MEASURE ZERO: TWO CASE STUDIES

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The intimate relationship between the theory of integration and harmonic analysis is well-known [5; 6; 7; 8; 10; 11; 15; 16; 19; 26]. I shall focus on two aspects of their story: Vitali's role in the discovery of the Radon-Nikodym theorem and some recent developments concerning (Riemann's) sets of uniqueness (U-sets).

I'll trace the subject of unique representation of trigonometric series via the evolution of integration theory to Lebesgue's theory with its correct notion of measure zero (due to Borel); and then proceed to the present day when we now know that a characterization of U-sets depends heavily on diophantine and spectral synthesis properties of sets. The main issue here will be to point out the inadequacy of sets of measure zero to provide any insight when dealing with such questions in Fourier analysis.

For the Radon-Nikodym theorem, I'll begin with Carleson's recent (1966) spectacular solution to the (Luzin) L^2 convergence problem (and the Hunt extension for the L^p, $p > 1$, case); and indicate the manner in which ideas centering around the Radon-Nikodym theorem were crucial for his result. In fact, a key technique emanates from problems in Fourier series which were false for all the Lebesgue points of a given function but true except on sets of measure zero; the positive value (sic) of measure zero is the main point here. From this stage it is a tiny and tempting step back in time to see actually the birth of the Radon-Nikodym theorem (by Vitali and Lebesgue!)

The concept of measure zero, then, is the theme in following these two arteries winding through the viscera of analysis. The broad outline sketched above and elaborated on below makes no claim to any historical insight. On the other hand, Vitali's contributions to integration theory and the arithmetic properties of U-sets are fascinating tales. The paths which I want to follow leading to these topics are both crucially illuminated by the concept of measure zero. From my point of view the approach is a convenience, but one which follows along existing roads; and I hope, at least, that the reader finds the potpourri of information entertaining along the way.

0. Measure Zero

The mathematical need for measure zero in tracing the development of integration theory from Riemann to Lebesgue is known; for example, Vito Volterra constructed functions f whose derivative exists everywhere but such that f' is not Riemann integrable. He did this in his second paper in 1881 when he was Ulisse Dini's student at the Scuola Normale Superiore (SNS) in Pisa (there will be more about SNS when we discuss both Riemann and Vitali). In fact, H.J. Smith had solved the same problem in 1875; but Lebesgue was apparently unaware of Smith's result in his thesis [9] and mentions Volterra's example prominently there.

On the other hand, Norbert Wiener has made a case concerning a physical motivation for creating the notion of measure zero in his paper [26, p. 63] on the history of harmonic analysis, presented in 1938 at the American Mathematical Society semicentennial commemoration. Basically the complete justification of Maxwell and Gibb's statistical mechanics demands a theory of measure zero; and "the ideas of statistical randomness and phenomena of zero probability were current among the physicists and mathematicians in Paris around 1900 and it was in a medium heavily ionized by these ideas that Borel and Lebesgue solved the mathematical problem of measure" [26].

1. Sets of Uniqueness and the Inadequacy of Measure Zero

1.1 B. Riemann

Bernhard Riemann's life (September 17, 1826-July 20, 1866), so tragic in its briefness and transcendental in its brevity brilliance, has been documented by his friend Dedekind, popularized by several others since then, and, even today, provides an exciting scene in the past to fathom (e.g., a study of the Betti-Riemann correspondence).(1) Our interest for the sequel focuses on his Habilitationsschrift [19]; here he begins with an important historical note on Fourier series, defines the Riemann integral to provide a broader setting for an analytically precise theory of Fourier series, and develops the Riemann localization principle which is a key technique in the study of U-sets. $E \subseteq [0, 2\pi) = \mathbf{T}$ is U if

$$\lim_{N\to\infty} \sum_{|n| \le N} c_n e^{inx} = 0 \text{ off } E \text{ implies } c_n = 0 \text{ for all } n.$$

The problem to determine U-sets is important since one would like to know if the representation of a function by a trigonometric series is unique or not. The first explicit results in this direction were given by Cantor (e.g., §1.2) although the following fundamental theorem (for uniqueness questions), first proved by Cantor, was apparently known by Riemann [19; 11, p. 110]:

(C-L) If $\lim_{N\to\infty} \sum_{|n| \le N} c_n e^{inx} = 0$ on [a, b] then $\lim_{|n|\to\infty} c_n = 0$.

It is interesting to observe the overlap between Dini and Riemann. Riemann convalesced and toured in Italy during the winter of 1862~~,~~ and then came to Pisa during 1863. He became quite friendly with Betti and Beltrami (Betti was director of SNS from 1865-1892). During that time, Dini was a student at SNS graduating in 1864 at scarcely 19 years old; he then spent a year in Paris with Bertrand, and returned to SNS where he spent the next 52 years! Dini became one of the 19th century giants in real variable and Fourier analysis, and, of course, includes Volterra and Vitali among his students at SNS. Riemann returned to Germany for the winter of 1864-65, but then came back again to Pisa. He died and was buried at Biganzolo in the northern part of Verbania (the Italian resort town on the western banks of Lago Maggiore just 15 miles south of the Swiss border).

1.2 G. Cantor

Georg Cantor (March 3, 1845-January 6, 1918) wrote several important papers on U-sets during the early 1870's (Crelle's Journal, volumes 72 and 73, and Math. Ann., volume 5). In the first, he proved (C-L) (§1.1) and using this fact proved (in the second) that ϕ is U. The subsequent papers gave simplifications of proof and extensions of the basic result, showing finally that certain countable infinite sets are U. The study of special types of infinite sets in this work certainly influenced his later research activity, and it was in 1874 that he gave his famous and controversial proof of the theorem that there are only countably many algebraic numbers (an algebraic number is a root of a polynomial with integer coefficients). The remainder of his life was devoted to the study of the infinite, not only in a mathematical milieu, but often delving into various philosophical notions of infinity due to the Greeks, the scholastic philosophers, and his contemporaries. He certainly did not dote on all philosophers. In a letter to Bertrand Russell (who was then at Trinity College, Cambridge), Cantor, in London at the time (1911), writes:

"...and I am quite an adversary of Old Kant, who, in my eyes has done much harm and mischief to philosophy, even to mankind; as you easily see by the most perverted development of metaphysics in Germany in all that followed him, as in Fichte, Schelling, Hegel(2), Herbart, Schopenhauer, Hartman, Nietzche, etc. etc. on to this very day. I never could understand why...reasonable..peoples...could follow yonder sophistical philistine, who was so bad a mathematician." It is interesting to note that the great Dedekind joined Cantor along the slalom to conquer the powers (sic) that be. Recently (1966), F.A. Medvedev has studied Dedekind's set theoretic contributions in the long (27 years) Cantor-Dedekind correspondence. The marvelous Dedekind, by the way, was a high school teacher in Brunswick for fifty years (from 1862)!

Cantor's U-set papers were preceded by H. Heine's uniqueness theorem (Crelle's Journal, volume 71) in 1870 which assumed that the given trigonometric series were uniformly convergent off arbitrary neighborhoods of a fixed finite number of points. Heine was at University of Halle with Cantor and attributes this approach to Cantor.

Cantor, of course, tried to prove all countable sets are U; and this was finally achieved by F. Bernstein (1908) and W.H. Young (1909). Actually Bernstein proved somewhat more, showing that E is U if it does not contain any non-ϕ perfect subsets (every countable set satisfies this property and closed sets with this property are countable (in $\mathbb{T}$)).

1.3 D. Men'shov

Dmitrii Men'shov (April 18, 1892-) proved a key result on U-sets in 1916 by finding a non-U-set E with Lebesgue measure $m(E) = 0$. He did this just after graduating from Moscow University, where he wrote his thesis under N. Luzin. His example has stimulated a great deal of study about sets of measure zero; and research about specific sets of measure zero now forms a significant part of modern Fourier analysis and potential theory. Actually, on the basis of Men'shov's example, Luzin and Bari defined the notion of "U-set" as such. Earlier, de la Vallée-Poussin had proved that if a trigonometric series converges to $f \in L^1$ off a countable set E then the series is the Fourier series of f; and it was generally felt that the same would be true if $m(E) = 0$. Consequently, Men'shov's example had a certain amount of shock value, to say the least.

Since we will be discussing Carleson's result a little in §2 it is interesting to note that Men'shov solved the analogue for measurable functions in 1940-41. Luzin, in 1915, had noted that if f is measurable on $\mathbb{T}$ and finite a.e. then there is a trigonometric series which converges to f by both Riemann and Abel summation. The problem was to show if such a series exists which converges pointwise a.e. to f; Men'shov showed precisely this! Thus with the Carleson-Hunt theorem and Kolmogorov's example of $f \in L^1$ with Fourier series diverging everywhere (1926), "all that remains" (in the broad sense) of the Luzin problem is an investigation of the analogous situation for f measurable but taking infinite values on a set of positive measure. Actually Men'shov has an affirmative answer on this latter problem for the case of convergence in measure instead of convergence a.e.

1.4 N. Bari and A. Rajchman

What with Men'shov's example, Alexander Rajchman (who died at Dachau in 1940) "seems to have been the first to realize that for sets of measure zero that occur in the theory of trigonometric series it is not so much the metric as the arithmetic properties that matter" [21 (from Zygmund's biography of Salem)]. Rajchman [17 (1922)] proved the existence of perfect (in particular, uncountable and closed) U-sets. He was motivated by some work of Hardy and Littlewood (Acta Math. 37(1914)), and later (1920) Steinhaus, on diophantine approximation to introduce "H-sets" and proved that such sets are U. Rajchman, in a letter to Luzin, thought that any U-set is contained in a countable union of H-sets, and it was only in 1952 that Pyatetskii-Shapiro [illegible] proved this conjecture false. The Cantor set is H and therefore U! By the way, it is easy to verify that if $m(E) > 0$ then E is not U.

Actually, Nina Bari had proved the existence of perfect U-sets in 1921 and presented her results at Luzin's seminar (at the University of Moscow); they were unpublished at the time of Rajchman's paper, although they were communicated to

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who refered to them in

Rajchman in [17 (1923)]. This does not minimize the importance of Rajchman's results since he established a large class of perfect U-sets and illustrated the need for diophantine properties in constructing such sets.

Nina Bari (November 19, 1901-July 15, 1961) established her first results on U-sets as an undergraduate, and throughout her life, although she engaged in several other research areas, was an outstanding expositor and contributor on the tricky business of uniqueness. One of her major results is that the countable union of closed U-sets is U — although the problem is open for the finite union of arbitrary U-sets. Another, which was proven in 1936-37 and which has an interesting sequel (e.g., §1.6) shows that if α is rational and $E(\alpha)$ is the Cantor set with ratio of dissection α, then $E(\alpha)$ is U if and only if $1/\alpha$ is an integer; her theorem depends heavily on diaphantine considerations.

1.5 Number Theoretic and Spectral Synthesis Remarks

~~is this right~~

1.5a Kronecker Sets

Kronecker proved: if $\{x_1, \cdots, x_n, \pi\} \subseteq \mathbb{R}$ is linearly independent over the rationals, $\{y_1, \cdots, y_n\} \subseteq \mathbb{R}$, and $\varepsilon > 0$, then there is an integer m such that for each j, stet

$$|e^{ix_j m} - e^{iy_j}| < \varepsilon.$$

Because of this we say that closed $E \subseteq \mathbb{T}$ is a Kronecker set if for each $\varepsilon > 0$ and continuous $f\colon E \to \mathbb{C}$, $|f| = 1$, there is an integer m for which

$$\sup_{x \in E} |f(x) - e^{imx}| < \varepsilon.$$

1.5b <u>The Spectral Synthesis Problem</u>

Let $A(\mathbb{T})$ be the absolutely convergent Fourier series $f(x) = \sum a_n e^{inx}$ on $\mathbb{T}$ normed by $\|f\| = \sum |a_n|$, and let its dual be $A'(\mathbb{T})$. For each closed set $E \subseteq \mathbb{T}$ let $A'(E)$ be those elements of $A'(\mathbb{T})$ with support contained in E and let $A_S'(E)$ be those $T \in A'(E)$ such that $\langle T, f \rangle = 0$ for all $f \in A(\mathbb{T})$ vanishing on E. E is a <u>spectral synthesis set</u> (S) if $A'(E) = A_S'(E)$.

Norbert Wiener (and Arne Beurling) posed the problem to determine if a given closed subset of $\mathbb{T}$ is S or not, and in 1930 Wiener proved (in his Tauberian theorem) that the empty set is S. In 1949-52, Kaplansky, Segal, and Helson proved that a one-point set is S (in fact, it is what is called a Ditkin set — Ditkin sets are S and it is not known if the two notions are equivalent). These two results, together with a technique introduced by Ditkin in 1939, were combined by Shilov to get the general form of Wiener's Tauberian theorem (in fact, Shilov's result dates from the early forties and his theorem is valid for algebras which have the "one-point" property that Kaplansky, etc. proved for $A(\mathbb{T})$). This general Wiener theorem tells us, in particular, that if E is closed and ∂E is countable then E is S; it is basically a result about Ditkin sets. Using a different idea, Carl Herz proved that the Cantor set is S.

In the other direction, Laurent Schwartz (1947) showed that the sphere's surface is non-S in $\mathbb{R}^3$; and Paul Malliavin (1959) proved that every non-discrete locally compact abelian group has non-S-sets.

Since I have defined S-sets in such cold-blooded fashion you may wonder from where they come. That story is exciting and complicated, but an amusing start would be a visit to the "Harmonic Analyzers and Synthesizers" exhibit at the Smithsonian's Museum of History and Technology. Although there was much mathematical stimulation for the S-set problem, Wiener convincingly claims some physical motivation. From a purist's point of view the following from Levinson's biography of Wiener is interesting: "G.H. Hardy once asked me whether Wiener's claims about the applied origin of his work was not a 'pose'." Levinson argues against this.

1.5c <u>Pisot Numbers</u>

An <u>algebraic integer</u> is an algebraic number (<u>§1.2</u>) where the corresponding polynomial is monic. A <u>Pisot number</u> is a real algebraic integer $\alpha > 1$ with the property that all of the other roots of its minimal polynomial have modulus less than 1. Pisot numbers come into the picture for the investigation of badly distributed sequences, as opposed to Weyl's uniform distribution (Kronecker's theorem can be proved using Weyl's results on uniform distribution).

Independently of each other, Thue (1912 in Norske Vid. Selsk. Skr.!) and Hardy (1919 in J. of Ind. Math. Soc.) observed that

$$\alpha \text{ Pisot implies } \lim_{n\to\infty} \alpha^n \pmod 1 = 0,$$

and proved (what is still one of the key properties of Pisot numbers): if $\alpha > 1$ is an algebraic integer and $\lim_n \alpha^n \pmod 1 = 0$ then α is Pisot. The work of both Thue and Hardy does not seem to have been properly advertised until the early 1960's. Pisot numbers have been most extensively studied by Pisot beginning with his thesis in 1938 and a good bibiliography on the subject up to 1962 is in Crelle's Journal 209(1962).

Note that $\alpha_1 = 1/2(1 + \sqrt{5})$ is Pisot since $|1/2(1 - \sqrt{5})| < 1$. For quadratic Pisot numbers α_1 with conjugate α_2 we trivially verify the above Thue-Hardy observation as follows:

$$(\alpha_1{}^n + \alpha_2{}^n) = (\alpha_1{}^{n-1} + \alpha_2{}^{n-1})(\alpha_1 + \alpha_2) - \alpha_1\alpha_2(\alpha_1{}^{n-2} + \alpha_2{}^{n-2}),$$

$$\alpha_1 + \alpha_2 = 1, \quad \alpha_1\alpha_2 = -1,$$

and an induction agrument show

$$\alpha_1{}^n + \alpha_2{}^n \in \mathbb{Z};$$

and so $|\alpha_2| < 1$ therefore implies

$$\lim_{n\to\infty} \alpha_1{}^n \pmod 1 = 0.$$

1.6 R. Salem and the French School

Why Pisot numbers in Fourier analysis? Well, Salem proved the following brilliant result: let $E(\alpha)$ be a Cantor set with constant ratio of dissection $\alpha \in (0, 1/2)$; $E(\alpha)$ is U if and only if $1/\alpha$ is a Pisot number! Salem announced this result in 1943 (while an instructor at MIT) and an error in the sufficiency conditions was found by members of the theory of functions seminar at University of Moscow in 1945. In 1948 Salem published some special cases in which the sufficiency is true, and in 1954 Pyatetskii-Shapiro proved that if β is Pisot of degree n and $\beta > 2^n$ then $E(\alpha)$ is U where $\alpha = 1/\beta$. Finally, in 1955, Salem and Zygmund, using the Pyatetskii-Shapiro method, proved the full generality of the originally stated result.

Raphaël Salem (November 7, 1898-June 20, 1963) was the key figure in the revival of the now flourishing Paris (Orsay) school of Fourier analysis led by Jean-Pierre Kahane. Salem returned to Paris after the war and his lectures in 1948 on unsolved problems in Fourier series were certainly the catalyst for this present activity. Salem's career is warmly sketched by Zygmund [21] from his birth in Saloniki, his banking profession (manager of the Banque de Paris et des Pays-Bas by 1938!), to the days on Brattle Street (ah, the banks of the Charles in spring), and to Paris.

Two of the striking results that have evolved from the study of U-sets and the notions of §1.5 are:

a. (Malliavin, 1962) If every closed subset of closed $E \subseteq \mathbb{T}$ is S then E is U.

b. (Varopoulos, 1965) Measures are the only pseudo-measures supported by Kronecker sets; and so if E is Kronecker then the hypothesis of a is satisfied and E is U.

2. The Radon-Nikodym Theorem and the Importance of Measure Zero

2.1 L. Carleson, and Luzin's Problem

In his dissertation of 1915 (actually he published a Comptes Rendus Acad. Sci., Paris note in 1913 on the relevant material), Luzin gave necessary and sufficient conditions that $f \in L^2(\mathbb{T})$ have a Fourier series convergent a.e.; and at the time essentially posed the problem as to whether every $f \in L^2(\mathbb{T})$ has Fourier series converging a.e. Actually, men such as Fatou (1906), Jerosch and Weyl (1908), Weyl (1909), W.H. Young (1912), Hobson (1913), Plancherel (1913), and Hardy (1913) had worked specifically on such issues. Refined "log-estimates" by Kolmogorov-Seliverstov (1925), Plessner (1926), and Littlewood-Paley (1931) kept interest in the problem at a fine pitch. Finally, in 1966 (Acta Math. 116(1966) 135-157), Carleson proved that if $f \in L^2(\mathbb{T})$ then its Fourier series converges a.e. (to f); and in 1968, using the method of Carleson's proof and the theory of interpolation of operators, R.A. Hunt extended Carleson's result to the $L^p(\mathbb{T})$, $p > 1$, case (recall Kolmogorov's example in §1.3).

An important lemma for Carleson, and that aspect of Carleson's proof that allows us to trace back in time to Vitali, is (his Lemma 5, p. 140):

> Let $\{I_k\}$ be a disjoint cover of $(0, 1)$ by open intervals where $m(I_k) = d_k$ and the center of I_k is t_k. Define
>
> $$D(x) = \sum_k \frac{d_k^{\,2}}{(x - t_k)^2 + d_k^{\,2}}, \quad x \in (0, 1),$$
>
> and
>
> $$U_M = \{x \in (0, 1): \; D(x) > M\}.$$
>
> Then there are $C, K > 0$ such that for all M
>
> $$m(U_M) \leq Ce^{-KM}.$$

2.2 J. Marcinkiewicz.

Józef Marcinkiewicz died in a prison camp in 1940 at the age of 30.

His interest to us right now concerns certain of his techniques which helped him to go deeper than what the Radon-Nikodym theorem (or fundamental theorem of calculus (FTC), e.g., §2.3) seems to allow; these techniques are closely tied in with Carleson's lemma of §2.1, and, in fact, using Marcinkiewicz's methods, Zygmund has deduced Carleson's lemma [27].

At the risk of being turgid as a substitute for being technical in a discussion of this sort, let me try to describe how the FTC comes into the picture. If $f \in L^1(\mathbb{R})$ then a key part of FTC is that

$$\lim \frac{1}{m(I)} \int_I f(t)\,dt = f(x), \quad \text{a.e.},$$

where I is an interval containing x and the limit indicates that we let the $m(I)$'s tend to 0. Thus

$$\lim \frac{1}{m(I)} \int_I (f(t) - f(x))\,dt = 0, \quad \text{a.e.}$$

In fact, a lot more can be said:

$$\lim \frac{1}{m(I)} \int_I |f(t) - f(x)|\,dt = 0, \quad \text{a.e.};$$

and points at which this is true form the set L_f of <u>Lebesgue points</u> of f (noting that for each I, $m(I \cap L_f) = m(I)$).

Now there are certain classical theorems in Fourier analysis whose conclusions (C) hold for subspaces X of $L^1(T)$; and by the nature of their proofs (C) is valid for each $x \in L_f$ of the given $f \in X$. During the 1920's and 1930's when it was popular to try to extend such results to all of $L^1(\mathbb{T})$, the difficulties that arose frequently culminated in some ingenious counterexample that showed the existence of $f \in L^1(\mathbb{T})$ such that (C) failed for some $x \in L_f$. This situation did not preclude the possibility that (C) might hold a.e. for each $f \in L^1(\mathbb{T})$; and in fact this was the type of result Marcinkiewcz attained with the techniques mentioned above. Consequently, measure zero was precisely the correct notion for such questions. Since wordy descriptions as above often tend to obfuscate the issue, I refer to [27] for some concrete examples on the matter.

2.3 The Vitali-Lebesgue-Radon-Nikodym Theorem

Let us begin by stating FTC and indicating that the Radon-Nikodym theorem generalizes FTC to $\mathbb{R}^n$ and more general measure spaces. $F: [a, b] \longrightarrow \mathbb{R}$ is absolutely continuous if for each $\varepsilon > 0$ there is $\delta > 0$ such that

$$\sum_1^n (b_j - a_j) < \delta \quad \text{implies} \quad \sum_1^n |F(b_j) - F(a_j)| < \varepsilon,$$

for any disjoint collection $\{(a_j, b_j) \subseteq [a, b]: \ j = 1, \cdots, n\}$. FTC states: F is absolutely continuous if and only if F is the indefinite integral of its derivative. Once FTC was established the problem was to generalize it to $\mathbb{R}^n$; and thus emerged the characterization of measures absolutely continuous with respect to a given bounded measure μ in terms of L_μ^1. The Radon-Nikodym theorem's basic formula, to tide you over into the next paragraph, is

$$\text{(R-N)} \qquad \nu(A) = \int_A f d\mu, \quad f \in L_\mu^1,$$

and f is the (Radon-Nikodym) "derivative" of ν with respect to μ. The point is that Radon-Nikodym is the natural analogue of FTC once we depart from the straight and narrow (i.e., $\mathbb{R}$).

In [10, 1st edition, p. 94], Lebesgue considered the following definition:

(D-R) A bounded function f is <u>integrable</u> if there is a function F with bounded derived numbers such that $F' = f$ a.e. The <u>integral of f in (a, b)</u> is $F(b) - F(a)$.

Such a definition generalizes the integrals of Riemann and Duhamel. Lebesgue introduced (D-R) by saying: "Je ne m'occuperai pas, pour le moment du moins, de la suivante." And, he keeps his word (and reticence) until on the very last page of text (p. 129), in a footnote no less, we are dealt the following thermonuclear device: "Pour qu'une fonction soit intègrable indéfinie, il faut de plus que sa variation totale dans une infinité dénombrable d'intervalles de longueur totale L tende vers zéro avec L. Si, dans l'énoncé de la page 94 (i.e., (D-R) above), on n'assiyettit pas f à ètre borneé, ni F à ètre à nombres dérivés bornés, mais seulement à la condition précédente, on a une definition de l'intégrale équivalente à celle développée dans ce Chapitre et applicable à toutes les fonctions sommables, bornées ou non." Thus, in an obtuse presentation and as a footnote and without proof, we are handed the fundamental theorem of calculus!

In 1904, Vitali [23] defined absolute continuity exactly as we have above and went on to state and prove FTC. The idea of absolute continuity had been used by Axel Harnack in 1884; and Harnack, in turn, was definitely influenced at this point by Dini's work on Fourier series (1880) [8, pp. 77-78].

Vitali's next step in this business is [24]. He begins by proving the Vitali covering theorem and uses the covering theorem to prove an FTC in $\mathbb{R}^2$. He also deduces FTC (on $\mathbb{R}$) with the covering theorem. Because of the importance of set functions in the development of integration theory we note that in <u>§5</u> of [24] Vitali considers families of rectangles and the formula (R-N) for A, a "rettangolo coordinato."

Essentially, in his major work of 1910 [12], Lebesgue relies on the Vitali covering theorem and "les travaux de M. Volterra(3), à définir la dérivée de la fonction $F(A)$ en un point P comme la limite du rapport $F(A)/m(A)$, A étant un ensemble contenant P et dont on fait tendre toutes les dimensions vers zéro" [12, p. 361].

In [12], Lebesgue begins by quoting Vitali's FTC in $\mathbb{R}^2$; and notes that an "inadvertance" by Vitali in his proof is corrected by considering a <u>regular family</u> of rectangles in $\mathbb{R}^2$. This constraint to define Radon-Nikodym derivatives by taking limits over regular families is necessary. Further, by proving FTC in $\mathbb{R}^2$ in terms of set functions (not depending on rectangular coordinates as Vitali had done), Lebesgue set the stage for the synthesis and generalization of Radon in 1913, where he (Radon) incorporated Stieltjes integrals into the scheme of things, and Nikodym.

Concerning the above mentioned footnote in [10, 1st edition], Lebesgue [12, p. 365] writes: "J'avais, dans mes

Leçons, tout à fait incidemment et sans démonstration, fait connaître" the FTC.

A perusal of [12] indicates the crucial dependence of Lebesgue on the Vitali covering theorem ("un théorème capital" [12, p. 390]) and Vitali's original proof for Lebesgue's setting ("La démonstration qu'on lira plus loin est presque copiée sur celle de M. Vitali" [12, p. 390]).

2.4 Vitali: Luzin's Theorem and Vitali-Hahn-Saks Theorem

2.4a Luzin's Theorem

In 1912 Luzin proved (what is now known as Luzin's theorem): if f is Lebesgue measurable on $[a, b]$ and $\varepsilon > 0$ there is g continuous on $[a, b]$ such that $m\{x: f(x) \neq g(x)\} < \varepsilon$. Now as early as 1903, Borel and Lebesgue had studied the topological properties (such as continuity) of measurable functions. In 1905 (in Rend. Istit. Lombardo 38(1905) 599-603) Vitali proved Luzin's theorem (on pp. 601-602)! He then used this "lemma" to prove what is now known as the Vitali-Carathéodory theorem. We note that Bourbaki defines measurable functions in terms of the Vitali criterion: f (defined on $[a, b]$, say) is Lebesgue measurable if there is a set of measure zero $E \subseteq [a, b]$ and a partition of $[a, b] \setminus E$ formed by a sequence (finite or infinite) of compact sets $\{K_n\}$ such that f restricted to K_n is continuous.

2.4b <u>The Vitali-Hahn-Saks-Nikodym (-Dieudonné-Grothendieck) Theorem</u>

$\{f_n\} \subseteq L_m{}^1[a, b]$ is <u>uniformly absolutely continuous</u> if for each $\varepsilon > 0$ there is $\delta > 0$ such that for all Lebesgue measurable A satisfying $m(A) < \delta$ and for all n

$$\left|\int_A f(t)dm(t)\right| < \varepsilon.$$

Using this notion Vitali proved (<u>Rend. Circolo Mat. Palermo</u> 23(1907)): let $\{f_n\} \subseteq L_m{}^1[a, b]$; f_n converges to $f \in L_m{}^1[a, b]$ in L^1-norm if and only if $f_n \longrightarrow f$ pointwise a.e. and $\{f_n\}$ is uniformly absolutely continuous. An immediate corollary is the Lebesgue dominated convergence theorem

This result of Vitali's has been the source of one of the deepest results in measure theory, which we now describe. In 1933, after some work of Hahn in 1922, Saks proved that if a sequence of bounded measures $\{\nu_n\}$ is absolutely continuous with respect to a given measure μ and $\nu_n(A)$ converges for each $A \in \mathcal{Q}$ (given the measure space $(X, \mathcal{Q}, \mu)$) then $\{\nu_n\}$ is uniformly absolutely continuous and so (by Vitali's theorem) ν_n converges on $\mathcal{Q}$ to a bounded measure ν. Earlier in 1933, Nikodym had shown (he announced the results in 1931) that if $\{\nu_n\}$ converges on $\mathcal{Q}$ then it converges (on $\mathcal{Q}$) to a bounded measure; and using his (Saks) sufficient conditions for uniform absolute continuity and a standard trick for buying the absolute continuity hypothesis, Saks derived Nikodym's result.

It is an important application (in Fourier analysis, for example) to extend Nikodym's theorem to read—

> If for all $B \in \mathcal{B}$, $\lim \nu_n(B)$ exists, then there is a measure ν for which $\lim \nu_n(B) = \nu(B)$,

where $\mathcal{B} \subseteq \mathcal{A}$ is as small as possible. If X is a compact space, Dieudonné (1951) and Grothendieck (1953) have shown that we can choose $\mathcal{B}$ as the family of open sets in X.

2.5 Vitali's Life

Giuseppe Vitali (August 26, 1875-February 29, 1932) was born in Ravenna, the oldest of five children. After graduating from the "liceo" in Ravenna, he studied mathematics at the University of Bologna in 1895; then he received a scholarship to SNS (Dini and Bianchi were there at the time). He graduated in 1899 with a thesis in which he extended to Riemann surfaces a theorem of Mittag-Leffler. His next work was devoted to abelian integrals. In 1901 he was an assistant to Dini, and then he began his career as a high school teacher! After two brief appointments in Sassari and Voghera he taught at the Liceo "Colombo" in Genova from 1904-1922. This, of course, does not match Dedekind's record (e.g., §1.2) but outdoes Weierstrass' 14 year stint as a high school teacher; of course, Weierstrass' "defenders" would point out that his (Weierstrass) service in teaching penmanship and gymnastics (besides mathematics) should count for something extra.

In any case, as we have seen in §2.3 and §2.4, Vitali was not spending all of his time in PTA and chaperoning dances. Finally in 1923, Vitali received a position at the University of Modena (a weak counterexample to one of the fundamental theorems of life that "you can keep a good man down"). In 1924 he went to the University of Padova, where at the end of 1926, he was struck with hemiplegia (a paralysis resulting from injury to the motor center of the brain). Fortunately his intellectual powers were unaffected. He left Padova in 1930 for the University of Bologna. He died there suddenly after class one afternoon in the company of his colleague, Bortolotti.

Footnotes:

(1) Generally I shall not discuss well-known facts about well-known mathematicians which can be found in well-known biographies.

(2) Hegel of Ceres infamy!

(3) 1889.

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