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The Orbit of an Equatorial Satellite

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FINAL REPORT

THE ORBIT
OF AN
EQUATORIAL SATELLITE

Task No. 0232-G

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ABSTRACT

Exact integration of the equations of motion of a secondary body in the equatorial plane of a rotationally symmetric central body is obtained by means of Weierstrass elliptic functions.

In this solution, if some simplifications are made, the analytical expression for the radius vector as a function of the true anomaly resembles that of a revolving conic. The validity of such simplifications is thoroughly discussed.

The solution of the inverse problem, the true anomaly as a function of the radius vector, has also been found and presented in terms of Legendre's elliptic integral of the first kind.

The results obtained are used to evaluate the shift of the orbital apsidal line at each revolution of the secondary body.

The amount of this shift is in agreement with that observed in artificial earth satellites, as well as in the orbits of some natural satellites close to their respective planets. Particular cases are the fifth satellite of Jupiter and the satellites Phobos and Deimos of Mars.

In the case of the planet Mercury, a difference of one second of arc between equatorial and polar radius of the sun has been assumed; then a shift of about 14 seconds per century can be derived, which is 34 percent of the observed value. The question of the real value of the sun's flattening is yet open and awaits experimental confirmation. It is believed, however, that a part of the observed perihelion shift of Mercury is due to the oblateness of the sun.

NOTE

In this paper, for the Weierstrass P-function, the Greek letter γ (gamma) is used throughout.

CONTENTS

Section I

INTRODUCTION

1.1 OBJECTIVE

The main objective of this task is the exact integration of the equations of motion of a secondary body of negligible mass around a rotationally symmetric central body, such as an oblate spheroid. This motion takes place in the equatorial plane of the central body, whose external potential in this plane is approximated by

$$\mathcal{U} = \frac{\mu}{r} + \frac{\alpha}{r^3} , \qquad (r > 1), \tag{1}$$

where the equatorial radius of the central body has been taken as the unit of length,

$$\mu = k^2 M , \tag{2}$$

$$\alpha = \frac{1}{2} \mu J , \tag{3}$$

where

k^2 = the Gaussian gravitational constant,

M = the total mass of the central body,

J = a constant depending on the flattening of the surface delimiting the central body.

It will be shown that this integration can be performed rigorously, using Weierstrassian elliptic functions. The results obtained are then used to evaluate the apsidal line shift of the secondary body orbit. They are also applied to the cases of equatorial artificial earth satellites and to the natural satellites of Mars and Jupiter. In particular, assuming that the external surface of the sun is that of an oblate spheroid, the results are also applied to the case of the planet Mercury. It is well known that the observed perihelion shift of this planet, which amounts to about 42 seconds of arc per century, cannot be explained by a pure Newtonian formulation. On the contrary, the relativistic theory claims to explain this shift quite accurately. Note, however, that in the relativistic theory, the oblateness of the sun is completely disregarded.

At present there is scanty experimental information about the difference between equatorial and polar radius of the sun. It is believed that this difference is definitely under a second of arc. R. H. Dicke[1*] of Princeton has initiated experiments to reveal the exact value of this difference.

This study is directed at developing a rigorous formulation which would give the Newtonian effect of the solar bulge on the perihelion of Mercury, once the difference between equatorial and polar radius of the sun is known.

1.2 BACKGROUND

The evaluation of the apsidal line shift in the secondary body orbit due to the oblateness of the central body was first made by Tisserand[2] (1889).

Burgatti[3] (1927) elaborated Tisserand's formulation after Armellini's proposal[4] (1923) to investigate the shift of the apsidal line in the orbit of the fifth satellite of Jupiter. Sconzo[5] (1937) also evaluated the flattening of the sun required to produce the observed perihelion shift of Mercury; he confirmed that it would be necessary to have a difference of one second of arc or larger between equatorial and polar radius of the sun in order to explain the observed phenomenon in the orbit of Mercury.

Later, Brouwer[6] (1946) rigorously integrated the equations of motion, using Delaunay's canonical variables under the assumption of a potential more general than that expressed by equation (1). He added, in fact, another term of the form β/r^5 to the right hand side of equation (1). After the integration, he explained the motion of the fifth satellite of Jupiter.

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* Superscript numbers within the text refer to references in the Bibliography.

In the recent past, several papers have been devoted to the study of equatorial orbits of artificial earth satellites. Anthony and Perko[7] (1961) presented an approximate analytical solution derived from the Poincaré method. Brenner[8] (1962) described the motion by means of a rapidly converging series of ordinary trigonometric functions. The argument of these functions, however, depends on the Jacobian elliptic function sn^{-1}. Vinti[9] (1962) treated the equatorial orbit as a particular case of his general solution known as the intermediary orbit for satellite astronomy. Beletski[10] (1963) used the method of osculating elements, replacing the time by the true anomaly. Anderson and Lorell[11] (1963) added relativistic perturbation terms to the Newtonian equations and found the variations in the classical orbital elements from which, in turn, the perihelion shift can be evaluated.

1.3 OUTLINE OF THE TECHNICAL APPROACH

The scope of the present task, according to the objective described above in Subsection 1.1, is to establish analytical relationships between the orbital radius vector and true anomaly and the time.

The technical approach is subdivided into four sections. In the first two sections (Sections 2 and 3) the motion of the secondary body in the gravitational field described by equation (1) is discussed. In Section 2, certain mathematical and physical assumptions which facilitate the integration are made. This approach is similar to that used by many authors who have dealt with the two-body problem in the theory or relativity. (Refer to, for example, Chazy[12] and Droste[13].) This particular case is referred to as the Approximate Solution. This approach leads to the "revolving conic" concept first conceived by Newton. The apsidal line shift and the period of the orbit are also evaluated.

The analytical solution is presented in Section 3. The radius vector and the time are expressed in terms of the true anomaly by means of Weierstrass elliptic functions.

Section 4 is designated as The Inverse Problem. In it, the true anomaly is expressed as a function of the radius vector by means of Legendre's elliptic integral of the first kind. Gaussian hypergeometric series and Legendre's functions with fractionary index are also used for the evaluation of the periods of the Weierstrass γ - function. As a by-product, a new set of formulas has been derived, one of which expresses the incomplete elliptic integral F in terms of Legendre's polynomials.

Finally, in Section 5, the results obtained in the analytical solution and in the inverse problem are used to evaluate the shift of the orbital apsidal line at each revolution of the secondary body. The amount of this shift is in agreement with that observed in close earth-satellite orbits, as well as in the orbits of the fifth satellite of Jupiter and of the two Martian satellites Phobos and Deimos.

In the case of Mercury, the perihelion shift of this planet can be explained, partially, with an assumed flattening of the Sun.

SECTION 2

APPROXIMATE SOLUTION

2.1 BASIC EQUATIONS

According to the force function expressed by equation (1), the motion of the secondary body is subject to the central acceleration

$$f(r) = -\frac{\partial U}{\partial r} \tag{4}$$

The center of the primary body is taken as the origin of a polar coordinate system. Let r and v, respectively, be the radius vector and the true anomaly. Then, the Newton-Binet formula provides the differential equation of the orbit

$$f(r) = \frac{h^2}{r^2}\left[\frac{1}{r} + \frac{d^2\left(\frac{1}{r}\right)}{dv^2}\right] , \tag{5}$$

where h is twice the areal velocity.

Now, combining (4) and (5) by means of (1), and using the law of areas with respect to the origin, one obtains

$$\frac{d^2\rho}{dv^2} + \rho = A + B\rho^2 , \tag{6}$$

$$\frac{1}{\rho^2}\frac{dv}{dt} = h , \tag{7}$$

where

$$\rho = \frac{1}{r} \quad , \quad A = \frac{\mu}{h^2} \quad , \quad B = \frac{3}{2} \frac{\mu}{h^2} J \ . \tag{8}$$

Equations (6) and (7) are the basic equations of the problem.

2.2 THE REVOLVING CONIC AS AN APPROXIMATION SOLUTION

Given a conic

$$r = \frac{p}{1 + e \cos v} \tag{9}$$

where p is the semi-latus rectum and e the eccentricity, any other orbit of the form

$$r = \frac{p}{1 + e \cos \frac{v}{\nu}} \quad , \tag{10}$$

where ν is a positive constant, is called a conic transformed by a proportional true anomaly.

Under certain assumptions, the orbit defined by (6) resembles the curve represented by equation (10). In fact, consider ρ varying in a neighborhood of a fixed ρ_o. Thus,

$$\rho^2 = \rho_o^2 + 2(\rho - \rho_o)\rho_o + (\rho - \rho_o)^2 \tag{11}$$

and disregarding terms of the order $(\rho - \rho_o)^2$, make the following approximation

$$\rho^2 = -\rho_o^2 + 2\rho\rho_o \tag{12}$$

Substituting (12) into (6) yields

$$\frac{d^2\rho}{dv^2} = -(1 - 2B\rho_o)\,\rho + (A - B\rho_o^2). \tag{13}$$

If the restriction is made that

$$1 - 2B\rho_o > 0 \tag{14}$$

and define

$$p = \frac{1 - 2B\rho_o}{A - B\rho_o^2}, \tag{15}$$

$$\nu^2 = \frac{1}{p(A - B\rho_o^2)}, \tag{16}$$

equation (13) becomes

$$\frac{d^2\rho}{dv^2} = -\frac{1}{\nu^2}\left(\rho - \frac{1}{p}\right). \tag{17}$$

Integrating (17), one obtains

$$p\rho = 1 + e\cos\frac{v - v_o}{\nu}, \tag{18}$$

where v_o and e are two arbitrary constants. By a convenient choice of polar axis, let $v_o = 0$, and (18) takes the form of (10). Thus, in the neighborhood of any ρ_o satisfying (14), the trajectory characterized by (6) is a conic transformed by a proportional true anomaly.

The curve (10) first conceived by Newton is sometimes called a revolving conic. Newton[14] demonstrated that the result expressed by equation (18) is valid without restrictions when the acceleration f (r) contains two terms, one inversely proportional to the square and the other to the cube of the radius vector.

2.3 THE APSIDAL ANGLE

The apsidal angle, or angle between the minimum and maximum values of r_1 is equal to π for the conic (9).

It shall now be shown, that for a revolving conic, the apsidal angle is $\nu\pi$. In fact, the roots of

$$\frac{d\rho}{dv} = 0 \tag{19}$$

are $v = 0, \nu\pi, 2\nu\pi, \ldots$; further,

$$\frac{d^2\rho}{dv^2} = -\frac{e}{p\nu^2}\cos\frac{v}{\nu} .$$

Since $\rho_o = \frac{1}{r_o}$ should be less than unity, and J is a small quantity such that $\frac{3}{2} J < 1$, one obtains

$$A - B\rho_o^2 > 0 .$$

Thus, because of (14), $p > 0$, and it is easy to check that $e > 0$. Hence, the only roots of (19) for which $\frac{d^2\rho}{dv^2} < 0$ are $v = 0, 2\nu\pi, \ldots$; and the roots of (19) for which $\frac{d^2\rho}{dv^2} > 0$ are $v = \nu\pi, 3\nu\pi, \ldots$. Therefore, a minimum r is at $v = 0$ and a maximum is at $v = \nu\pi$, and the apsidal angle is $\nu\pi$.

Define

$$\delta = (\nu - 1)\,\pi \tag{20}$$

and call $2\,\delta$ the apsidal line shift during a revolution.

Equation (10) can now be rewritten as

$$r = \frac{p}{1 + e\cos\left(\frac{\pi}{\pi + \delta}\,v\right)} \quad . \tag{21}$$

Next, evaluate 2δ, using the first integral of equation (6) and making certain assumptions. It will be observed that this value of $2\,\delta$ conforms to the definition (20) and to the value of ν given by equation (16).

The first integral of (6) is

$$\left(\frac{d\rho}{dv}\right)^2 = \frac{2}{3}\,B\,\rho^3 - \rho^2 + 2\,A\,\rho + C\ , \tag{22}$$

where C is an arbitrary constant.

Equation (22), except for the meaning of the coefficients A and B, is formally identical with the equation of motion encountered in the relativistic theory of the two-body problem by Schwarzschild[15], Eddington[16], DeSitter[17], Kopff[18], and many others.

Noting that $\frac{2}{3}$ B is a small quantity, the major assumption is that the orbit is a low-eccentricity ellipse locally.

Denoting by $P^{(3)}(\rho)$, the third degree polynomial in the right-hand member of equation (22)

$$P^{(3)}(\rho) = \frac{2}{3}\,B\,\rho^3 - \rho^2 + 2\,A\,\rho + C\ , \tag{23}$$

on the basis of this assumption, equation

$$P^{(3)}(\rho) = 0$$

has two almost equal positive roots ρ_2 and ρ_3. Since e is close to zero, for a given $\rho_o = \frac{1}{r_o}$, set

$$\rho_2 = \frac{1}{r_o(1+e)}, \quad \rho_3 = \frac{1}{r_o(1-e)},$$

The third real root ρ_1 of $P^{(3)}(\rho) = 0$ is also positive and much greater than ρ_2 and ρ_3, because

$$\rho_1 + \rho_2 + \rho_3 = \frac{3}{2B}$$

and

$$\rho_2 + \rho_3 = \frac{2}{r_o(1-e^2)} .$$

Note also that $P^{(3)}(\rho) = \frac{2}{3} B (\rho - \rho_2)(\rho_3 - \rho)(\rho_1 - \rho)$ is always positive for $\rho_2 < \rho < \rho_3$.

Then, equation (22) may be written as

$$\frac{d\rho}{dv} = \sqrt{P^{(3)}(\rho)} . \tag{24}$$

This equation is often encountered in mathematical analysis and in mechanical problems. It was first investigated by Weierstrass[19]. The solution of (24) is a real periodic function $\rho(v)$ with period

$$P = 2 \int_{\rho_2}^{\rho_3} \frac{d\rho}{\sqrt{P^{(3)}(\rho)}} \tag{25}$$

In order to compute P, change the integration variable by putting

$$\rho = \frac{1 + e \cos \theta}{r_o (1 - e^2)} \tag{26}$$

and obtain

$$P = 2 \int_0^{\pi} \frac{d\theta}{\sqrt{1 - \frac{2B}{r_o (1 - e^2)} - \frac{2}{3} B \frac{e \cos \theta}{r_o (1 - e^2)}}} \tag{27}$$

Now, since B is a small quantity, retaining only the first power of B, equation (27) yields

$$P = 2 \int_0^{\pi} \left[1 + \frac{B}{r_o (1 - e^2)} + \frac{1}{3} B \frac{e \cos \theta}{r_o (1 - e^2)} \right] d\theta$$

which is easily integrated to give

$$P = 2 \pi \left[1 + \frac{B}{r_o (1 - e^2)} \right] \quad . \tag{28}$$

Therefore,

$$2 \delta = \frac{2 \pi B}{r_o (1 - e^2)} \tag{29}$$

On the other hand, from (16), first deduce

$$\nu - 1 = B\rho_o$$

and taking into account the definition (20), it is obtained that

$$2\delta = 2\pi B\rho_o$$

which coincides with (29), when $e = 0$.

2.4 ANOMALISTIC PERIOD

Even though this analysis has been local, one can define that the time for a complete revolution (anomalistic period) T as the time elapsed between two consecutive points of the trajectory for which ρ and $\frac{d\rho}{dv}$ are equal in absolute value and signs.

In order to determine this period T, obtain from equations (7) and (22)

$$dt = \frac{1}{h\rho^2} \frac{d\rho}{\sqrt{P^{(3)}(\rho)}} .$$

Hence, with the same approximation as in Subsection 2.3

$$T = \frac{r_o (1 - e^2)^2}{h} \int_0^{2\pi} \left[1 + \frac{B}{r_o (1 - e^2)} + \frac{B\, e \cos\theta}{3\, r_o (1 - e^2)} \right] \frac{d\theta}{(1 + e\cos\theta)^2} . \quad (30)$$

The integration of (30) yields

$$T = r_o^{3/2} \left(P - \frac{2}{3}\delta e^2 \right) .$$

Section 3

ANALYTIC SOLUTION

3.1 RADIUS VECTOR r AS A FUNCTION OF TRUE ANOMALY v.

The integral of equation (22), (which is the first integral of (6)), can be found directly.

After changing the variables as follows

$$v = \sqrt{2}\,\theta \quad , \quad \frac{1}{3} B \rho = s + \frac{1}{6} \, , \tag{31}$$

equation (22) becomes

$$\left(\frac{ds}{d\theta}\right)^2 = P^{(3)}(s) \, , \tag{32}$$

where

$$P^{(3)}(s) = 4 s^3 - g_2 s - g_3$$

$$g_2 = \frac{1}{3} (1 - 4 AB)$$

$$g_3 = \frac{1}{27} (1 - 6 AB) - \frac{2}{9} B^2 C.$$

Again assume, adjusting the value of the integration constant C conveniently, that the roots s_1, s_2, s_3 of $P^{(3)}(s) = 0$ are real and satisfy

$$s_2 < s_3 < s_1 \quad .$$

Since

$$s_1 + s_2 + s_3 = 0 \quad ,$$

it is deduced that s_2 is negative and s_1 positive. Write

$$P^{(3)}(s) = 4\,(s - s_1)\,(s - s_2)\,(s - s_3) \tag{33}$$

and obtain

$$P^{(3)}(s) < 0 \quad \text{for} \quad -\infty < s < s_2 \quad \text{and} \quad s_3 < s < s_1 \ ,$$

$$P^{(3)}(s) > 0 \quad \text{for} \quad s_2 < s < s_3 \quad \text{and} \quad s_1 < s < \infty \ .$$

As is well known, the general solution of (32) is the doubly-periodic Weierstrass γ - function (more usually known as the P-function)

$$s = \gamma\,(\theta + L,\, \omega_1,\, \omega_2) \ ,$$

where L is an arbitrary constant and the two periods $2\omega_1$ and $2\omega_2$ are

$$2\omega_1 = 2\int_{s_2}^{s_3} \frac{ds}{\sqrt{P^{(3)}(s)}} \quad , \tag{34}$$

$$2\omega_2 = 2\int_{s_3}^{s_1} \frac{ds}{\sqrt{P^{(3)}(s)}} \quad . \tag{35}$$

The constant L can be determined with the condition that

$$s_2 = \gamma\,(0 + L,\, \omega_1,\, \omega_2)$$

which yields

$$L = \omega_2 \tag{36}$$

The evaluation of the two periods $2\omega_1$ and $2\omega_2$ will be made later, in Subsection 3.3. For the time being, consider them as well-determined constants, the first of which is real and the second purely imaginary. In conclusion, the solution of (32) may be written simply

$$s = \gamma(\theta + \omega_2) \quad . \tag{37}$$

Taking into account the relationships (31) and the first of (8), the radius vector r is found as a function of the true anomaly v

$$r = \frac{2B}{1 + 6\gamma\left(\frac{v}{\sqrt{2}} + \omega_2\right)} \quad . \tag{38}$$

The result of (38) can be expressed in terms of the Jacobi elliptic function to give

$$r = \frac{2B}{1 + 6s_2 + 6(s_3 - s_2)\mathrm{sn}^2 M\theta} \quad , \tag{39}$$

where

$$M = \sqrt{s_1 - s_2} \quad . \tag{40}$$

In fact, it is well known[20] that

$$\gamma(\theta + \omega_2) = s_2 + \frac{M^2}{\mathrm{sn}^2(M\theta + \Omega_2)} \quad ,$$

where

$$\Omega_2 = M\omega_2 \quad ;$$

but, if it is noted that

$$\mathrm{sn}^2(M\theta + \Omega_2) = \frac{1}{k^2 \mathrm{sn}^2 M\theta} \quad ,$$

where

$$k^2 = \frac{s_3 - s_2}{s_1 - s_2} \tag{41}$$

then, one obtains

$$\gamma(\theta + \omega_2) = s_2 + (s_3 - s_2)\mathrm{sn}^2 M\theta \tag{42}$$

and (39) follows immediately.

3.2 TIME t AS A FUNCTION OF TRUE ANOMALY v.

With the substitution $v = \sqrt{2}\theta$, the integration of equation (7) provides

$$t - t_o = \sqrt{\frac{2}{h}} \int_0^\theta r^2 d\theta \quad . \tag{43}$$

In order to perform this integration, use the following two results

$$\gamma'(u) \int \frac{dz}{\gamma(z) - \gamma(u)} = 2z\zeta(u) + ln\frac{\sigma(z - u)}{\sigma(z + u)} \quad , \tag{44}$$

$$\frac{1}{2}\gamma'^2(u) \int \frac{dz}{(\gamma(z) - \gamma(u))^2} = -\frac{1}{2}[\zeta(u + z) - \zeta(u - z)] - \zeta(u)$$

$$- z\gamma(u) - \frac{1}{2}\gamma''(u) \int \frac{dz}{\gamma(z) - \gamma(u)} \quad , \tag{45}$$

where as usual

$$\gamma(z) = -\frac{d}{dz}\zeta(z) = -\frac{d^2}{dz^2} ln\sigma(z)$$

and ζ and σ are Weierstrass functions.

The derivation of equations (44) and (45) is given in Appendix A. In the same appendix, the proof is also given that a z_o exists for which

$$\gamma(z_o) = -\frac{1}{6} \tag{46}$$

$$\gamma'(z_o) \neq 0 \tag{47}$$

Now, because of (38), (43) becomes

$$t - t_o = \frac{4\sqrt{2}\, B^2}{h} \int_0^{\theta} \frac{d\theta}{[1 + 6\,\gamma(\theta + \omega_2)]^2} \quad ,$$

and by letting

$$u = \theta + \omega_2 \quad ,$$

one obtains

$$t - t_o = \frac{B^2\sqrt{2}}{9h} \int_{\omega_2}^{\theta+\omega_2} \frac{du}{[\gamma(u) + \frac{1}{6}]^2}$$

Using (46) and then (44) and (45), a straightforward calculation yields

$$t - t_o = \frac{2B^2\sqrt{2}}{9h\gamma'^2(z_o)} \Bigg\{ -\frac{1}{2}\Big[\zeta(z_o + \theta + \omega_2) - \zeta(z_o + \omega_2)\Big]$$

$$+ \frac{1}{2}\Big[\zeta(z_o - \theta - \omega_2) - \zeta(z_o - \omega_2)\Big] - \frac{1}{6}\theta \tag{48}$$

$$+ \frac{1}{2}\frac{\gamma''(z_o)}{\gamma'(z_o)}\left[2\theta\zeta(z_o) + \ell n\frac{\sigma(\theta + \omega_2 - z_o)}{\sigma(\omega_2 - z_o)} + \ell n\frac{\sigma(\omega_2 + z_o)}{\sigma(\omega_2 + \theta + z_o)}\right]\Bigg\} .$$

3.3 THE PERIODS $2\omega_1$ AND $2\omega_2$

Referring to (34), put

$$s = s_2 + (s_3 - s_2)\sin^2\varphi \quad , \tag{49}$$

then, obtained is

$$\omega_1 = \frac{1}{\sqrt{s_1 - s_2}}\int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1 - k^2\sin^2\varphi}} = \frac{1}{\sqrt{s_1 - s_2}} F(k, \frac{\pi}{2}) \quad , \tag{50}$$

where k has been defined by (41) and $F(k\ \frac{\pi}{2})$ is Legendre's complete elliptic integral of the first kind. For this integral, the following series expansion may be used

$$F(k, \frac{\pi}{2}) = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}k\right)^2 + \left(\frac{1\cdot3}{2\cdot4}k^2\right)^2 + \left(\frac{1\cdot3\cdot5}{2\cdot3\cdot6}k^3\right)^2 + \ldots\right] \tag{51}$$

Similarly, using the same transformation (49), the following is obtained from (35)

$$\omega_2 = \frac{iF(k', \frac{\pi}{2})}{\sqrt{s_1 - s_2}} \quad , \tag{52}$$

where

$$k'^2 = 1 - k^2 = \frac{s_1 - s_3}{s_1 - s_2} \tag{53}$$

SECTION 4

THE INVERSE PROBLEM

4.1 TRUE ANOMALY v AS A FUNCTION OF RADIUS VECTOR r

Although the formulation given in the previous section is analytically rigorous, it is not practical for numerical computations. A more suitable expression for the equation of the trajectory can be derived from (39) if use is made of Legendre's elliptic integral[21].

Comparing (42) with (49), it is first deduced that

$$\mathrm{sn} M\theta = \sin \varphi \qquad (54)$$

and inverting

$$\theta = \frac{1}{M} F(k, \varphi) \quad , \qquad (55)$$

where F is the symbol for Legendre's elliptic integral of the first kind.

More explicitly, equations (55) may be written as follows

$$\theta(s) = \frac{1}{\sqrt{s_1 - s_2}} F\left(\left[\frac{s_3 - s_2}{s_1 - s_2}\right]^{1/2}, \text{arc sin}\left[\frac{s - s_2}{s_3 - s_2}\right]^{1/2}\right)$$

or, by virtue of (39) and the first of (31)

$$v(r) = \frac{\sqrt{2}}{\sqrt{s_1 - s_2}} F\left(\left[\frac{s_3 - s_2}{s_1 - s_2}\right]^{1/2}, \text{arc sin}\left[\frac{2B - r(1 + 6s_2)}{6r(s_3 - s_2)}\right]^{1/2}\right) \qquad (56)$$

4.2 USE OF LEGENDRE'S FUNCTIONS AND GAUSSIAN HYPERGEOMETRIC SERIES

Legendre's functions with fractionary index or Gaussian hypergeometric series can be used for the computation of ω_1 and ω_2, as well as for the computation of the associated quantities η_1 and η_2 defined by

$$\eta_1 = -\int_{s_2}^{s_3} \frac{sds}{\sqrt{P^{(3)}(s)}} , \tag{57}$$

$$\eta_2 = -\int_{s_3}^{s_1} \frac{sds}{\sqrt{P^{(3)}(s)}} , \tag{58}$$

and satisfying Legendre's relationship

$$\eta_1\omega_2 - \eta_2\omega_1 = \frac{i\pi}{2} . \tag{59}$$

Halphen[22], in treating the Gaussian problem[23] of the attraction of an elliptic ring, stated that the computation of ω_1 does not require the actual solution of the equation $P^{(3)}(s) = 0$.

This statement is first confirmed by expressing ω_1 in terms of Legendre's function with index $-\frac{1}{6}$.

Putting

$$s = \left(\frac{1}{3}g_2\right)^{1/2} \cos x ,$$

$$g_s = \left(\frac{1}{3}g_2\right)^{3/2} \cos 3\sigma ,$$

it is easily found that

$$P^{(3)}(s) = f(x) = \left(\frac{1}{3}g_2\right)^{3/2} (\cos 3x - \cos 3\sigma) .$$

The roots of f(x) = 0 are given by

$$3x = \pm 3\sigma + 2j\pi, \quad \text{(j is zero or an integer)}.$$

The three roots of f(x) = 0, which correspond to s_1, s_2 and s_3, are

$$x_1 = \sigma < \frac{1}{3}\pi \quad ,$$

$$x_2 = \frac{2\pi}{3} + \sigma \quad ,$$

$$x_3 = \frac{2\pi}{3} - \sigma \quad .$$

Then, it follows from (34) that

$$\omega_1 = -\left(\frac{1}{3}g_2\right)^{-1/4} \int_{x_2}^{x_3} \frac{\sin x \, dx}{\sqrt{\cos 3x - \cos 3\sigma}} \tag{60}$$

A remarkable relationship found by Dirichlet[24] and modified by Mehler[25] is recalled, by means of which the Legendre function $X_\nu(\cos 3\sigma)$ can be expressed in a definite integral form

$$X_\nu(\cos 3\sigma) = \frac{1}{\pi\sqrt{2}} \int_{-3\sigma}^{+3\sigma} \frac{e^{\left(\nu+\frac{1}{2}\right)i\phi}}{\sqrt{\cos\phi - \cos 3\sigma}} d\phi \quad . \tag{61}$$

Putting

$$\phi = 3x - 2\pi$$

the following is obtained

$$X_\nu(\cos 3\sigma) = \frac{3}{\pi\sqrt{2}} \int_{x_3}^{x_2} \frac{e^{\left(\nu+\frac{1}{2}\right)i\left(3x-2\pi\right)}}{\sqrt{\cos 3x - \cos 3\sigma}} dx \quad ,$$

from which it is deduced

$$\frac{\sqrt{2}\pi}{3} e^{(2\nu+1)\pi} \cdot X_{\nu}(\cos 3\sigma) = -\int_{x_2}^{x_3} \frac{e^{3\left(\nu+\frac{1}{2}\right)ix}}{\sqrt{\cos 3x - \cos 3\sigma}} dx \quad . \tag{62}$$

Equating coefficients of imaginary parts in (62),

$$\frac{\sqrt{2}\pi}{3} \sin (2\nu + 1)\pi \cdot X_{\nu}(\cos 3\sigma) = -\int_{x_2}^{x_3} \frac{\sin 3\left(\nu + \frac{1}{2}\right)x}{\sqrt{\cos 3x - \cos 3\sigma}} dx \quad , \tag{63}$$

and making $\nu = -\frac{1}{6}$,

$$\left(\frac{1}{6}\right)^{1/2} \pi X_{-\frac{1}{6}}(\cos 3\sigma) = -\int_{x_2}^{x_3} \frac{\sin x}{\sqrt{\cos 3x - \cos 3\sigma}} dx \quad . \tag{64}$$

By virtue of (60), it is concluded that

$$\omega_1 = \left(\frac{1}{12g_2}\right)^{1/4} \pi X_{-\frac{1}{6}}(\cos 3\sigma) \quad , \tag{65}$$

which has the advantage over equation (50) of not requiring knowledge of the roots s_1, s_2, and s_3.

Similarly, one may also find

$$\eta_1 = \frac{1}{12} \pi (12g_2)^{1/4} X_{\frac{1}{6}}(\cos 3\sigma) \tag{66}$$

and analogous expressions for ω_2 and η_2.

On the other hand, $X_\nu(\lambda)$ can be expressed in general by means of a Gaussian hypergeometric series[26]

$$X_\nu(\lambda) = F\left(-\nu, \nu+1, 1, \frac{1-\lambda}{2}\right) . \tag{67}$$

Putting again, $\nu = -\frac{1}{6}$ and $\lambda = \cos 3\sigma$, one obtains

$$X_{-\frac{1}{6}}(\cos 3\sigma) = F\left(\frac{1}{6}, \frac{5}{6}, 1, \sin^2 \frac{3\sigma}{2}\right) , \tag{68}$$

which inserted in (65) provides another expression for ω_1.

Similarly, putting $\nu = \frac{1}{6}$ one obtains

$$X_{\frac{1}{6}}(\cos 3\sigma) = F\left(-\frac{1}{6}, \frac{7}{6}, 1, \sin^2 \frac{3\sigma}{2}\right) , \tag{69}$$

which can be inserted in (66) to provide the expression for η_1 in terms of a Gaussian hypergeometric series.

By contrast, the expression for η_1 in terms of Legendre's complete elliptic integrals of first and second kind, $F\left(k, \frac{\pi}{2}\right)$ and $E\left(k, \frac{\pi}{2}\right)$, which may be obtained from (57) by using the transformation (49), is

$$\eta_1 = \sqrt{s_1 - s_2}\, E\left(k, \frac{\pi}{2}\right) - \frac{s_1}{\sqrt{s_1 - s_2}} F\left(k, \frac{\pi}{2}\right) , \tag{70}$$

where $F\left(k, \frac{\pi}{2}\right)$ has already been defined by (51), and

$$E\left(k, \frac{\pi}{2}\right) = \frac{\pi}{2}\left[1 - \left(\frac{1}{2}k\right)^2 - \frac{1}{3}\left(\frac{1\cdot 3}{2\cdot 4}k^2\right)^2 - \frac{1}{5}\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}k^3\right)^2 - \dots\right] . \tag{71}$$

4.3 DIGRESSION ON LEGENDRE'S FUNCTIONS WITH FRACTIONARY INDEX

It has been shown in the previous section how ω_1 and η_1 can be expressed in terms of Legendre's functions with index $-\frac{1}{6}$ and $\frac{1}{6}$, respectively. These fractionary indices can, however, be avoided if $X_\nu(\lambda)$ is expanded into a power series of $X_n(\lambda)$, where n is now an integer.

Fortunately, this expression exists already in the mathematical literature[26]

$$X_\nu(\lambda) = \frac{\sin \nu\pi}{\pi} \sum_{n=0}^{\infty} (-1)^n \left[\frac{1}{\nu - n} - \frac{1}{\nu + n + 1}\right] X_n(\lambda) \quad . \tag{72}$$

For $\nu = -\frac{1}{6}$ and $\lambda = \cos 3\sigma$, obtained is

$$X_{-\frac{1}{6}}(\cos 3\sigma) = \frac{18}{\pi} \sum_{n=0}^{\infty} (-1)^n \frac{2n + 1}{(6n + 1)(6n + 5)} X_n(\cos 3\sigma) \quad , \tag{73}$$

while for $\nu = \frac{1}{6}$,

$$X_{\frac{1}{6}}(\cos 3\sigma) = \frac{18}{\pi} \sum_{n=0}^{\infty} (-1)^{n+1} \frac{2n + 1}{(6n - 1)(6n + 7)} X_n(\cos 3\sigma) \quad . \tag{74}$$

It will be shown that the incomplete elliptic integral $F(k, \varphi)$ which appears in equation (55), can also be expanded into a series of Legendre polynomials. In fact, if

$$\tan \frac{1}{2}\varphi = \xi \quad , \tag{75}$$

then

$$F(k, \varphi) = 2\int_0^{\xi} \frac{d\xi}{\sqrt{1 + 2\lambda\xi^2 + \xi^4}} \quad , \tag{76}$$

where λ is now defined by

$$\lambda = 1 - 2k^2 \quad . \tag{77}$$

On the other hand, one can write

$$(1 + 2\lambda\xi + \xi^4)^{-1/2} = \sum_{n=0}^{\infty} (-1)^n X_n(\lambda) \xi^{2n} \tag{78}$$

which is uniformly convergent if $|\lambda| \leq a$, $|\xi| \leq b$ and a and b are two positive constants satisfying the condition $2ab^2 + b^4 < 1$.

Substituting (78) into (76), integrating term by term, and replacing the variable φ defined by (75),

$$F(k, \varphi) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} X_n(\lambda) \tan^{2n+1}\left(\frac{1}{2}\varphi\right) \quad . \tag{79}$$

In particular for $\varphi = \frac{\pi}{2}$, the expression for Legendre's complete elliptic integral is obtained

$$F\left(k, \frac{\pi}{2}\right) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1} X_n(\lambda) \quad . \tag{80}$$

Other interesting formulas may be deduced from (72), for instance, the following

$$X_{-\frac{1}{2}}(\lambda) + X_{\frac{1}{2}}(\lambda) = \frac{16}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n - 1)(2n + 1)(2n + 3)} X_n(\lambda) \quad . \tag{81}$$

Since, as it may be shown that

$$E\left(k, \frac{\pi}{2}\right) = \frac{\pi}{4}[X_{-\frac{1}{2}}(\lambda) + X_{\frac{1}{2}}(\lambda)] \quad , \tag{82}$$

by using (81) it follows

$$E\left(k, \frac{\pi}{2}\right) = 4\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n - 1)(2n + 1)(2n + 3)} X_n(\lambda) \quad , \tag{83}$$

which is the analog of (80).

The formulas derived in this section are believed to be new in the mathematical literature.

SECTION 5

NUMERICAL EVALUATIONS

In this section, it will be shown that the trajectory found analytically in Section 3 and defined by

$$r = \frac{2B}{1 + 6\gamma(\theta + \omega_2)} \tag{84}$$

has the character of a revolving conic. To this end, some assumptions must be made in order to find approximate formulas for the evaluation of $\gamma(\theta + \omega_2)$, the three roots s_1, s_2, s_3 of $P^3(s) = 0$, and the quantities ω_1 and η_1 involved in the computation of the apsidal shift per revolution 2δ.

5.1 EVALUATION OF $\gamma(\theta + \omega_2)$

The following series expansion[27] will be used

$$\gamma(\theta + \omega_2) = \frac{\eta_1}{\omega_1} - 2\left(\frac{\pi}{\omega_1}\right)^2 \sum_{m=1}^{\infty} \frac{mq^m}{1 - 2q^{2m}} \cos\frac{m\pi\theta}{\omega_1} \quad , \tag{85}$$

where

$$q = \exp\left\{i\pi\frac{\omega_2}{\omega_1}\right\} \quad . \tag{86}$$

In particular, for $\theta = 0$,

$$s_2 = \gamma(\omega_2) = -\frac{\eta_1}{\omega_1} - 2\left(\frac{\pi}{\omega_1}\right)^2 \sum_{m=1}^{\infty} \frac{mq^m}{1-2q^{2m}} \quad . \tag{87}$$

Now, noting that

$$s_3 = \gamma(\omega_1 + \omega_2) \quad ,$$

it is deduced from equation (84),

$$\text{for} \quad \theta = 0, \quad r(0) = \frac{2B}{1 + 6s_2} \quad ;$$

$$\text{for} \quad \theta = \omega_1, \quad r(\omega_1) = \frac{2B}{1 + 6s_3} \quad .$$

r(0) is evidently the maximum of r in the interval $(0, \omega_1)$ and $r(\omega_1)$ the minimum, comparable to $r_{max} = a(1 + e)$ and $r_{min} = a(1 - e)$ in an elliptic orbit.

Putting

$$\frac{1 + 6s_2}{1 + 6s_3} = \frac{1 - e}{1 + e} \tag{88}$$

where e is assumed to be a small quantity, the determination of s_1, s_2 and s_3 depends on the solution of the following system of equations

$$s_1 + s_2 + s_3 = 0$$

$$1 + 6s_2 = \frac{1 - e}{1 + e}(1 + 6s_3)$$

$$s_3 - s_2 = k^2(s_1 - s_2) \quad .$$

In terms of k^2 (assumed to be also a small quantity) and of ϵ defined as

$$\epsilon = \frac{3 - e}{2e} \tag{89}$$

the solution of the above system is

$$s_1 = \frac{1}{6}\frac{2 - k^2}{1 + k^2\epsilon}, \qquad s_2 = -\frac{1}{6}\frac{1 + k^2}{1 + k^2\epsilon}, \qquad s_3 = \frac{1}{6}\frac{2k^2 - 1}{1 + k^2\epsilon} \tag{90}$$

It is also deduced that

$$s_1 - s_2 = \frac{1}{2(1 + k^2\epsilon)} \tag{91}$$

Next, if it is observed that the difference between the true anomalies at the two radii $r(0)$ and $r(\omega_1)$ is very close to π, it can then be written that

$$\sqrt{2}\,\omega_1 = \pi + \delta, \tag{92}$$

where δ is a small quantity.

Now, using equation (50), the expression (51), and the expression of $(s_1 - s_2)^{-1/2} = \sqrt{2}\,(1 + k^2\epsilon)^{1/2}$, with $k^2\epsilon < 1$, from (92) it is deduced successively

$$\sqrt{2}\,\omega_1 = \sqrt{\frac{2}{s_1 - s_2}}\,F\left(k, \frac{\pi}{2}\right), \tag{93}$$

$$\frac{\delta}{\pi} = \frac{1}{2}\left(\epsilon + \frac{1}{2}\right)k^2 + \frac{1}{8}\left(-\epsilon^2 + \epsilon + \frac{9}{8}\right)k^4 + \frac{1}{16}\left(\epsilon^3 - \frac{\epsilon^2}{2} + \frac{9}{8}\epsilon + \frac{25}{16}\right)k^6 + \dots \tag{94}$$

and k^2 can be evaluated. In fact, inverting (94) and neglecting powers of δ higher than the second,

$$k^2 = \frac{2\delta}{\pi\left(\epsilon + \frac{1}{2}\right)}. \tag{95}$$

In this order of approximation it is deduced

from (92) $$\omega_1 = \frac{\pi}{\sqrt{2}}\left(1 + \frac{\delta}{\pi}\right) = \frac{\pi}{\sqrt{2}}\left[1 + \frac{1}{2}\left(\epsilon + \frac{1}{2}\right)k^2\right] \tag{96}$$

from (90)

$$s_1 = \frac{1}{3}\left[1 - \left(\epsilon + \frac{1}{2}\right)k^2\right] \tag{97}$$

$$s_2 = \frac{1}{6}\left[1 - \left(\epsilon - 1\right)k^2\right] \tag{98}$$

$$s_3 = \frac{1}{6}\left[1 - \left(\epsilon + 2\right)k^2\right] \tag{99}$$

from (91) $$s_1 - s_2 = \frac{1}{2}\left(1 - \epsilon k^2\right) \tag{100}$$

from (70) $$\eta_1 = \frac{\pi}{6\sqrt{2}}\left[1 - \frac{1}{2}\left(\epsilon + \frac{1}{2}\right)k^2\right] \tag{101}$$

from (101) and (96)

$$\frac{\eta_1}{\omega_1} = \frac{1}{2}s_1 \tag{102}$$

$$\left(\frac{\pi}{\omega_1}\right)^2 = 6s_1 \tag{103}$$

Equation (87) can now be written as follows

$$\sum \frac{mq^m}{1 - 2q^{2m}} = -\frac{s_2 + \frac{\eta_1}{\omega_1}}{2\left(\frac{\pi}{\omega_1}\right)^2} = \frac{k^2}{48s_1} , \tag{104}$$

from which one obtains

$$q = \frac{k^2}{48s_1} - 2\left(\frac{k^2}{48s_1}\right)^2 + 4\left(\frac{k^2}{48s_1}\right)^3 - \ldots \tag{105}$$

Assuming $k^2 << s_1$, the first term only in the right-hand member of (105) can be retained

$$q = \frac{k^2}{48s_1} \tag{106}$$

and consequently, the series expansion (85) becomes

$$\gamma(\theta + \omega_2) = -\frac{1}{2}s_1 - \frac{k^2}{4}\cos\frac{\pi\theta}{\omega_1} \quad . \tag{107}$$

5.2 REVOLVING CONIC CHARACTER OF THE ANALYTICAL SOLUTION

Inserting the approximate expression (107) into equation (84), one obtains

$$r = \frac{2B}{1 - 3s_1 - \frac{3}{2}k^2\cos\frac{\pi\theta}{\omega_1}} \quad ,$$

and taking into account (92) and (98),

$$r = \frac{\frac{2B}{k^2}}{\left(\epsilon + \frac{1}{2}\right) - \frac{3}{2}\cos\left(\frac{\pi}{\pi + \delta}\sqrt{2}\theta\right)} \quad . \tag{108}$$

On the other hand, from (95) is obtained

$$\frac{1}{k^2} = \frac{\pi\left(\epsilon + \frac{1}{2}\right)}{2\delta} \quad , \tag{109}$$

and from (89)

$$\epsilon + \frac{1}{2} = \frac{3}{2e} \quad , \tag{110}$$

and it is also

$$v = \sqrt{2\theta} \tag{111}$$

Substituting (109) through (111) into (108), one concludes that

$$r = \frac{\frac{B\pi}{\delta}}{1 - e \cos\left(\frac{\pi}{\pi + \delta} v\right)} \tag{112}$$

which is a revolving conic of type (21). In (112), the change in the sign of the cosine function is explained by the fact that θ, hence v also, is reckoned from the radius vector maximum, r(0), which corresponds to $\theta = 0$.

Note that the validity of (112) is subject to the following conditions: e is a small quantity (implying that s_2 and s_3 are nearly equal); $k^2 << s_1$ (implying that δ is a small quantity), and finally $k^2 \epsilon < 1$.

5.3 SOME EXAMPLES

The semi-latus rectum p of the revolving conic (112) is

$$p = \frac{B\pi}{\delta}$$

Recalling the definition of B (see the third formula of (8)), it is deduced for the apsidal line shift per revolution

$$2\delta = \frac{3\mu J\pi}{h^2 p} \tag{113}$$

Following are some applications of (113) to some concrete cases:

a. Artificial Satellite—For a hypothetical satellite of low eccentricity orbit in the earth's equatorial plane and with a perigee distance of $r_o = 1.21$ (earth radii), assuming $J = 1082 \times 10^{-6}$, and putting $p = r_o$, the numerical evaluation of (113) provides

$$2\delta = 5.14 \text{ degrees per day.}$$

For comparison, applying the DOCET formulation[28] for the secular perturbation in the argument of perigee

$$\dot{\omega} = a^{-3/2} 4.98 \frac{I_4}{p^2} \text{ degrees/day}$$

where $I_4 = 3 \cos^2 i - 1 = 2$ (for $i = 0$), making again $p = a = r_o$, one would obtain

$$\dot{\omega} = 5.13 \text{ degrees/day.}$$

b. Satellites of Mars—Assumed value of J for Mars[29]: $J = 21 \times 10^{-4}$.

Satellite	r_o (Mars' Radii)	Eccentricity	2δ Degrees/Year Computed	2δ Degrees/Year Observed[30]
Phobos	2.8	0.021	159.5	158.5
Deimos	6.9	0.003	6.7	6.5

c. 5th Satellite of Jupiter—Assumed value of J for Jupiter[31]: $J = 1.49 \times 10^{-2}$.

r_o (Jupiter's Radii)	Eccentricity	2δ Degrees/Year Computed	2δ Degrees/Year Observed[6]
2.54	0.003	907°.5	917°.4

d. Planet Mercury—Assumed value of J for the Sun: $J = 0.001$, corresponding to a difference of one second of arc between equatorial and polar radius.

r_o (In Solar Unit)	Eccentricity	2δ Sec. of Arc/Century Computed	2δ Sec. of Arc/Century Observed[39]
0.387	0.21	14.3	42.2

In conclusion, it can be said that, except for Mercury, formula (113) provides values of the apsidal line shift which are in agreement with observation.

For the planet Mercury, the 2δ computed represents only 34 percent of the observed value. Even if the flattening of the Sun is smaller than that assumed in this computation, we still believe that the oblateness of the Sun could account for part of the revolving conic character of the orbit of Mercury.

Appendix A

INTEGRAL FORMULAS FOR THE γ-FUNCTION

Begin with the following well-known identity:

$$\gamma(z) - \gamma(u) = \frac{-\sigma(z-u)\sigma(z+u)}{\sigma^2(z)\sigma^2(u)} \tag{1}$$

Proposition 1:

$$\gamma'(u)\int \frac{dz}{\gamma(z) - \gamma(u)} = 2z\zeta(u) + \ln\sigma(z-u) - \ln\sigma(z+u) \quad .$$

Proof: Taking the logarithmic derivative of (1) with respect to u, the left-hand side becomes

$$\frac{-\gamma'(u)}{\gamma(z) - \gamma(u)} \quad , \tag{2}$$

and the right hand side is

$$\frac{-\frac{1}{\sigma^2(z)}\frac{d}{du}\left(\left[\sigma(z-u)\sigma(z+u)\right]\sigma^{-2}(u)\right)}{-\frac{\sigma(z-u)\ (z+u)}{\sigma^2(z)\sigma^2(u)}} = \frac{\sigma^2(u)}{\sigma(z-u)\sigma(z+u)}\left[-2\sigma(z-u)\sigma(z+u)\sigma^{-3}(u)\frac{d\sigma(u)}{du}\right.$$

$$\left. + \sigma^{-2}(u)\frac{d}{du}\left\{\sigma(z-u)\sigma(z+u)\right\}\right] \tag{3}$$

$$= -\frac{2}{\sigma(u)}\frac{d\sigma(u)}{du} + \frac{1}{\sigma(z+u)}\frac{d\sigma(z+u)}{du} + \frac{1}{\sigma(z-u)}\frac{d\sigma(z-u)}{du} \quad .$$

Using the definition of ζ and σ, (2) and (3) are equated to get

$$\frac{\gamma'(u)}{\gamma(z) - \gamma(u)} = 2\zeta(u) - \zeta(z+u) + \zeta(z-u) \quad . \tag{4}$$

Integrating (4) and again utilizing the definition of ζ and σ, the proposition follows qed

Proposition 2:

$$\frac{1}{2}(\gamma'(z))^2 \int \frac{du}{[\gamma(u) - \gamma(z)]^2} = -\frac{1}{2}\zeta(z+u) + \frac{1}{2}\zeta(z-u) - \zeta(z) - u\gamma(z)$$

$$-\frac{1}{2}\gamma'(z)\int \frac{du}{\gamma(u) - \gamma(z)} \quad .$$

Proof: Taking the logarithmic derivative of (1) with respect to z, one obtains

$$\frac{\gamma'(z)}{\gamma(z) - \gamma(u)} = \zeta(z+u) + \zeta(z-u) - 2\zeta(z) \tag{5}$$

since

$$\frac{-\frac{1}{\sigma^2(u)}\frac{d}{dz}\left(\left[\sigma(z-u)\sigma(z+u)\right]\sigma^{-2}(z)\right)}{-\frac{\sigma(z-u)\sigma(z+u)}{\sigma^2(z)\sigma^2(u)}} = \frac{\sigma^2(z)}{\sigma(z-u)\sigma(z+u)}\left(-2\sigma(z-u)\sigma(z+u)\sigma^{-3}(z)\frac{d\sigma(z)}{dz}\right.$$

$$\left. + \sigma^{-2}(z)\frac{d}{dz}\left[\sigma(z-u)\sigma(z+u)\right]\right)$$

$$= \frac{-2}{\sigma(z)}\frac{d\sigma(z)}{dz} + \frac{1}{\sigma(z+u)}\frac{d\sigma(z+u)}{dz} + \frac{1}{\sigma(z-u)}\frac{d\sigma(z-u)}{dz} \quad .$$

Using the definitions of ζ and σ, (5) is obtained immediately. Subtracting (4) from (5), one gets

$$\frac{\gamma'(z) - \gamma'(u)}{\gamma(z) - \gamma(u)} = \zeta(z + u) + \zeta(z - u) - 2\zeta(z) - 2\zeta(u) + \zeta(z + u) - \zeta(z - u),$$

implying

$$\frac{\gamma'(z) - \gamma'(u)}{2[\gamma(z) - \gamma(u)]} = \zeta(z + u) - \zeta(z) - \zeta(u) \tag{6}$$

Differentiating (6) with respect to z yields

$$\gamma(z + u) = \gamma(z) - \frac{1}{2}\frac{d}{dz}\left[\frac{\gamma'(z) - \gamma'(u)}{\gamma(z) - \gamma(u)}\right] .$$

Thus,

$$\gamma(z + u) = \gamma(z) - \frac{1}{2}\left(\frac{\gamma''(z)}{\gamma(z) - \gamma(u)} - \frac{\gamma'(z)[\gamma'(z) - \gamma'(u)]}{[\gamma(z) - \gamma(u)]^2}\right) .$$

Rearranging yields

$$\frac{1}{2}\frac{(\gamma'(z))^2}{[\gamma(z) - \gamma(u)]^2} = -\frac{1}{2}\frac{\gamma''(z)}{\gamma(u) - \gamma(z)} + \gamma(z + u) - \gamma(z) + \frac{1}{2}\frac{\gamma'(z)\gamma'(u)}{[\gamma(z) - \gamma(u)]^2} . \tag{7}$$

Taking the antiderivative of (7), with respect to u, one obtains

$$\frac{1}{2}(\gamma'(z))^2\int\frac{du}{[\gamma(u) - \gamma(z)]^2} = -\frac{1}{2}\gamma''(z)\int\frac{du}{\gamma(u) - \gamma(z)} - \zeta(z + u) - u\gamma(z)$$

$$+ \frac{1}{2}\gamma'(z)\int\frac{\gamma'(u)du}{[\gamma(z) - \gamma(u)]^2} . \tag{8}$$

Letting $v = \gamma(z) - \gamma(u)$ implies

$$\int \frac{\gamma'(u)du}{[\gamma(z) - \gamma(u)]^2} = -\int \frac{dv}{v^2} = \frac{1}{\gamma(z) - \gamma(u)} .$$

Hence, because of (5),

$$\frac{1}{2}\gamma'(z)\int \frac{\gamma'(u)du}{[\gamma(z) - \gamma(u)]^2} = \frac{1}{2}\zeta(z + u) + \frac{1}{2}\zeta(z - u) - \zeta(z) \quad (9)$$

Substituting (9) into (8), gives the desired result. qed

Proposition 3:

(a) Given any complex number c, there exists a complex number z_o so that $\gamma(z_o) = c$.

(b) If $P^{(3)}(c) \neq 0$, then $\gamma'(z_o) \neq 0$.

Proof:

(a) For any elliptic function, the number of roots in an arbitrary period-parallelogram is equal to the number of its poles in this parallelogram. γ is, of course, elliptic with double poles at the period points. Also, $\gamma(z) - c$ is clearly elliptic with double poles. Thus, $\gamma(z) - c = 0$ has two roots.

(b) Clear, for if $\gamma'(z_o) = 0$, then $4\gamma 3(z_o) - g_2\gamma(z_o) - g_3 = 0$ implying that c is a zero of $P^{(3)}(z)$. qed

Appendix B

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