Non-uniform sampling and spiral MRI reconstruction

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ABSTRACT

There is a natural formulation of the Classical Uniform Sampling Theorem in the setting of Euclidean space, and in the context of lattices, as sampling sets, and unit cells E, e.g., the Voronoi cell. For sampling at the Nyquist rate, the sampling function corresponds to the sinc function, and it is an integral over E. The set E is a tile for Euclidean space under translation by elements of the reciprocal lattice.

We have a constructive, implementable non-uniform sampling theorem in the context of uniformly discrete sampling sets and sets E, corresponding to the unit cells of the uniform sampling result. The set E has the property that the translates by the sampling set of the polar set of E is a covering of Euclidean space. The theorem depends on the theory of frames, and can be viewed as a modest generalization of a theorem of Beurling.

The application herein is to fast magnetic resonance imaging (MRI) by direct signal reconstruction from spectral data on spirals.

Keywords: Non-uniform sampling, Fourier frames, Beurling density, MRI

1. INTRODUCTION

1.1. Background and Outline

Let R^d be d-dimensional Euclidean space, and let $\widehat{R}^d$ denote R^d when it is considered as the domain the Fourier transforms of signals defined on R^d. The *Paley-Wiener* space PW_E is

$$PW_E = \left\{ \varphi \in L^2(\widehat{R}^d) : \operatorname{supp} \varphi^\vee \subseteq E \right\},$$

where $E \subseteq R^d$ is closed, $L^2(\widehat{R}^d)$ is the space of finite energy signals φ on $\widehat{R}^d$, i.e.,

$$\|\varphi\|_{L^2(\widehat{R}^d)} = \left(\int_{\widehat{R}^d} |\varphi(\gamma)|^2 \, d\gamma \right)^{1/2} < \infty,$$

$\varphi^\vee$ is the inverse Fourier transform of φ defined as

$$\varphi^\vee(x) = \int_{\widehat{R}^d} \varphi(\gamma) e^{2\pi i <x,\gamma>} d\gamma,$$

and $\operatorname{supp} \varphi^\vee$ denotes the support of $\varphi^\vee$.

Beurling[1] proved the following theorem for the case that E is the closed ball $\overline{B(0,R)} \subseteq R^d$ centered at $0 \in R^d$ and with radius R.

Theorem 1.1 (Beurling's Theorem). *Let $\Lambda \subseteq \widehat{R}^d$ be uniformly discrete, and define*

$$\rho = \rho(\Lambda) = \sup_{\zeta \in \widehat{R}^d} \operatorname{dist}(\zeta, \Lambda),$$

where $\operatorname{dist}(\zeta, \Lambda)$ is the Euclidean distance between the point ζ and the set Λ. If $R\rho < 1/4$, then Λ is a Fourier frame for $PW_{\overline{B(0,R)}}$.

Recall that a set Λ is *uniformly discrete* if there is $r > 0$ such that

$$\forall \lambda, \gamma \in \Lambda, \quad |\lambda - \gamma| \geq r,$$

where $|\lambda-\gamma|$ is the Euclidean distance between λ and γ. Fourier frames will be defined in Section 3, but a consequence of their definition in the assertion of Beurling's Theorem is that every finite energy signal f defined on E has the representation

$$f(x) = \sum_{\lambda \in \Lambda} a_\lambda(f) e^{2\pi i <x,\lambda>} \tag{1}$$

in L^2-norm on E, where $\sum_{\lambda\in\Lambda} |a_\lambda(f)|^2 < \infty$. Beurling used the term "set of sampling" instead of "Fourier frame".

We shall reformulate Beurling's Theorem in Theorem 7.2, in terms of a covering condition; and we shall refer to Theorem 7.2 as the *Beurling Covering Theorem*. Beurling's Theorem stated above becomes an obvious corollary. Our interest in this topic goes back to a problem about fast magnetic resonance imaging (MRI) posed to us by Dennis Healy. Besides the present approach to non-uniform sampling using the power of Beurling's results, there are other approaches[2].[3]

In Section 2, we shall discuss the Classical Uniform Sampling Theorem for perspective with the result in Section 7 and for ultimately comparing lattice and tiling ideas with analogous notions from non-uniform sampling. Fourier frames are introduced Section 3 in order to have a convenient structure in which to develop non-uniform sampling formulas. Section 4 is devoted to a discussion of density criteria in the setting of completeness results, and such criteria are essential for understanding effective signal reconstruction from non-uniformly spaced sampled values. We state the fundamental characterization of Fourier frames in terms of density in Section 5.

The notion of balayage is defined in Section 6; and Beurling's theorem (1959),[4] describing the fundamental relationship between balayage and Fourier frames, is stated. This relationship is an essential component of the Beurling Covering Theorem in Section 7. We also take the opportunity to make some preliminary comments about related on-going work relating dimensionality, tilings, lattices, and coverings.

In Section 8 we shall use the Beurling Covering Theorem to solve a mathematical version of the aforementioned problem concerning fast MRI.[5] Basically, spectral (Fourier transform) data of an unknown signal f is given on a discrete subset of finitely many interleaving spirals contained in $\widehat{R}^2$; and the problem is to extract the original signal $f \in L^2(R^2)$ from this data whenever possible. Our solution is contained in Examples 2 and 3. In these examples, we shall provide a proof and algorithm for characterizing and constructing a uniformly discrete spectral subset $\Lambda \subseteq \widehat{R}^2$ on given interleaving spirals with the property that Λ is a Fourier frame in the sense of (1).

Complete proofs of the aforementioned results, along with an analysis and history of the various relevant concepts of density, will appear in a forthcoming research tutorial,[6] which also contains a complete bibliography.

1.2. Notation

We shall use standard notation from harmonic analysis[7].[8]

Further, $M(\widehat{R}^d)$ is the convolution algebra of bounded Radon measures on $\widehat{R}^d$; and, if $\Lambda \subseteq \widehat{R}^d$ is closed, then $M(\Lambda)$ denotes the closed subspace of $M(\widehat{R})^d$ consisting of those elements supported by Λ. Integration over Euclidean space will be denoted by "$\int$"; and the Fourier transform $\widehat{f}$ of f defined on R^d is formally given by

$$\varphi(\gamma) = \widehat{f}(\gamma) = \int f(x) e^{-2\pi i <x,\gamma>}\, dx,$$

where $\gamma \in \widehat{R}^d$. $A(R^d)$ is the space of absolutely convergent inverse Fourier transforms on R^d, and $A'(R^d)$, its dual space when $A(R^d)$ is normed by

$$\|\varphi^\vee\|_{A(R^d)} = \|\varphi\|_{L^1(R^d)},$$

is the space of *pseudo-measures* on R^d. $\mathcal{B}_b(E)$ is the space of bounded continuous functions φ on $\widehat{R}^d$ for which $\operatorname{supp}\varphi^\vee \subseteq E$, where E is closed and $\varphi^\vee$ is the distributional inverse Fourier transform of φ. Clearly, $\mathcal{B}_b(E)^\wedge \subseteq A'(R^d)$. Finally, we write $e_\lambda(x) = e^{2\pi i <x,\gamma>}$ for $x \in R^d$ and $\gamma \in \widehat{R}^d$.

2. CLASSICAL UNIFORM SAMPLING THEOREM

We begin by stating the Classical Uniform Sampling Theorem on R.

Theorem 2.1 (Classical Uniform Sampling Theorem).

Let $T, \Omega > 0$ satisfy the condition that $0 < 2T\Omega \leq 1$, and let $s \in PW_{1/(2T)}$ satisfy the condition that $\widehat{s}$ is a bounded function on R which equals 1 on $[-\Omega, \Omega]$. Then

$$\forall f \in PW_{\Omega}, \quad f = T \sum f(nT) \tau_{nT} s, \tag{2}$$

where the convergence of the sum is in L^2-norm and uniformly on R, and where $(\tau_{nT}s)(t)$ designates the translation $s(t - nT)$.

It should be emphasized that there are non-bandlimited versions of Theorem 2.1, which can be interpreted as modeling aliasing, including such decompositions in terms of Gabor and wavelet systems.[2]

The proof of Theorem 2.1 is elementary, and it depends essentially on periodization.[7] The extension to R^d requires the following definition.

Definition 2.2 (Lattices).

A *lattice* $H \subseteq R^d$ is the image of Z^d under some nonsingular linear transformation, i.e., H is a discrete subgroup of Euclidean space R^d consisting of integral linear combinations of elements $v_1, v_2, \ldots, v_d \in R^d$, which form a basis for R^d. The *reciprocal lattice* $\Lambda \subseteq \widehat{R}^d$ of H is the lattice consisting of all $\gamma \in \widehat{R}^d$ with the property that the inner product $\langle x, \gamma \rangle$ is an integer n_x for each $x \in H$.

A *unit cell* or *fundamental region* of a lattice $\Lambda \subseteq \widehat{R}^d$ is a set $U \subseteq \widehat{R}^d$, not necessarily connected, such that $\mathcal{T} = \{\gamma + U : \gamma \in \Lambda\}$ is a *tiling* or *partition* of $\widehat{R}^d$, i.e., the elements of $\mathcal{T}$ are pairwise disjoint and

$$\bigcup_{\gamma \in \Lambda} (\gamma + U) = \widehat{R}^d. \tag{3}$$

There are many possible choices for the unit cell of a given lattice. For example, the Voronoi cell or Brillouin zone is the unit cell of Λ defined as the set of all points in $\widehat{R}^d$ closer to the origin than to any other lattice point.

Theorem 2.3 (A d-dimensional Uniform Sampling Theorem).

Let $H \subseteq R^d$ be a lattice and let $U \subseteq \widehat{R}^d$ be a unit cell of the reciprocal lattice Λ. Define the sampling function

$$\forall x \in R^d, \quad s_U(x) = \frac{1}{|U|} \int_U e^{2\pi i \langle x, \gamma \rangle} \, d\gamma,$$

where $|U|$ is the Lebesgue measure of U, and let $f \in L^2(R^d)$ have the property that $\widehat{f} = 0$ a.e. off of U.

a. There is a continuous function f_c on R^d such that $f = f_c$ a.e.

b. If f is continuous on R^d, then

$$f = \sum_{y \in H} f(y) \tau_y s_U,$$

where the convergence of the sum is in L^2-norm and uniformly on R^d.

Note that Theorem 2.3, as a d-dimensional version of Theorem 2.1, is only stated for the sampling function s_U corresponding to the *sinc* function.

Remark 1 (Early Applications Motivating d-Dimensional Uniform Sampling).

a. In the mid-1950s, Brillouin (1956) discussed 3-dimensional uniform sampling with regard to some crystallographic problems (see the terminology "Brillouin zone" before Theorem 2.3), and Bracewell (1956) used 2-dimensional uniform sampling with regard to issues in radio astronomy. Miyakawa's basic formulation (1959), in terms of the "Nyquist relationships" between lattice and unit cell, was in the context of multivariate stochastic processes, and Sasakawa (1960-61) provided applications of Miyakawa's Theorem. Petersen and Middleton (1962) completed Miyakawa's approach, both theoretically and with many important examples. There is also Prosser's sampling theorem and analysis of truncation error (1966). Although not "early" relative to the 1950s, we also mention Dubois' application (1985) to video systems and Mersereau's work (1979) on hexagonal sampling. In this discussion as well in other parts of this paper, we have not provided specific references due to space considerations. We more than compensate for these omissions in our research tutorial.[6]

b. Motivating-applications from mathematics go back to J.M. Whittaker's uniform sampling theorem (1935) for entire functions f of order less than two. The sampled values in Whittaker's formula are $f(m+in)$, $m, n \in Z$; and, as noted by Pólya, the formula yielded a positive solution to Littlewood's conjecture that if $\{f(m+in)\}$ is bounded then f is a constant. There are many other papers on d-dimensional uniform sampling, and Higgins' exposition (1985) is a reasonable place to start (but not to finish!).

c. The proofs of Theorem 2.3 are conceptually similar to those of Theorem 2.1. They depend essentially on periodization in the guise of the proper Poisson Summation Formula (PSF) or of the canonical Fourier expansions associated with R^d and its discrete subgroups. The Nyquist hypothesis $2T\Omega \leq 1$ and bandlimited hypothesis $f \in PW_\Omega$ of Theorem 2.1 are replaced in Theorem 2.3 by the pairing H, Λ (where Λ is essential for choosing some unit cell $U \subseteq \widehat{R}^d$) and the hypothesis that $\widehat{f} = 0$ off of U, respectively.

d. In 1965 Igor Kluvánek proved Theorem 2.3 in the general setting of locally compact abelian groups. There are applications of this result to saving bandwidth (even on R) and relations to the construction of single dyadic orthonormal wavelets on R^d.[6]

3. FOURIER FRAMES

Definition 3.1 (Frames).

Let H be a separable Hilbert space with inner product $\langle x, y\rangle$ and norm $\|x\| = \langle x, x\rangle^{1/2}$. a. A sequence $\{x_n : n \in Z^d\} \subseteq H$ is a *frame* for H if there exist $A, B > 0$ such that

$$\forall y \in H, \quad A\|y\|^2 \leq \sum |\langle y, x_n\rangle|^2 \leq B\|y\|^2. \tag{4}$$

A and B are *frame bounds*, and a frame is *tight* if $A = B$. A frame is *exact* if it is no longer a frame whenever any one of its elements is removed.

The *frame operator* of the frame $\{x_n\}$ is the function $S : H \longrightarrow H$ defined as $Sy = \sum\langle y, x_n\rangle x_n$ for all $y \in H$.

The theory of frames was developed by Duffin and Schaeffer.[9]

b. An exact frame is a bounded unconditional basis and vice-versa. In particular, orthonormal bases (ONBs) are exact frames and there are frames which are not exact frames.

An essential feature of frames $\{x_n\} \subseteq H$ is that they provide the harmonics for signal reconstruction formulas. Frames may not be ONBs, but ONBs are not necessarily an advantage when it comes to noise reduction and stable decompositions.

Theorem 3.2 (Frame Decomposition Theorem).

Let $\{x_n : n \in Z^d\} \subseteq H$ be a frame for H with frame bounds A and B.

a. The frame operator S is a topological isomorphism with inverse $S^{-1} : H \longrightarrow H$. $\{S^{-1}x_n\}$ is a frame with frame bounds B^{-1} and A^{-1}, and

$$\forall y \in H,\ y = \sum\langle y, S^{-1}x_n\rangle x_n = \sum\langle y, x_n\rangle S^{-1}x_n \ \text{in } H. \tag{5}$$

b. If $\{x_n\}$ is a tight frame for H, if $\|x_n\| = 1$ for all n, and if $A = B = 1$, then $\{x_n\}$ is an orthonormal basis for H.

c. If $\{x_n\}$ is an exact frame for H, then $\{x_n\}$ and $\{S^{-1}x_n\}$ are biorthonormal, i.e.,

$$\forall m, n, \quad \langle x_m,\ S^{-1}x_n\rangle = \delta(m, n) = \begin{cases} 0 & \text{if } m \neq n, \\ 1 & \text{if } m = n. \end{cases}$$

$\{S^{-1}x_n\}$ is the unique sequence in H which is biorthonormal to $\{x_n\}$.

d. If $\{x_n\}$ is an exact frame for H, then the sequence resulting from the removal of any one element is not complete in H, i.e., the linear span of the resulting sequence is not dense in H.

Definition 3.3 (Fourier Frames).

a. Let $R > 0$, and assume that the sequence $\{e_\lambda : \lambda \in \Lambda\}$ is a frame for $H = L^2[-R, R]$. This is clearly equivalent to the assertion that there exist $A, B > 0$ such that

$$\forall F \in PW_R, \quad A\|F\|^2_{L^2(\widehat{R})} \leq \sum_{\lambda\in\Lambda} |F(\lambda)|^2 \leq B\|F\|^2_{L^2(\widehat{R})}. \tag{6}$$

As such we say that $\{e_\lambda : \lambda \in \Lambda\}$ is a *Fourier frame* for $L^2[-R, R]$, and by (5) we have

$$\forall f \in L^2[-R, R], \quad f = \sum_{\lambda\in\Lambda} a_\lambda(f)e_\lambda \ \text{ in } \ L^2[-R, R]. \tag{7}$$

(7) is a *non-harmonic Fourier series*, see Chapter VII of Paley and Wiener.[10]

b. The *frame radius* $R_f(\Lambda)$ of Λ is

$$R_f(\Lambda) = \sup\{R \geq 0 : \{e_\lambda\} \ \text{ is a Fourier frame for } \ L^2[-R, R]\}.$$

Example 1. Let us switch the notation R^d and $\widehat{R}^d$ in Theorem 2.3. We let $\Lambda \subseteq \widehat{R}^d$ be a lattice, and we let $E \subseteq R^d$ be a unit cell of the reciprocal lattice. Then, if $F \in L^2(\widehat{R}^d)$ is a continuous function with the property that $f = F^\vee = 0 a.e.$ off of E, we have the sampling formula

$$f(x) = \frac{1}{|E|}\left(\sum_{\lambda\in\Lambda} F(\lambda)e^{2\pi i<x,\gamma>}\right)\mathbf{1}_E(x) \tag{8}$$

in $L^2(R)$. More generally, if $\Lambda \subseteq \widehat{R}^d$ is a Fourier frame for PW_E, then

$$\forall f \in L^2(E), \quad f(x) = \sum_{\lambda\in\Lambda} c_\lambda e^{2\pi i<x,\gamma>}$$

in $L^2(E)$, which can be compared with (8).

4. DENSITY CRITERIA FOR COMPLETENESS

In order to gain insight into the structure of sets $\Lambda \subseteq \widehat{R}^d$ for which $\{e_\lambda : \lambda \in \Lambda\}$ is a Fourier frame for PW_E for some $E \subseteq R^d$, it is reasonable to consider first criteria for which $\{e_\lambda : \lambda \in \Lambda\}$ is complete in the case $\Lambda \subseteq \widehat{R}$. To be precise we let $\Lambda = \{\lambda_k : k \in Z,\ \lambda_{-1} < 0 \leq \lambda_0, \text{ and } \lim_{k\to\pm\infty} \lambda_k = \pm\infty\} \subseteq \widehat{R}$ be a strictly increasing sequence, which is *uniformly discrete* or *separated* in the sense that

$$\exists r > 0 \ \text{ such that } \ \forall k \in Z, \quad \lambda_{k+1} - \lambda_k \geq r.$$

Further, for each $R > 0$, we let

$$X_R = \overline{\text{span}}\{e_\lambda : \lambda \in \Lambda\}$$

denote the closed linear span of $\{e_\lambda : \lambda \in \Lambda\}$ in $L^2[-R, R]$. Paley and Wiener[10](Chapter VI) refer to X_R as the "closure" of the set $\{e_\lambda : \lambda \in \Lambda\}$ of complex exponential functions. If $X_R = L^2[-R, R]$, then $\{e_\lambda : \lambda \in \Lambda\}$ is *complete* in $L^2[-R, R]$.

Definition 4.1 (The Closure of Sets of Complex Exponential Functions).

a. The *radius of completeness* $R_c(\Lambda)$ of Λ is

$$R_c(\Lambda) = \sup\{R \geq 0 : X_R = L^2[-R, R]\}.$$

$R_c(\Lambda)$ is well-defined since it is clear that if $R_1 < R_2$ and $X_{R_2} = L^2[-R_2, R_2]$, then $X_{R_1} = L^2[-R_1, R_1]$.

b. An essential *problem* is to compute $R_c(\Lambda)$, and in particular to find an intrinsic property of Λ so as to conclude that $X_R = L^2[-R, R]$ for a given R. Note that $X_R = L^2[-R, R]$ for each $R < R_c(\Lambda)$ and $X_R \neq L^2[-R, R]$ for each

$R > R_c(\Lambda)$. $R_c(\Lambda)$ is equal to the radii of completeness for the L^p-spaces $L^p[-R,R]$, $1 \le p < \infty$, as well as the space $C[-R,R]$ of continuous functions on $[-R,R]$.

The following theorem is an early, fundamental, and deep result due to Paley and Wiener[10](Section 26).

THEOREM 4.2 (A PALEY-WIENER COMPLETENESS THEOREM).

Assume that Λ has the property that $\lambda_0 = 0$ and $\lambda_{-k} = -\lambda_k$ for $k \ge 1$. For each $\gamma > 0$ let $n(\gamma)$ be the cardinality of $\{\lambda_k : k \ge 1 \text{ and } \lambda_k \le \gamma\}$. If

$$\overline{\lim_{\gamma\to\infty}} \frac{n(\gamma)}{\gamma} > 2R, \tag{9}$$

then $X_R = L^2[-R,R]$.

REMARK 2 (COMPLETENESS AND DENSITY).

a. The "$\overline{\lim}$" on the left side of (9) is a density condition, and such conditions are essential hypotheses, not only for completeness theorems such as Theorem 4.2 and Equation (15) below, but also for non-uniform sampling formulas.

In engineering terms and in the context of non-uniform sampling, we can expect completeness if the number of samples per unit time exceeds *on average* twice the largest frequency in the given signal, i.e., if the average sampling rate exceeds the Nyquist rate. This sampling criteria is a density condition, and accurately quantifying the correct density to obtain completeness is difficult.

Further, there are genuine engineering applications of some of these completeness theorems in uniquely determining signals from their non-uniformly spaced samples, e.g., Beutler's work (1966) using results of Levinson.

b. Paley and Wiener's book (1934)[10] is the progenitor and driving force for an extensive and deep theory relating refinements of Theorem 4.2 with various notions of density, see Definition 4.3. Some of the many notable works since then are due to Levinson (1940), Duffin-Eachus (1942) and Duffin-Schaeffer[9](1952), Kahane (1962), Beurling and Landau, e.g., Landau[11](1967), and Beurling and Malliavin (1962 and 1967).

There are also world class expositions due to Koosis (1970 and 1996) and Redheffer (1977) reflecting the authors' profound understanding of the problems and their own seminal contributions from the 1960s, cf., Boas' book (1954) and a more recent and justly influential book due to Young (1980).

This material is expanded as background in a forthcoming tutorial on multidimensional non-uniform sampling.[6]

DEFINITION 4.3 (DENSITY CRITERIA).

Let $\Lambda = \{\lambda_k : k \in Z,\ \lambda_{-1} < 0 \le \lambda_0, \text{ and } \lim_{k\to\pm\infty} \lambda_k = \pm\infty\} \subseteq \widehat{R}$ be a strictly increasing uniformly discrete sequence, and define the function

$$n_\Lambda = \sum k \mathbf{1}_{[\lambda_k,\lambda_{k+1})}$$

whose distributional derivative is $n'_\Lambda = \sum_{\lambda\in\Lambda} \delta_\lambda$. Clearly, if $n : (0,\infty) \to \{0,1,\ldots\}$ is defined by $n(\gamma) = \text{card}\{\lambda_k : |\lambda_k| \le \gamma\}$, where "card" is cardinality, then $n(\gamma) = n_\Lambda(\gamma) - n_\Lambda(-\gamma)$.

a. A reasonable definition of the *density* of Λ is

$$\lim_{\gamma\to\infty} \frac{n(\gamma)}{2\gamma}$$

when this limit exists. As such, and since $n_\Lambda(\gamma_k) = k$, we shall define the *natural density* of Λ as

$$D_n(\Lambda) = \lim_{|k|\to\infty} \frac{k}{\lambda_k} \tag{10}$$

when the limit in (10) exists. Otherwise, we consider the *upper*, resp., *lower*, *natural densities*

$$D_n^+(\Lambda) = \overline{\lim_{|k|\to\infty}} \frac{k}{\lambda_k}, \quad \text{resp.,} \quad D_n^-(\Lambda) = \underline{\lim_{|k|\to\infty}} \frac{k}{\lambda_k}.$$

b. Λ has *uniform density* $D_u(\Lambda) > 0$ if

$$\exists C > 0 \text{ such that } \forall |\gamma| > 0,\ |n_\Lambda(\gamma) - D_u(\Lambda)\gamma| \le C. \tag{11}$$

Since $n_\Lambda(\lambda_k) = k$, the uniform density inequality (11) is equivalent to the condition that

$$\exists C > 0 \ \text{ such that } \ \forall k, \quad |\frac{k}{\lambda_k} - D_u(\Lambda)| \le \frac{C}{|\lambda_k|},$$

or, equivalently,

$$|\lambda_k - \frac{k}{D_u(\Lambda)}| \le \frac{C}{D_u(\Lambda)} = L.$$

In particular, if $D_u(\Lambda)$ exists, then $D_n(\Lambda)$ exists and equals $D_u(\Lambda)$. In fact, uniform density can be viewed as natural density constrained by a convergence rate of $1/|\lambda_k|$.

c. For each $\gamma > 0$ and each interval $I \subseteq \widehat{R}$ of length γ, let $n_I(\gamma) = \text{card}\,\{\lambda_k \in I\}$. Define

$$n^-(\gamma) = \inf_I n_I(\gamma) \quad \text{and} \quad n^+(\gamma) = \sup_I n_I(\gamma).$$

The *lower* and *upper Beurling densities* of Λ are

$$D_b^-(\Lambda) = \lim_{\gamma\to\infty} \frac{n^-(\gamma)}{\gamma} \quad \text{and} \quad D_b^+(\Lambda) = \lim_{\gamma\to\infty} \frac{n^+(\gamma)}{\gamma},$$

respectively. These limits exist since n^- is superadditive and n^+ is subadditive, although it is more common to replace the limits by a $\underline{\lim}$ and $\overline{\lim}$, respectively.

d. If Λ has uniform density $D_u(\Lambda) \in (0,\infty)$, then

$$D_b^-(\Lambda) = D_b^+(\Lambda) = D_u(\Lambda).$$

e. If $D_b^-(\Lambda) = D_b^+(\Lambda) = D(\Lambda)$, then the natural density $D_n(\Lambda)$ exists and

$$D_n(\Lambda) = D(\Lambda).$$

Remark 3 (Beurling-Malliavin Density).

a. By definition of $R_c(\Lambda)$, it is clear that Paley and Wiener's Theorem 4.2 is equivalent to the assertion that

$$\frac{1}{2}D_n^+(\Lambda) \le R_c(\Lambda). \tag{12}$$

Pólya (1929) introduced the notion that is called the *Pólya maximum density* $D_p^+(\Lambda)$ of Λ, and it has the property that $D_p^+(\Lambda) \ge D_n^+(\Lambda)$. In 1935 Levinson proved that if Λ is a positive sequence, then

$$\frac{1}{2}D_p^+(\Lambda) \le R_c(\Lambda). \tag{13}$$

Further, Paley and Wiener[10](Theorem XXXIV on page 94) proved that if $\Lambda = \{\lambda_k : \lambda_{-k} = -\lambda_k, k = 0,1,\ldots\}$ has uniform density $D_u(\Lambda) > 0$ with small enough bound, then

$$R_c(\Lambda) \le \frac{1}{2}D_u(\Lambda). \tag{14}$$

Other upper bounds on $R_c(\Lambda)$ are due to Koosis (1958) and Redheffer (1954).

b. In light of (12), (13), and (14), it is not unreasonable to conjecture that $\frac{1}{2}D_n^+(\Lambda) = R_c(\Lambda)$ for symmetric sequences Λ, e.g., Schwartz (1943). Kahane (1958) constructed a symmetric sequence with the properties that $D_n(\Lambda) = 0$ and $R_c(\Lambda) = \infty$. Kahane's sequence Λ is not uniformly discrete, cf., the results of Koosis (1960) and Redheffer (1968).

c. In one of the highlights of 20th century analysis, Beurling and Malliavin (1962 and 1967) in 1960–1961 devised a notion of density, denoted by $D_{bm}(\Lambda)$, allowing them to prove

$$\frac{1}{2}D_{bm}(\Lambda) = R_c(\Lambda). \tag{15}$$

The direction, $R_c(\Lambda) \geq \frac{1}{2}D_{bm}(\Lambda)$, is the easier to prove; and the direction, $R_c(\Lambda) \leq \frac{1}{2}D_{bm}(\Lambda)$, requires a deep study of the canonical product,

$$\prod\left(1 - \frac{z^2}{\lambda_k^2}\right),$$

using potential theory. The explanation par excellence is in Koosis' book (1996).

5. CHARACTERIZATION OF FOURIER FRAMES

The following theorem is a characterization of Fourier frames in terms of density. Part a is due to Duffin and Schaeffer[9] (1952); part b is due to Landau[11] (1967), although not using the term "frame"; and part c is due to Jaffard[12] (1991). Jaffard also characterized sets Λ of frequencies giving rise to Fourier frames as finite unions of iniformly discrete sets at least one of which is uniformly dense. There is another deep characterization of such sets by Ortega-Cerda and Seip (2000).

Theorem 5.1 (Fundamental Theorem of Fourier Frames).

a. If Λ has uniform density $D_u(\Lambda) > 2R$, then $\{e_\lambda : \lambda \in \Lambda\}$ is a Fourier frame for $L^2[-R, R]$.

b. If $\{e_\lambda : \lambda \in \Lambda\}$ is a Fourier frame for $L^2[-R, R]$, then $D_b^-(\Lambda) \geq 2R$.

c. If $R_f(\Lambda) \in (0, \infty)$, then

$$R_f(\Lambda) = \frac{1}{2}\sup\{D_u(\Lambda')\},$$

where $\Lambda' \subseteq \Lambda$ has finite uniform density.

6. BALAYAGE

Definition 6.1. a. A convex, compact subset $E \subseteq R^d$ is a *symmetric body* if it is symmetric about the origin in the sense that if $x \in E$ then $-x \in E$.

b. Let $E \subseteq R^d$ be a symmetric body. Define

$$\forall x \in R^d, \quad \|x\|_E = \inf\{r : x \in rE, \quad r \geq 0\},$$

where $rE = \{ry : y \in E\}$. It is elementary to prove that $\|\cdot\|_E$ is a norm on R^d which is equivalent to the Euclidean norm.

c. Let $E \subseteq R^d$ be a symmetric body. The set

$$E^* = \{\gamma \in \widehat{R}^d : \forall x \in E, \quad < x, \gamma > \leq 1\},$$

is the *polar set* of E. It is clear that E^* is a symmetric body, and E^* is the unit sphere with respect to the norm $\|\cdot\|_{E^*}$. In fact,

$$\|\gamma\|_{E^*} = \sup_{x \in E} | < x, \gamma > |$$

and

$$\|x\|_E = \sup_{\gamma \in E^*} | < x, \gamma > |. \tag{16}$$

d. Let $z \in C^d$. Because of (16) it is natural to define $\|z\|_E$ as

$$\|z\|_E = \sup_{\gamma \in E^*} | < z, \gamma > |.$$

Definition 6.2. Let $E \subseteq R^d$ be a symmetric body with polar set E^*. An entire function φ on C^d is of *exponential type* E^* if

$$\forall \epsilon > 0, \quad \exists A_\epsilon > 0$$

such that

$$\forall z \in C^d, \quad |\varphi(z)| \leq A_\epsilon e^{2\pi(1+\epsilon)\|z\|_E}.$$

The classical Plancherel-Pólya Theorem, originally proved on R, has the following formulation on R^d, see pages 108–114 of Stein and Weiss.[8]

Theorem 6.3. *(Plancherel-Pólya Theorem) Let E be a symmetric body and let $p > 1$. If φ is an entire function of exponential type E^*, then*

$$\forall \xi \in \widehat{R}^d, \quad \left(\int_{\widehat{R}^d} |\varphi(\gamma + i\xi)|^p \, d\gamma \right)^{1/p} \leq e^{2\pi\|\xi\|_{E^*}} \|\varphi\|_{L^p(\widehat{R}^d)}.$$

The Plancherel-Pólya Theorem can be used to prove the following Paley-Wiener Theorem. Generally, we shall use the elementary sufficient conditions of the Paley-Wiener Theorem in order to obtain a function of exponential type; and then we use this property of a given function in order to invoke Plancherel-Pólya.

Theorem 6.4. *(Paley-Wiener Theorem) Let $E \subseteq R^d$ be a symmetric body, and let $\varphi \in L^2(\widehat{R}^d)$. Then* $\operatorname{supp} \varphi^\vee \subseteq E$ *if and only if φ is the restriction to $\widehat{R}^d$ of an entire function of exponential type E^*.*

Definition 6.5. a. Let $E \subseteq R^d$, and let $\mu, \nu \in M(\widehat{R}^d)$. The notation $\mu \sim_E \nu$ denotes the property that $\mu^\vee = \nu^\vee$ on E.

b. Let $E \subseteq R^d$, and let $\Lambda \subseteq \widehat{R}^d$ be a closed set. *Balayage* is possible for (E, Λ) if

$$\forall \mu \in M(\widehat{R}^d), \ \exists \nu \in M(\Lambda) \ \text{ such that } \ \mu^\vee = \nu^\vee \ \text{ on } \ E,$$

i.e., for each $\mu \in M(\widehat{R}^d)$, there is a $\nu \in M(\Lambda)$ such that $\mu \sim_E \nu$.

The following is a consequence of the Open Mapping Theorem.

Let $E \subseteq R^d$, and let $\Lambda \subseteq \widehat{R}^d$ be a closed set. Assume balayage is possible for (E, Λ). There is $K > 0$ such that

$$\forall \mu \in M(\widehat{R}^d), \qquad \inf_{\nu \in M(\Lambda), \mu \sim_E \nu} \|\nu\|_{M(\Lambda)} \leq K \|\mu\|_{M(\widehat{R}^d)}. \tag{17}$$

The infimum of those K for which (17) holds is designated $K(E, \Lambda)$.

Beurling considered the following two properties on a closed set $E \subseteq R^d$.

(α) For each $x_0 \in E$ and each $\epsilon > 0$, there is $\mu_\epsilon \in M(\overline{B(x_0, \epsilon)} \cap E)$ such that $\lim_{|\gamma| \to \infty} \widehat{\mu}_\epsilon(\gamma) = 0$;

(β) E is a set of spectral synthesis, i.e., $T(f) = 0$ for all $f \in A(R^d)$ and all $T \in A'(R^d)$ with the properties that $f = 0$ on E and $\operatorname{supp} T \in A'(R^d)$.

Clearly, if $int\, E \neq 0$, then condition (α) is satisfied. There are other equivalent formulations of condition (β). For example, condition (β) is satisfied if and only if $\int \varphi(\gamma) \, d\mu(\gamma) = 0$ for all $\varphi \in \mathcal{B}_b(E)$ and all $\mu \in M(\widehat{R}^d)$ with the property that $\mu^\vee = 0$ on E. We shall use the fact that *closed, convex sets $E \subseteq R^d$ are sets* of *spectral synthesis*.

The following result was proved by Beurling.[4]

Lemma 6.6. *Let $E \subseteq R^d$ be a compact set satisfying properties (α) and (β), and let $E_\epsilon = \{x : dist(x, E) \leq \epsilon\}$. If $K(E, \Lambda) < \infty$, then there exists ϵ_0 such that*

$$\forall\, 0 < \epsilon < \epsilon_0, \quad K(E_\epsilon, \Lambda) < \infty.$$

We shall also need the following result due to Ingham (1934). This lemma has major extensions and is intimately related to the uncertainty principle in the context of Fourier analysis.

Lemma 6.7. *Let Ω be a positive, increasing, continuous function on $[0, \infty)$, and assume it satisfies the growth conditions,*

$$\int_0^\infty \frac{\Omega(\gamma)}{\gamma^2}\, d\gamma < \infty$$

and

$$\int_{\widehat{R}^d} e^{-2\Omega(|\xi|)}\, d\xi < \infty.$$

For any $\epsilon > 0$, there exists an entire function h on C^d with the properties that $h(0) = 1$,

$$\forall \xi \in \widehat{R}^d, \quad |h(\xi)| \le c e^{-\Omega(|\xi|)},$$

and supp $\widehat{h} \subseteq \overline{B(0, \epsilon)}$.

Using these results as well as Theorems 6.3 and 6.4, Beurling provided the ideas and methods for proving the following theorem.

Theorem 6.8. *Let $E \subseteq \widehat{R}^d$ be a closed set satisfying properties (α) and (β), assume E is symmetric with respect to the origin, and let $\Lambda \subseteq \widehat{R}^d$ be uniformly discrete. If balayage is possible for (E, Λ), then Λ is a Fourier frame for PW_E.*

7. BEURLING COVERING THEOREM

The following result was proved for the unit ball $E = \overline{B(0,1)} \subseteq R^d$ by Beurling.[1] As with Theorem 6.8, the proof of the following result is a consequence of Beurling's methods and ideas.[6]

Lemma 7.1. *Let $E \subseteq R^d$ be a symmetric body, and let $\xi, \eta \in \widehat{R}^d$ be points such that $\xi - \eta \in \{\gamma : \|\gamma\|_{E^*} \le 1/4\}$. There exists a discrete measure $\nu \in M(\widehat{R}^d)$ with the properties that ν has support contained in the line passing through ξ and η, $\nu(\{\xi\}) = \nu(\{\eta\}) = 0$, and*

$$\delta_\xi \sim_E \cos(2\pi\|\xi - \eta\|_{E*})\delta_\eta + \nu.$$

This balayage of the Dirac measure is then used in conjunction with Theorem 6.8 to prove the following Beurling Covering Theorem.

Theorem 7.2. *Let $E \subseteq R^d$ be a symmetric body, and let $\Lambda \subseteq \widehat{R}^d$ be uniformly discrete set satisfying the covering property*

$$\bigcup_{\lambda \in \Lambda} \tau_\lambda E^* = \widehat{R}^d.$$

If $r < 1/4$, then Λ is a Fourier frame for PW_{rE}.

The example at the end of Section 3 and Theorem 7.2 leads to the problem of relating coverings $\Lambda + E^*$ and tilings $H + E$ when comparing the non-uniform and uniform sampling cases. Since tilings on R^d do not give rise coverings on $\widehat{R}^d$ for $d > 4$, when analyzing uniform sampling as a special case of Theorem 7.2, it is natural to construct appropriate reciprocal sets H corresponding to uniformly discrete Λ so that the hypothesis in Theorem 7.2 generalizes tiling on R^d. This represents current work with Cabrelli and Molter.

8. A SPIRAL MRI RECONSTRUCTION ALGORITHM

As mentioned in the Introduction, we shall use the Beurling Covering Theorem to give a constructive irregular sampling signal reconstruction method in the setting of interleaving spirals, see Example 3. Our method is much more general than the geometry of interleaving spirals, and the particular examples which follow can in fact be calculated using Beurling's original formulation stated in the Introduction.

Example 2. Fix $c > 0$. We shall show how to choose a uniformly discrete subset Λ_R of the spiral

$$A_c = \{(c\theta \cos 2\pi\theta, c\theta \sin 2\pi\theta) : \theta \ge 0\} \subseteq \widehat{R}^2$$

such that Λ_R is a Fourier frame for $PW_{\overline{B(0,R)}}$, for some $R > 0$.

i. Let $(\lambda_0, \gamma_0) \in \widehat{R}^2$. There exists $\theta_0 \in [0, 1)$ and $r_0 \geq 0$ such that $(\lambda_0, \gamma_0) = r_0(\cos 2\pi\theta_0, \sin 2\pi\theta_0) \in \widehat{R}^2$. We first observe that

$$\mathrm{dist}((\lambda_0, \gamma_0), A_c) \leq c/2, \tag{18}$$

and, in fact, $\sup_{(\lambda_0,\gamma_0)\in\widehat{R}^2} \mathrm{dist}((\lambda_0, \gamma_0), A_c) = c/2$. To see this, note that either $0 \leq r_0 < c\theta_0 < c$ or that there is $n_0 \in N \cup \{0\}$ for which

$$c(n_0 + \theta_0) \leq r_0 < c(n_0 + 1 + \theta_0).$$

Thus,

$$\mathrm{dist}((\lambda_0, \gamma_0), A_c) \leq \min\{|c(n_0 + \theta_0) - r_0|, \, |c(n_0 + \theta_0 + 1) - r_0|\}$$

since

$$|(\lambda_0, \gamma_0) - (\theta_0 + n_0)e^{2\pi i(\theta_0+n_0)}| = |r_0 - c(\theta_0 + n_0)|.$$

Also,

$$c = c(n_0 + \theta_0 + 1) - c(n_0 + \theta_0) = c(n_0 + \theta_0 + 1) - r_0 + r_0 - c(n_0 + \theta_0) \geq 2\min\{|c(n_0 + \theta_0) - r_0|, \, |c(n_0 + \theta_0 + 1) - r_0|\},$$

and (18) is obtained.

ii. Now, choose R for which $Rc < 1/2$, and then take $\delta > 0$ such that $(c/2 + \delta)R < 1/4$. Next, choose a uniformly discrete set of points Λ_R along the spiral A_c having curve distance between consecutive points less than 2δ, and beginning within 2δ of the origin. Then the curve distance, and consequently the ordinary distance, from any point on the spiral A_c to Λ_R is less than δ. Further, as we showed in part i, the distance from any point in $\widehat{R}^2$ to the spiral A_c is less than or equal to $c/2$. Thus, by the triangle inequality,

$$\forall \xi \in \widehat{R}^2, \quad \mathrm{dist}(\xi, \Lambda_R) \leq \frac{c}{2} + \delta = \rho.$$

Hence, $R\rho < 1/4$ by our choice of R and δ; and so we can invoke Beurling's Covering Theorem, Theorem 7.2, or the original formulation stated in the Introduction, to conclude that Λ_R is a Fourier frame for $PW_{\overline{B(0,R)}}$.

Example 3. Given *any* $R > 0$ and $c > 0$. We shall show how to construct a finite interleaving set $B = \cup_{k=1}^{M-1} A_k$ of spirals

$$A_k = \left\{ c\theta e^{2\pi i(\theta - k/M)} : \theta \geq 0 \right\}, \quad k = 0, 1, \ldots, M-1,$$

and a uniformly discrete set $\Lambda_R \subseteq B$ such that Λ_R is a Fourier frame for $PW_{\overline{B(0,R)}}$. Thus, all of the elements of $L^2(\overline{B(0,R)})$ will have a decomposition in terms of the Fourier frame $\{e_\lambda : \lambda \in \Lambda_R\}$.

We begin by choosing M such that $cR/M < 1/2$. We shall write any given $\xi_0 \in \widehat{R}^2$ as $\xi_0 = r_0 e^{2\pi i\theta_0}$, where $r_0 \geq 0$ and $\theta_0 \in [0, 1)$. Then either $0 \leq r_0 < c\theta_0 < c$ or there is $n_0 \in N \cup \{0\}$ for which

$$c(n_0 + \theta_0) \leq r_0 < c(n_0 + 1 + \theta_0).$$

In this latter case, we can find $k \in \{0, \ldots, M-1\}$ such that

$$c(n_0 + \theta_0 + \frac{k}{M}) \leq r_0 < c(n_0 + \theta_0 + \frac{k+1}{M}).$$

Thus,

$$\mathrm{dist}(\xi_0, B) \leq \frac{c}{2M}.$$

Next, we choose $\delta > 0$ such that $R\rho < 1/4$, where $\rho = (c/2M + \delta)$.

Then, for each $k = 0, 1, \ldots, M-1$, we choose a uniformly discrete set Λ_k of points along the spiral A_k having curve distance between consecutive points less than 2δ, and beginning within 2δ of the origin. The curve distance, and consequently the ordinary distance, from any point on the spiral A_k to Λ_k is less than δ. Finally, we set $\Lambda_R = \cup_{k=0}^{M-1} \Lambda_k$. Thus, by the triangle inequality,

$$\forall \xi \in \widehat{R}^2, \quad \text{dist}(\xi, \Lambda_R) \leq \text{dist}(\xi, B) + \text{dist}(B, \Lambda_R) \leq \frac{c}{2M} + \delta = \rho.$$

Hence, $R\rho < 1/4$ by our choice of M and δ; and so we can invoke Beurling's Covering Theorem, Theorem 7.2, or the original formulation state in the Introduction, to conclude that Λ_R is a Fourier frame for $PW_{\overline{B(0,R)}}$.

Note that since we are reconstructing signals on a space domain having area about R^2, we require essentially R interleaving spirals. On the other hand, if we are allowed to choose the spiral(s) after we are given $PW_{\overline{B(0,R)}}$, then we can choose Λ_R contained in a *single* spiral A_c for $c > 0$ small enough.

REFERENCES

1. A. Beurling, "Local harmonic analysis with some applications to differential operators," in *Proc. Annual Science Conference*, pp. 109–125, Belfer Graduate School of Science, 1966.
2. J. J. Benedetto, "Irregular sampling and frames," in *Wavelets–A Tutorial in Theory and Applications*, C. K. Chui, ed., pp. 445–507, Academic Press, Boston, 1992.
3. K. Grochenig, "Non-uniform sampling in higher dimensions: from trigonometric polynomials to bandlimited functions," in *Modern Sampling Theory: Mathematics and Applications*, J. Benedetto and P. Ferreira, eds., p. Chapter 7, Birkhauser, Boston, 2000.
4. A. Beurling, "On balayage of measures in Fourier transforms (Seminar, Inst. for Advanced Studies, 1959–60, unpublished)," in *Collected Works of Arne Beurling*, L. Carleson, P. Malliavin, J. Neuberger, and J. Wermer, eds., Birkhauser, Boston, 1989.
5. M. Bourgeois, F. T. A. W. Wajer, D. van Ormondt, and D. Graveron-Demilly, "Reconstruction of MRI images from non-uniform sampling, application to intrascan motion correction in functional MRI," in *Modern Sampling Theory: Mathematics and Applications*, J. Benedetto and P. Ferreira, eds., p. Chapter 16, Birkhauser, Boston, 2000.
6. J. Benedetto and H.-C. Wu, "The Beurling-Landau theory of Fourier frames and applications," *J. Fourier Analysis and Applications* **6**, 2000.
7. J. J. Benedetto, *Harmonic Analysis and Applications*, CRC Press, Inc., Boca Raton, FL, 1997.
8. E. Stein and G. Weiss, *An Introduction to Fourier Analysis on Euclidean Space*, Princeton University Press, Princeton, NJ, 1971.
9. R. J. Duffin and A. C. Schaeffer, "A class of nonharmonic Fourier series," *Trans. Amer. Math. Soc.* **72**, pp. 341–366, 1952.
10. R. Paley and N. Wiener, *Fourier transforms in the complex domain*, Amer. Math. Soc. Colloquium Publ. 19, New York, 1934.
11. H. J. Landau, "Necessary density conditions for sampling and interpolation of certain entire function," *Acta Math.* **117**, pp. 37–52, 1967.
12. S. Jaffard, "A density criterion for frames of complex exponentials," *Michigan Math. J.* **38**, pp. 339–348, 1991.