Hilbert space $\mathcal{H}_1$ to the Hilbert space $\mathcal{H}_2$. The *norm* of an element $K \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is $\|K\| = \sup_{x\in\mathcal{H}_1} \frac{\|Kx\|_{\mathcal{H}_2}}{\|x\|_{\mathcal{H}_1}} < \infty$. If $K \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ then the range of K is $K(\mathcal{H}_1) \triangleq \{Kf : f \in \mathcal{H}_1\}$. With $\mathcal{H}'$ a subspace of a Hilbert space $\mathcal{H}$ the operator $P_{\mathcal{H}'} : \mathcal{H} \mapsto \mathcal{H}'$ is the orthogonal projection operator onto $\mathcal{H}'$. The *Fourier transform* of $f \in L^2(\mathbf{R})$ is $\widehat{f}(\gamma) = \int f(t)e^{-2\pi it\gamma}dt$, for $\gamma \in \widehat{\mathbf{R}}(\equiv \mathbf{R})$, where convergence of the integral to $\widehat{f}$ is in the L^2-sense. The *dilation* operator $D_s : L^2(\mathbf{R}) \mapsto L^2(\mathbf{R})$ is a unitary map given by $(D_s f)(t) \triangleq s^{\frac{1}{2}} f(st)$. and the *translation* operator $\tau_a : L^2(\mathbf{R}) \mapsto L^2(\mathbf{R})$ is a unitary map given by $(\tau_a f)(t) \triangleq f(t-a)$.

3 FRAMES

We review the theory of (global) Hilbert space frames in $L^2(\mathbf{R})$ and develop some necessary tools. The theory of frames is due to Duffin and Schaeffer [DS52], cf., [Dau92], [DGM86], [HW89], [You80]. Let $\mathcal{H}$ be a Hilbert space contained in $L^2(\mathbf{R})$, and with norm $\|\ldots\| \triangleq \|\ldots\|_2$ induced from $L^2(\mathbf{R})$.

As a concept, frames provide an intermediate ground between the two related notions of *completeness* in a space and an orthonormal *basis* for a space. In fact the statements that a set $\{\phi_n\}$ is (a) complete in $\mathcal{H}$, (b) a frame for $\mathcal{H}$, and (c) an orthonormal basis for $\mathcal{H}$ are progressively stronger. In other words c $\Longrightarrow$ b $\Longrightarrow$ a.

Definition 3.1 *a. A sequence $\{\phi_n\} \subseteq \mathcal{H}$ is a frame for $\mathcal{H}$ if there exist frame bounds $A, B > 0$ such that*

$$\forall f \in \mathcal{H}, \quad A\|f\|^2 \leq \sum |\langle f, \phi_n\rangle|^2 \leq B\|f\|^2, \tag{3.1}$$

where summation is over $\mathbf{Z}$.

b. The frame operator of the frame $\{\phi_n\}$ is the function $S : \mathcal{H} \mapsto \mathcal{H}$ defined as $Sf = \sum \langle f, \phi_n\rangle \phi_n$.

The following result exhibits some fundamental properties of frames, e.g., [DS52], [Dau90], [Ben93].

Theorem 3.2 *a. If $\{\phi_n\} \subseteq \mathcal{H}$ is a frame with frame bounds A, B, then S is a topological isomorphism with inverse S^{-1}, $\{S^{-1}\phi_n\}$ is a frame with frame bounds B^{-1} and A^{-1}, and*

$$\forall f \in \mathcal{H}, \quad f = \sum \langle f, S^{-1}\phi_n\rangle \phi_n = \sum \langle f, \phi_n\rangle S^{-1}\phi_n \tag{3.2}$$

in $\mathcal{H}$.

b. If $\{\phi_n\} \subseteq \mathcal{H}$, let $L : \mathcal{H} \mapsto \ell^2(\mathbf{Z})$ be defined as $Lf = \{\langle f, \phi_n\rangle\}$, cf., (3.3). If $\{\phi_n\}$ is a frame then $S = L^*L$, where L^* is the adjoint of L.

Since the frame operator S may be factored [DGM86, Dau90] as L^*L an immediate consequence is that

$$\langle f, Sf\rangle = \langle f, L^*Lf\rangle = \langle Lf, Lf\rangle = \|Lf\|^2.$$

Since $\{\phi_n\}$ is a frame with frame bounds A and B this implies that $A\|f\|^2 \leq \|Lf\|^2 \leq B\|f\|^2$. Thus,

$$\|L\| \leq B^{\frac{1}{2}} \quad \text{and} \quad \|L^{-1}\| \leq A^{-\frac{1}{2}},$$

where L^{-1} is defined on the range $L(\mathcal{H})$.

It is clear that if A and B are frame bounds for a frame $\{\phi_n\}$ then any other pair A_1 and B_1 such that $0 < A_1 < A$ and $\infty > B_1 > B$ are also valid frame bounds for $\{\phi_n\}$. It is of interest to know the smallest upper bound and the largest lower bound which serve as frame bounds for a frame. This motivates the notion of the *best* frame bounds. Given a frame $\{\phi_n\}$ for a Hilbert space $\mathcal{H}$ with frame operator S, the *best bounds* A and B are

$$A = \inf_{f \in \mathcal{H}} \frac{\langle f, Sf \rangle}{\|f\|^2}, \quad B = \sup_{f \in \mathcal{H}} \frac{\langle f, Sf \rangle}{\|f\|^2}.$$

Since $\|Lf\|^2 = \langle f, Sf \rangle$ it follows that the best bounds A, B are also $A = \|L^{-1}\|^{-2}$ and $B = \|L\|^2$.

3.1 Frame Representation

Let us introduce the notion of a *frame representation* operator L. L is a mapping from $\mathcal{H}$ to $\ell^2(\mathbb{Z})$ and is defined as

$$L: \quad \begin{array}{ll} \mathcal{H} \rightarrow & \ell^2(\mathbb{Z}) \\ f \mapsto & \{\langle f, \phi_n \rangle\}. \end{array} \tag{3.3}$$

Figure 1 depicts the mapping L and its adjoint L^*. If $\{\phi_n\}$ is a frame for $\mathcal{H}$ then the mapping defined in (3.3) is called the *frame representation* operator. The frame representation operator L plays a central role in Theorem 3.2. Part a of the theorem describes one method to recover a signal $f \in \mathcal{H}$ from its frame representation $Lf \in \ell^2(\mathbb{Z})$. In part b, the theorem indicates that the frame operator S has factors L and L^*. In addition, Theorem 3.3 below states that the frame representation operator L has an inverse when considered on the range $L(\mathcal{H})$. These facts form the basis for the iterative reconstruction scheme given in Proposition 3.10 and, in turn, the notion of the *frame correlation* operator discussed in Section 3.2.

Consider the operator L of Theorem 3.2 and its adjoint. The theorem asserts that L and its adjoint L^* are factors of the frame operator S. Explicitly, the operators $L : \mathcal{H} \mapsto \ell^2(\mathbb{Z})$, and $L^* : \ell^2(\mathbb{Z}) \mapsto \mathcal{H}$ are

$$Lf = \{\langle f, \phi_n \rangle\} \quad \text{and} \quad L^* c = \sum c_n \phi_n, \tag{3.4}$$

where $f \in \mathcal{H}$ and $c \in \ell^2(\mathbb{Z})$. Hence,

$$Sf = \sum \langle f, \phi_n \rangle \phi_n = L^* L f.$$

A characterization of frame representation operators is given in the following theorem.

Theorem 3.3 ([Ben93, Theorem 3.6]) *The sequence $\{\phi_n\}$ is a frame for $\mathcal{H}$ if and only if the mapping L given in (3.3) is a well defined topological isomorphism onto a closed subspace of $\ell^2(\mathbb{Z})$.*

Thus, if L is a frame representation operator then L is injective (one to one) and L^* is surjective (onto).

3.2 Frame Correlation

A concept which arises naturally in frame theory is the notion of *frame correlation* given in Definition 3.4. In Proposition 3.9 the importance of the frame correlation in the reconstruction process is underscored.

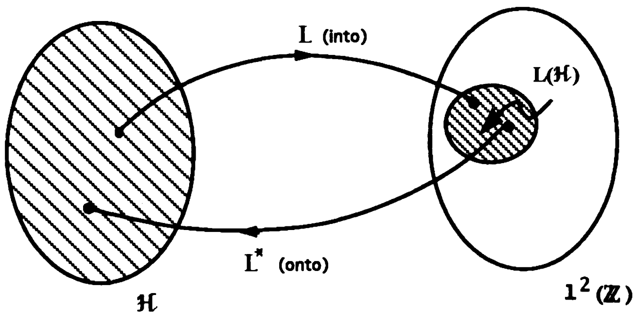

Figure 1: The mappings L and L^*

Definition 3.4 (Frame Correlation) *Let $\{\phi_n\}$ be a frame for the Hilbert space $\mathcal{H}$ with frame representation operator L. The frame correlation operator is defined as $R \stackrel{\Delta}{=} LL^*$.*

The frame operator $S = L^*L$ and the frame correlation $R = LL^*$ are similar objects and play similar roles in the theory of frames. In fact, a reconstruction theory may be developed without ever introducing the frame correlation, cf.,[FG92]. Hence, we may ask why the frame correlation is an important object to study? To answer this note that

$$S : \mathcal{H} \mapsto \mathcal{H}, \quad \text{while} \quad R : L(\mathcal{H}) \mapsto L(\mathcal{H}).$$

In many cases of interest $\mathcal{H}$ will be an infinite dimensional Hilbert space having elements which can not be directly processed by a digital machine while $L(\mathcal{H})$ will consist of *discrete* elements, i.e., countable sets, which (if truncated) may be processed digitally. For example, such an $\mathcal{H}$ is the space of bandlimited functions PW_Ω, and $L(\mathcal{H}) = \{f(t_n) : f \in PW_\Omega\}$ for some sequence $\{t_n\}$. Thus the operator S does not admit a digital implementation while R does.

From Theorem 3.2a a frame $\{\phi_n\}$ has an associated dual frame $\{\psi_n\}$, where $\psi_n \stackrel{\Delta}{=} S^{-1}\phi_n$ and S is the frame operator. As a frame, $\{\psi_n\}$ also has a frame representation operator L_ψ, where $L_\psi f \stackrel{\Delta}{=} \{\langle f, \psi_n\rangle\} = \{\langle f, S^{-1}\phi_n\rangle\}$. As a matter of notation we may write both L and L_ϕ to indicate the frame representation with respect to the frame $\{\phi_n\}$. With this notation, Equation (3.2) may be written as

$$\forall f \in \mathcal{H}, \quad f = L_\phi^* L_\psi f = L_\psi^* L_\phi f. \tag{3.5}$$

Proofs of the following propositions and Theorem 3.7 may be found in [Teo93].

Proposition 3.5 *Suppose $\{\phi_n\}$ is a frame for the Hilbert space $\mathcal{H}$ with frame representation operator L, correlation R and bounds A and B.*

a. If $\mathcal{H}$ is infinite dimensional then L is not compact. In particular, R is not compact.

b. Elements of R must decay away from the diagonal, i.e., $|m - n| \to \infty$ implies that $R_{m,n} \to 0$.

c. If the set $\{\phi_n\}$ is an orthonormal basis for $\mathcal{H}$ then the frame correlation operator is the identity.

d. $\ker R = L(\mathcal{H})^\perp$.

e. $R = P_{L(\mathcal{H})}R = RP_{L(\mathcal{H})}$

f. R is a non-negative real self-adjoint operator which maps $L(\mathcal{H})$ bijectively to itself.

Proposition 3.6 *Given a frame $\{\phi_n\}$ for the Hilbert space $\mathcal{H}$, the frame correlation matrix R has the matrix representation*

$$R = (\langle \phi_m, \phi_n \rangle). \tag{3.6}$$

Proposition 3.5e implies that R has an inverse on $L(\mathcal{H})$. For this inverse we shall write R^{-1} so that

$$\forall c \in L(\mathcal{H}) \quad c = R^{-1}Rc = RR^{-1}c.$$

To extend the inverse to all of $\ell^2(\mathbf{Z})$ we use the *pseudo inverse* $R^{\dagger} = R^{-1}P_{L(\mathcal{H})}$ where $P_{L(\mathcal{H})}$ is the orthogonal projection operator onto the image of L. Using Proposition 3.5e we have

$$\forall c \in \ell^2(\mathbf{Z}) \quad P_{L(\mathcal{H})}c = R^{\dagger}Rc = RR^{\dagger}c.$$

The following theorem shows that the best frame bounds of a frame are directly related to the operator norms of the frame correlation R and its pseudo inverse $R^{\dagger}$.

Theorem 3.7 *Let $\{\phi_n\}$ be a frame for a Hilbert Space $\mathcal{H}$ with best frame bounds A and B and frame correlation R. Then the best frame bounds are $A = \|R^{\dagger}\|^{-1}$ and $B = \|R\|$.*

3.3 Iterative Reconstruction

Assuming a frame $\{\phi_n\}$ for $\mathcal{H}$ with frame bounds A,B we have $\|I - \frac{2}{A+B}S\| \leq \frac{B-A}{A+B} < 1$, so that by the Neumann expansion,

$$S^{-1} = \frac{2}{A+B}\sum_{j=0}^{\infty}(I - \frac{2}{A+B}S)^j, \tag{3.7}$$

where I is the identity operator, e.g., [Ben92, Algorithm 50], [Ben93, Section 6.6]. For any $f_* \in \mathcal{H}$ applying (3.7) to Sf_* yields

$$f_* = \sum_{j=0}^{\infty}(I - \lambda S)^j(\lambda S)f_*, \tag{3.8}$$

where $\lambda = 2/(A+B)$.

An iterative procedure for the recovery of f_* from Sf_* could be constructed by (3.8) as a difference equation. With a view toward digital implementation, we instead wish to construct an iterative algorithm for the recovery of f_* from Lf_*. To do this we will first show that $I - \lambda R$ is a contraction on $L(\mathcal{H})$.

Lemma 3.8 *Let $\{\phi_n\}$ be a frame for $\mathcal{H}$ with frame representation operator L, correlation R, and bounds A, B. If $0 < \lambda < 2/B$ then $\|I - \lambda R\|_{L(\mathcal{H})} < 1$ and R^{-1} exists on $L(\mathcal{H})$. Moreover, $\|I - \lambda R\|_{\ell^2(\mathbf{Z})} = 1$ if $\ker L^* \neq \{0\}$. In particular we may take $\lambda = 2/(A+B)$ to ensure that $\|I - \lambda R\|_{\ell^2(\mathbf{Z})} = 1$.*

As a consequence of Lemma 3.8 we may write

$$R^{-1} = \lambda\sum_{j=0}^{\infty}(I - \lambda R)^j, \tag{3.9}$$

where R^{-1} is defined on the range of $L(\mathcal{H})$ and $0 < \lambda < 2/B$.

Proposition 3.9 *The signal f_* may be recovered from its frame representation Lf_* as*

$$f_* = \lambda \sum_{j=0}^{\infty} L^*(I - \lambda R)^j L f_*, \tag{3.10}$$

where $L^* c = \sum c_n \phi_n$ for $c = \{c_n\}$.

Proof: *Because of (3.8) and the fact that $S = L^* L$, it is sufficient to prove*

$$\lambda \sum_{j=0}^{\infty} L^*(I - \lambda R)^j L f_* = \sum_{j=0}^{\infty} (I - \lambda L^* L)^j (\lambda L^* L) f_*. \tag{3.11}$$

The $j = 0$ terms are clearly the same in (3.11). Assume $\lambda L^*(I - \lambda R)^j L f_* = (I - \lambda L^* L)^j (\lambda L^* L) f_*$. Then we compute

$$\begin{aligned}
\lambda L^*(I - \lambda R)^{j+1} L f_* &= \lambda L^*(I - \lambda R)^j L f_* - \lambda L^*(I - \lambda R)^j \lambda R L f_* \\
&= \lambda (I - \lambda L^* L)^j L^* L f_* - \lambda (I - \lambda L^* L)^j L^* L (\lambda L^* L f_*) \\
&= \lambda (I - \lambda L^* L)^j (I - \lambda L^* L) L^* L f_* = \lambda (I - \lambda L^* L)^{j+1} L^* L f_*,
\end{aligned}$$

and the result follows by induction. ∎

Proposition 3.9 leads directly to Algorithm 3.10 which details an iterative reconstruction procedure for the recovery of the signal f_* from its frame representation Lf_*. Moreover, this iterative procedure will converge at an exponential rate.

Algorithm 3.10 *Let $\{\phi_n\}$ be a frame for a Hilbert space $\mathcal{H}$ with frame representation L, correlation R and bounds A, B. Suppose we are given the frame representation $c_0 \triangleq Lf_*$ of a signal $f_* \in \mathcal{H}$. Set $f_0 = 0$. If $\lambda = 2/(A + B)$ and h_n, c_n and f_n are defined recursively as*

$$h_n \triangleq \lambda L^* c_n, \qquad c_{n+1} \triangleq c_n - L h_n, \qquad f_{n+1} \triangleq f_n + h_n,$$

then $\lim f_n = f_*$ in $\mathcal{H}$, and $\frac{\|f_n - f_*\|}{\|f_*\|} < \alpha^n$, where $\alpha \triangleq \|I - \lambda R\|_{L(\mathcal{H})} < 1$.

Proof: *An elementary induction argument shows that*

$$\forall n, \quad f_{n+1} = \lambda L^* \left(\sum_{j=0}^{n} (I - \lambda R)^j \right) c_0.$$

Consequently, by Proposition 3.9, we have $\lim f_n = f_*$. To prove the rate of convergence write $\|f_n - f_*\| = \|h_n\| = \|\lambda L^*(I - \lambda R)^n L f_*\|$. Noting that $\|(I - \lambda R)^n L f_*\| = \|(I - \lambda R)^n P_{L(\mathcal{H})} L f_*\| \leq \left(\|I - \lambda R\|_{L(H)}\right)^n \|Lf_*\|$ we have that

$$\|f_n - f_*\| \leq \lambda \|L^*\| \left(\|I - \lambda R\|_{L(H)}\right)^n \|Lf_*\| = \frac{B}{A + B} \alpha^n \|f_*\|.$$

Since $\frac{B}{A+B} < 1$ the result is obtained.

∎

Algorithm 3.10 underscores the importance of the correlation frame operator R in the reconstruction process. Formally we may rewrite (3.10) as

$$f_* = L^* R^{-1} L f_*. \tag{3.12}$$

We note that if R is known apriori the inverse frame correlation R^{-1} can be computed once (off-line) and stored for future reconstruction computations via (3.12).

4 LOCAL FRAMES

Let $\mathcal{H}$ be a Hilbert space, $\{\phi_n\}$ a frame for $\mathcal{H}$ and $f_* \in \mathcal{H}$. Define

$$\mathcal{H}_\delta(f_*) \stackrel{\triangle}{=} \text{span}\{\phi_n \ : \ |\langle f, \phi_n\rangle| > \delta\} \tag{4.13}$$

We think of the space $\mathcal{H}_\delta(f_*)$ as the localized space around the signal f_* with respect to the frame $\{\phi_n\}$. To further develop this idea we introduce the notion of a *frame localization* operator.

Example 4.11 *Consider the following operators defined with respect to a specific* $f_* \in \mathcal{H}$ *and* $\delta \in (0,1)$. *A family of truncations* $\{F_{f_*,\delta}\}$ *is given as*

$$(F_{f_*,\delta}c)_n = \begin{cases} c_n, & |(c_*)_n| \geq \delta \\ 0, & \textit{otherwise}, \end{cases} \tag{4.14}$$

where $c_* = Lf_*$.

This F has the following properties: F is a linear operator, $\|F\| = 1$, F is self adjoint, $F = F^2$, and F is the orthogonal projection operator onto the subspace $\mathcal{H}_\delta(f_*)$.

For all $c \in L(\mathcal{H})$ and $\delta > 0$, the truncation $F_{f_*,\delta}$ provides an orthogonal decomposition of c as

$$c = F_{f_*,\delta}c + (I - F_{f_*,\delta})c \qquad \text{and} \qquad \|c\|^2 = \|F_{f_*,\delta}c\|^2 + \|(I - F_{f_*,\delta})c\|^2.$$

Such an $F_{f_*,\delta}$ partitions c into two segments: one for which c_* has elements larger than δ and one for which c_* has elements less than or equal to δ. The two following lemmas show that (i) the former segment resides in a finite dimensional space and (ii) it is always possible to determine a δ which will ensure that an arbitrary percentage of the energy from the whole sequence c_* will be contained in this first finite dimensional segment.

Lemma 4.12 *Suppose* $F_{f_*,\delta}$ *is as in Example 4.11 for a fixed* $f_* \in \mathcal{H}$. *For all* $\delta > 0$ $\mathit{dim}\{F_{f_*,\delta}L(\mathcal{H})\} \leq \frac{\|c_*\|^2}{\delta^2} < \infty$.

Proof: *We have*

$$\|c_*\|^2 \geq \|F_{f_*,\delta}c_*\| \quad = \quad \sum_{|(c_*)_n| \geq \delta} |(c_*)_n|^2 \geq \delta^2 \, \mathit{card}\{n : |(c_*)_n| \geq \delta\} = \delta^2 \mathit{dim}\{F_{f_*,\delta}L(\mathcal{H})\},$$

so that $\mathit{dim}\{F_{f_*,\delta}L(\mathcal{H})\} \leq \frac{\|c_*\|^2}{\delta^2} < \infty$ *since* $c_* \in \ell^2(\mathbb{Z})$ *and* $\delta > 0$. ∎

Lemma 4.13 *Suppose* $F_{f_*,\delta}$ *is as in Example 4.11. Given* $\epsilon > 0$ *there is a* δ *so that*

$$\|(I - F_{f_*,\delta})c_*\|^2 < \epsilon\|c_*\|^2. \tag{4.15}$$

Proof: *Clearly,* $\lim_{\delta\to 0}\|F_{f_*,\delta}c_*\|^2 = \|c_*\|^2 < \infty$. *Therefore, for* ϵ *arbitrary there is some* $\delta > 0$ *so that* $|\|c_*\|^2 - \|F_{f_*,\delta}c_*\|^2| < \epsilon\|c_*\|^2$. *Since* $\|c_*\|^2 = \|F_{f_*,\delta}c_*\|^2 + \|(I - F_{f_*,\delta})c_*\|^2$ *we may conclude* $\|(I - F_{f_*,\delta})c_*\|^2 < \epsilon\|c_*\|^2$. ∎

Equation (4.15) quantifies the notion that the operator $F_{f_*,\delta}L$ extracts the most significant coefficients with respect to the specific signal f_*. Here the term "most significant" is quantified by the parameter $\epsilon \in (0,1)$. For example, a value of $\epsilon \approx 0$ indicates that almost every coefficient is significant, and a value of $\epsilon \approx 1$ indicates that almost every coefficient is insignificant. Via this lemma there is an interplay between the specified value of ϵ and

δ. In fact, Lemma 4.13 implies the existence of a *distribution* function $\nu(\epsilon)$ which serves as the boundary between acceptable and non-acceptable thresholds δ for a given ϵ. Given a particular $f_* \in \mathcal{H}$ define the distribution function $\nu_{f_*} : (0,1) \mapsto [0, \|c_*\|_\infty]$ as

$$\nu_{f_*}(\epsilon) = \inf\left\{\delta \ : \ \|(I - F_{f_*,\delta})c_*\|^2 < \epsilon\|c_*\|^2\right\}, \tag{4.16}$$

where $c_* = Lf_*$. A possible distribution function is shown in Figure 2.

Proposition 4.14 *Given a signal $f_* \in \mathcal{H}$, a distribution function ν_{f_*} as defined in (4.16) is a monotonically increasing function which is continuous from the left and $\lim_{\epsilon\to 0}\nu_{f_*}(\epsilon) = 0$ and $\lim_{\epsilon \to 1}\nu_{f_*}(\epsilon) = \|c\|_\infty$.*

Proof: *We show ν is monotonically increasing. Let $\epsilon_1 < \epsilon_2$ and define the sets S_1 and S_2*

$$S_i \overset{\triangle}{=} \left\{\delta \ : \ \|(I - F_{f_*,\delta})c_*\|^2 < \epsilon_i\|c_*\|^2\right\}, \quad i = 1,2.$$

Clearly $S_1 \subseteq S_2$ so that $\inf S_1 \leq \inf S_2$ and consequently $\nu_{f_*}(\epsilon_1) \leq \nu_{f_*}(\epsilon_2)$. ■

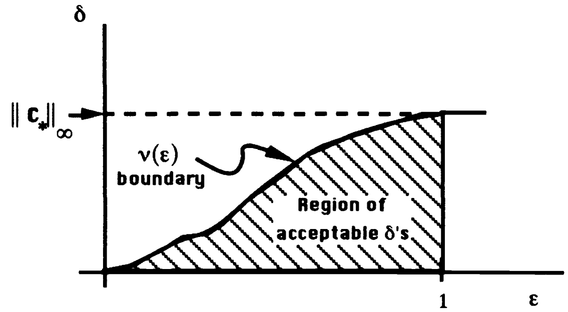

Figure 2: A possible truncation distribution function $\nu(\epsilon)$

The truncation distribution describes the relation between the necessary value for the truncation threshold δ and the desired percentage of energy preservation ϵ required after truncation. Typically, a value of ϵ is prescribed from which a compatible threshold δ is computed via the truncation distribution ν, i.e., $\delta = \nu(\epsilon)$. Suppose f_* is a particular signal in $\mathcal{H}$, L is the pertinent frame representation and ϵ is chosen as a fixed value between 0 and 1. With such a prescribed ϵ, if $\delta = \nu(\epsilon)$ it is assured that $\|(I - F_{f_*,\delta})Lf_*\|^2 < \epsilon\|Lf_*\|$. To see the ramifications of this requirement in the signal domain $\mathcal{H}$ let us first introduce the concept of "essential containment".

Definition 4.15 *A signal $f \in \mathcal{H}$ is ϵ-contained in a subspace $\mathcal{H}' \subseteq \mathcal{H}$ if $\|(I - P_{\mathcal{H}'})f\|^2 < \epsilon\|f\|^2$, where $P_{\mathcal{H}'}$ denotes the orthogonal projection operator onto the subspace $\mathcal{H}'$, and we may write $f \tilde{\in} \mathcal{H}'$ by ϵ.*

By the previous discussion if $\delta = \nu(\epsilon)$ we may say that f_* is ϵ-contained in $F_{f_*,\delta}L(\mathcal{H})$. Moreover, the essential containment property can be related back to the signal domain by Theorem 4.16.

4.1 Local Frame Representation

Assume that $\{\phi_n\}$ is a global frame for a Hilbert space $\mathcal{H} \subseteq L^2(\mathbb{R})$ with frame representation L and frame correlation R. Fix $\delta > 0$ and consider a particular element $f_* \in \mathcal{H}$ and the localization associated with the truncation operator $F_{f_*,\delta}$ given in Example 4.11. From Lemma 4.12 we see that localization by truncation has the property that the truncated space $\mathcal{H}_\delta(f_*)$ is finite dimensional. Here

$$\mathcal{H}_\delta(f_*) \triangleq \text{span}\{\phi_n \ : \ |\langle f_*, \phi_n\rangle| > \delta\} = \text{span}\{\phi_n \ : \ n \in J_\delta(f_*)\} = F_{f_*,\delta}L(\mathcal{H}),$$

where $J_\delta(f_*) = \{n \ : \ |\langle f_*, \phi_n\rangle| > \delta\}$ and $\text{card}J_\delta(f_*) < \infty$.

Because any finite collection of functions is a frame for its span we conclude that $\{\phi_n\}_{n\in J_\delta(f_*)}$ is a frame for $\mathcal{H}_\delta(f_*)$. Moreover the associated frame representation operator with respect to the truncated frame is $L_{f_*,\delta} = F_{f_*,\delta}L$ with local frame correlation $R_{f_*,\delta} = F_{f_*,\delta}RF_{f_*,\delta}$. A local reconstruction f_δ starting from an arbitrary $f \in \mathcal{H}$ is

$$f_\delta = L^*_{f_*,\delta}R^{-1}_{f_*,\delta}L_{f_*,\delta}f = (F_{f_*,\delta}L)^*(F_{f_*,\delta}RF_{f_*,\delta})^\dagger F_{f_*,\delta}Lf = L^*F_{f_*,\delta}R^{-1}F_{f_*,\delta}Lf$$

because $F_{f_*,\delta}$ is an orthogonal projection, viz. Example 4.11. Thus, a local reconstruction may be thought of in terms of the truncation of the correlation matrix $R_{f_*,\delta}$. The L^2-error associated with the local reconstruction is

$$\begin{aligned}\|f_* - f_\delta\| &= \|L^*R^{-1}Lf_* - L^*R^{-1}F_{f_*,\delta}Lf_*\| = \|L^*R^{-1}(I - F_{f_*,\delta})Lf_*\| \\ &\leq \|L^*\|\,\|R^{-1}\|\,\|(I - F_{f_*,\delta})Lf_*\| \leq \frac{B^{\frac{1}{2}}}{A}\|(I - F_{f_*,\delta})Lf_*\|.\end{aligned}$$

Thus, if f_* is ϵ-contained in $F_{f_*,\delta}L(\mathcal{H})$ then f_δ is $\frac{B}{A^2}\epsilon$-contained in $\mathcal{H}_\delta(f_*)$. In fact, Theorem 4.16 improves on this result with a tighter bound on the essential inclusion of $\frac{B}{A}$ instead of $\frac{B}{A^2}$. The theorem also provides controllable error bounds on the local frame representation of a signal. More than this, it provides a precise statement of the notion that a signal can be well represented by the most important (e.g., largest) coefficients in its frame expansion and implies a natural decomposition of the space $\mathcal{H}$ as $\mathcal{H}_\delta(f_*) \oplus \mathcal{H}_\delta(f_*)^\perp$.

Theorem 4.16 *Given a signal $f_* \in \mathcal{H}$, suppose $\{\phi_n\}$ is a frame for $\mathcal{H}$ with operator S, representation operator L, frame correlation R, and frame bounds A and B. Given $\epsilon > 0$, if $\delta = \nu(\epsilon)$ then*

$$\frac{\|f_* - f_\delta\|^2}{\|f_*\|^2} < \epsilon\frac{B}{A}, \tag{4.17}$$

where

$$f_\delta \triangleq S^{-1}L^*F_{f_*,\delta}Lf_* = L^*(F_{f_*,\delta}R^{-1}F_{f_*,\delta})Lf_*. \tag{4.18}$$

Proof: *First, we establish the formal identity $S^{-1}L^*F_{f_*,\delta}Lf_* = L^*(F_{f_*,\delta}R^{-1}F_{f_*,\delta})Lf_*$. We have*

$$\begin{aligned}S^{-1}L^*F_{f_*,\delta}Lf_* &= S^{-1}(F_{f_*,\delta}L)^*(F_{f_*,\delta}L)f_* = \sum\lambda(I - \lambda L^*L)^j(F_{f_*,\delta}L)^*(F_{f_*,\delta}L)f_* \\ &= \sum(F_{f_*,\delta}L)^*(I - \lambda LL^*)^j(F_{f_*,\delta}L)f_* = (F_{f_*,\delta}L)^*R^{-1}(F_{f_*,\delta}L)f_* = L^*(F_{f_*,\delta}R^{-1}F_{f_*,\delta})Lf_*.\end{aligned}$$

Now, write

$$f_* - f_\delta = S^{-1}Sf_* - S^{-1}L^*F_{f_*,\delta}Lf_* = S^{-1}L^*(I - F_{f_*,\delta})Lf_*. \tag{4.19}$$

Because S is a frame operator we have that $\forall g \in \mathcal{H},\ \ A\|g\|^2 \leq \langle Sg, g\rangle \leq B\|g\|^2$. In particular

$$\begin{aligned}A\|f_* - f_\delta\|^2 &\leq \langle L^*(I - F_{f_*,\delta})Lf_*, S^{-1}L^*(I - F_{f_*,\delta})Lf_*\rangle \\ &= \langle (I - F_{f_*,\delta})Lf_*, LS^{-1}L^*(I - F_{f_*,\delta})Lf_*\rangle \\ &\leq \|(I - F_{f_*,\delta})Lf_*\|\,\|LS^{-1}L^*(I - F_{f_*,\delta})Lf_*\| \\ &\leq \|LS^{-1}L^*\|\,\|(I - F_{f_*,\delta})Lf_*\|^2 \leq \|(I - F_{f_*,\delta})Lf_*\|^2 < \epsilon\|Lf_*\|^2 \leq \epsilon B\|f_*\|^2,\end{aligned}$$

from which the result follows. The manipulations are justified respectively as frame definition, adjoint operator property, Cauchy-Schwarz (and the fact that $\langle f_*, Sf_*\rangle$ is real and positive, i.e., S is a positive real operator), operator norm inequality, $\|LS^{-1}L^*\| \leq 1$ ($LS^{-1}L^*$ is the orthogonal projection onto the range of L), application of Lemma 4.13, and finally $\|L\|^2 \leq B$. ■

Theorem 4.16 shows that with $\mathcal{H}_\delta(f_*) \triangleq \text{span}\{\phi_n : |\langle f_*, \phi_n\rangle| > \delta\} = F_{f_*,\delta}L(\mathcal{H})$ we have $f_* \widetilde{\in} \mathcal{H}_\delta(f_*)$ by $\epsilon\frac{B}{A}$, where $\delta = \nu(\epsilon)$. We have decomposed the space $\mathcal{H}$ as $\mathcal{H} = \mathcal{H}_\delta(f_*) \bigoplus \mathcal{H}_\delta(f_*)^\perp$, where $f_* \widetilde{\in} \mathcal{H}_\delta(f_*)$ and f_* is essentially orthogonal to $\mathcal{H}_\delta(f_*)^\perp$.

4.2 Local Frame Correlation

Since local frame correlations are finite dimensional they are compact. This implies the existence of eigenvalues and allows the incorporation of standard matrix techniques such as singular value decompositions to determine the eigen-structure of a local frame correlation. From Proposition 3.6 local frame correlations $R_{f_*,\delta} = F_{f_*,\delta}RF_{f_*,\delta}$ are matrices given explicitly as $R_{f_*,\delta} = (R_{m,n})_{m,n\in J_\delta(f_*)}$ where $R = (R_{m,n})$ and $J_\delta(f_*) \triangleq \{n : |\langle f_*, \phi_n\rangle| > \delta\}$.

The following theorem relates the frame bounds of a local frame to the eigenvalues of the local frame correlation matrix. In particular it shows that the maximum and minimum eigenvalues associated with eigenvectors in the range $L(\mathcal{H})$ are the values of the frame bounds. This theorem may be proven as a corollary to Theorem 3.7.

Theorem 4.17 *Let $\{\phi_n\}_{n\in J}$ be a local frame for the finite dimensional Hilbert Space $H = \mathcal{H}_\delta(f_*)$ with local best frame bounds A and B and frame correlation $R_{f_*,\delta}$. Then the frame correlation $R = R_{f_*,\delta}$ is related to the local frame bounds A and B as $A = \min \sigma_L(R)$ and $B = \max \sigma_L(R)$, where $\sigma_L(\cdot)$ denotes the spectrum restricted to the range $L(H)$.*

5 LOCAL WAVELET FRAME

In this section we present an example of a local wavelet representation and illustrate the frame reconstruction of Algorithm 3.10.

A discrete wavelet representation of a signal f may be given as $Lf = \{\langle f, \tau_{t_{m,n}} D_{s_m} g\rangle\}$ where $\Gamma = \{(t_{m,n}, s_m)\}$ is a sampling set in the upper half plane $\mathbf{R} \times \mathbf{R}^+$ and g is the so called analyzing function.

We generate an analyzing function g which has good localization in both time and frequency as follows: let g_{ideal} be the real and even ideal bandpass filter specified as $\widehat{g}_{\text{ideal}} = 1_{[-b,-a]} + 1_{[a,b]}$, for $0 < a < b < \infty$. This filter is clearly well localized in frequency, however its decay in time follows $1/t$. To achieve better decay in time we convolve in frequency the ideal bandpass filter with the Dirichlet kernel $d_{2\pi c}(t) \triangleq \frac{sin(2\pi ct)}{\pi t}$ where $c > 0$ is small compared to $b - a$, i.e., $c < b - a$. This yields the trapezoidal analyzing function $\widehat{g}_{\text{trap}}$ given as $\widehat{g}_{\text{trap}} = 1_{[-c,c]} * \widehat{g}_{\text{ideal}}$. The specific values we have used are $a = 0.4$, $b = 0.5$ and $c = 0.05$ KHz. The sampling strategy which we use is regular with respect to the affine group structure, i.e., $\Gamma = \{(a_0^m nT, nT)\}$ where a_0 and T are discretization parameters.

For a given sampling structure $\{s_m\}_{m=1}^N$ on the scale axis s where N is a finite integer, the wavelet transform can be implemented as a bank of N linear filters. To see this note that $W_g f(t, s_m) = (f * D_{s_m}\widetilde{g})(t)$, so that the wavelet transform is the response of a bank of filters with impulse responses $\{D_{s_m}\widetilde{g}\}_{m=1}^N$.

Shown in Figure 3 is the regular discrete wavelet representation and its reconstruction for the signal "packet". The trapezoidal analyzing function g appears in the top upper right. To its left are the functions $G = \sum_{m=1}^N |D_{s_m}\widehat{g}|^2$ and $\widehat{g}$. The middle graph displays the reconstruction of Algorithm 3.10. The lower most graph displays the sampling set Γ. At the top of the bottom graph is the input signal and to its lower right the values of the reconstruction parameters.

Acknowledgment

This research was supported in part by the National Science Foundation's Engineering Research Centers Program: NSFD CDR 8803012 and the Office of Naval Research under grant N00014-91-J-1003.

References

[Ben92] J. Benedetto, Irregular sampling and frames, In C. Chui, editor, *Wavelets: A Tutorial in Theory and Applications*, pages 445–507. Academic Press, Boston, MA, 1992.

[Ben93] J. Benedetto, Frame decompositions, sampling and uncertainty principle inequalities, In J. Benedetto and M. Frazier, editors, *Wavelets: Mathematics and Applications*. CRC Press, Boca Raton, FL, 1993.

[BT] J. Benedetto and A. Teolis, A wavelet auditory model and data compression, Applied and Comp. Harmonic Analysis, submitted.

[BT92] J. Benedetto and A. Teolis, An auditory motivated time-scale signal representation, *IEEE-SP International Symposium on Time-Frequency and Time-Scale analysis*, October 1992.

[Dau90] I. Daubechies, The wavelet transform, time-frequency localization and signal analysis, *IEEE Trans. Information Theory*, 36(5):961–1005, 1990.

[Dau92] I. Daubechies, *Ten Lectures on Wavelets*, CBMS-NSF, SIAM, 1992.

[DGM86] I. Daubechies, A. Grossmann, and Y. Meyer, Painless nonorthogonal expansions, *J. Math. Phys.*, 27(5):1271–1283, May 1986.

[DS52] R. Duffin and S. Schaeffer, A class of nonharmonic Fourier series, *Trans. Amer. Math. Soc.*, 72:341–366, 1952.

[FG92] H. G. Feichtinger and K. Grochenig, Iterative reconstruction of multivariate band-limited functions from irregular sampling values, *SIAM J. Math Anal.*, 23(1):1–18, 1992.

[HW89] C. Heil and D. Walnut, Continuous and discrete wavelet transforms, *SIAM Review*, 31:628–666, 1989.

[Teo93] A. Teolis, *Discrete Representation of Signals from Infinite Dimensional Hilbert Spaces with Application to Noise Suppression and Compression*, PhD thesis, University of Maryland, College Park, MD, 1993.

[You80] R. M. Young, *An Introduction to Nonharmonic Fourier Series*, Academic Press, New York, 1980.

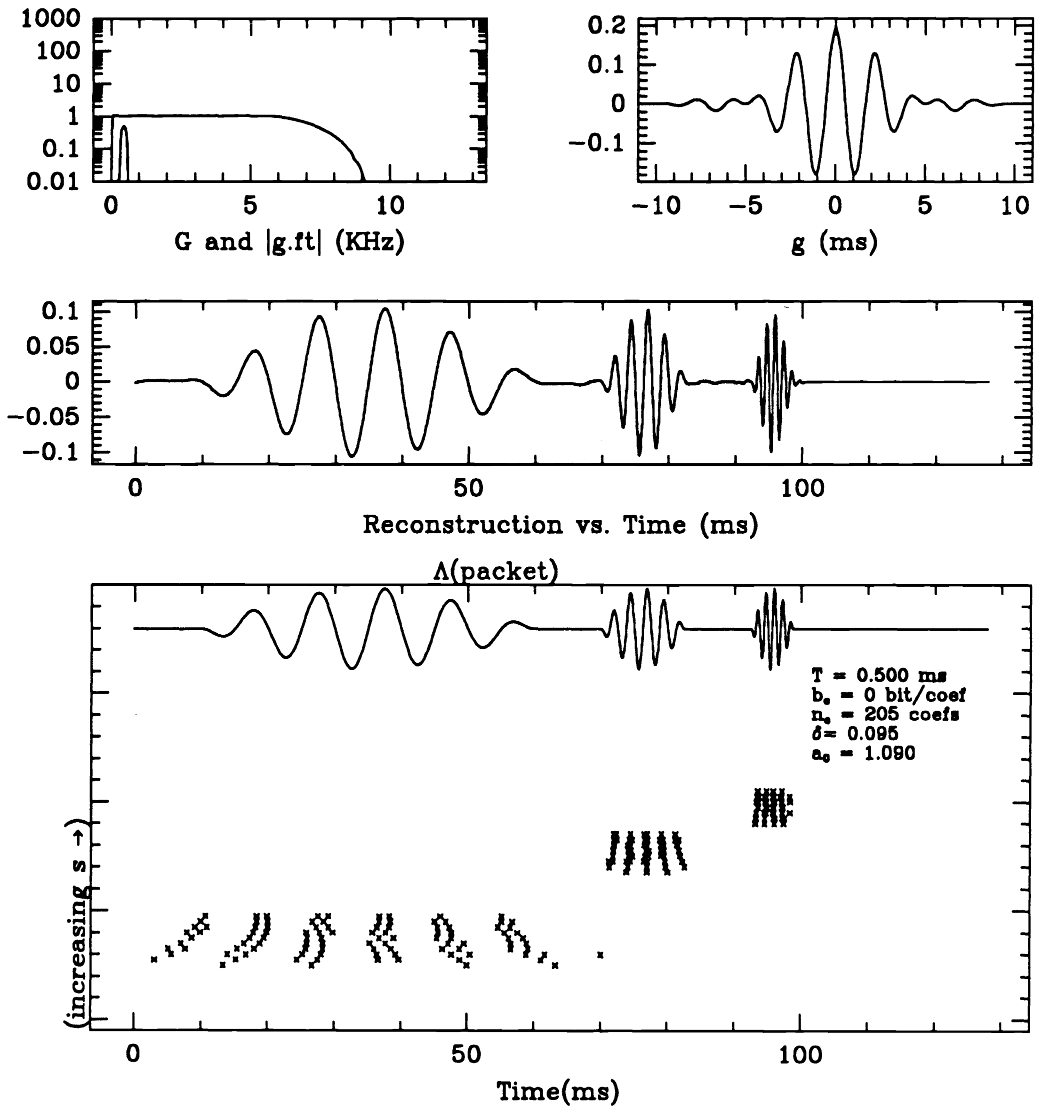

Figure 3: Local wavelet representation of the signal "packet".