MEASURE THEORETIC CHARACTERIZATION OF MULTIRESOLUTION FRAMES IN HIGHER DIMENSIONS

JOHN J. BENEDETTO AND JUAN R. ROMERO

Abstract. We extend some of the classical results of the theory of multiresolution analysis (MRA) frames to Euclidean space $\mathbb{R}^d, d > 1$, and provide relevant examples. In the process, we use the theory of shift-invariant subspaces to bring new insights to the theory of frame multiresolution analysis. In particular, we establish an analogue of the Mallat-Meyer algorithm for multidimensional MRA frames when the measure of a set related to the spectrum of the core subspace V_0 of the FMRA is zero.

1. Introduction

Frames were introduced in the 1950s to deal with problems in nonharmonic Fourier series [16]. They are an appropiate tool to deal with problems where redundancy, robustness, oversampling, and/or nonuniform sampling play a role. There are different techniques to construct wavelet frames (e.g., Generalized Multiresolution Analysis, Extension Principles, etc. [1, 2, 15, 19, 20, 25, 26, 27, 28]). We shall use the frame multiresolution analysis (FMRA) technique, which, although elementary and limited, allows us to accomplish our goal of constructing a multidimensional Mallat-Meyer algorithm for MRA frames by tensor products [3, 24, 14]. This theory depends on the measure theoretic properties of particular sets associated with natural periodizations. This measure theoretic point of view first appeared in [7] and [22], independently.

A feature that makes FMRAs potentially useful in signal processing is the fact that the perfect reconstruction filter banks associated to them can be narrow band, whence FMRA filter banks can achieve quantization noise reduction simultaneously with reconstruction of a given narrow-band signal [4, 5, 6, 7].

Section 2 gives some definitions and preliminary results. In section 3 we briefly discuss shift-invariant subspace theory.

In sections 4 and 5 we generalize the main results of the theory of FMRA proved in [6, 7]. The main results are Theorems 4-7 in section 4 and Theorems 8 in section 5. Section 6 is devoted to the construction of wavelet frames for $L^2\left(\mathbb{R}^d\right)$ in the spirit of Mallat-Meyer algorithm.

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Theorems 4-7 provide the equations neccesary to state quantitative sufficient conditions in order that an FMRA should give rise to a wavelet frame for $L^2\left(\mathbb{R}^d\right)$. In fact, Theorem 7, which summarizes Theorems 4-6, gives sufficient conditions for translates of a given finite set of functions to be a wavelet frame for a basic subspace W_0 of $L^2\left(\mathbb{R}^d\right)$.

Theorem 8 was proved independently by Benedetto and Treiber [7], and Kim et al. [21, 22]. It states that a neccesary and sufficient condition to obtain wavelets by a generalized Mallat-Meyer algorithm is that a set related to the spectrum of the core subspace V_0 of the FMRA should have measure zero.

2. Definitions and preliminaries

2.1. **Definitions.**

- The *Fourier transform* $\widehat{f}:\widehat{\mathbb{R}}^d \longrightarrow \mathbb{C}$ of $f \in L^1\left(\mathbb{R}^d\right)$ is defined by
$$\forall \gamma \in \widehat{\mathbb{R}}^d,\ \widehat{f}(\gamma) = \int_{\mathbb{R}^d} f(x)\, e^{-2\pi i x\cdot\gamma} dx.$$
$\widehat{\mathbb{R}}^d$ is $\mathbb{R}^d$ considered as the spectral domain of the Fourier transform, and $x\cdot\gamma$ denotes the standard inner product on $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$. The map $f \rightarrow \widehat{f}$ restricted to $L^1\left(\mathbb{R}^d\right) \cap L^2\left(\mathbb{R}^d\right)$ extends to a unitary map on $L^2\left(\mathbb{R}^d\right)$.
- A *frame* for a separable Hilbert space H is a sequence $\{f_i\}_{i\in\mathbb{I}} \subseteq H$, where $\mathbb{I}$ is a countable index set, for which there are constants $A, B > 0$ such that
$$\forall f \in H,\ A||f||^2 \leq \sum_{i\in\mathbb{I}} |\langle f, f_i\rangle|^2 \leq B||f||^2.$$
- A *Riesz basis* or *exact frame* is a sequence $\{f_i\}_{i\in\mathbb{I}} \subseteq H$, for which there are constants $A, B > 0$ such that
$$A\sum_n |c_n|^2 \leq \left\|\sum_n c_n f_n\right\|^2 \leq B\sum_n |c_n|^2.$$
for all sequences $\{c_n\}$ with finite number of nonzero entries.
- We define the map $\mathcal{X}$ in the following way: for $f \in L^2\left(\mathbb{R}^d\right)$,
$$\mathcal{X}(f)(\gamma) = \left\{\widehat{f}(\gamma + k)\right\}_{k\in\mathbb{Z}^d}.$$
Then $\mathcal{X}(f)(\gamma) \in l^2\left(\mathbb{Z}^d\right)$ for almost every γ in $\widehat{\mathbb{R}}^d$.
- The periodization of $|\widehat{\varphi}|^2$ is defined as $\Phi(\gamma) = \|\mathcal{X}(\varphi)(\gamma)\|^2_{l^2(\mathbb{Z}^d)}$.
- τ_y is the translation operator defined by $\tau_y f(x) = f(x-y)$.

Remark 1. *It is clear that if* $\varphi \in L^2\left(\mathbb{R}^d\right)$, *then* $\Phi \in L^1\left(\mathbb{T}^d\right)$; *and* $\|\Phi\|_{L^1(\mathbb{T}^d)} = \|\varphi\|^2_{L^2(\mathbb{R}^d)}$ *by the Parseval-Plancherel theorem.*

Definition 1. *A* frame multiresolution analysis *(FMRA)* $(V_j, \varphi)_{j\in\mathbb{Z}}$ *of* $L^2(\mathbb{R}^d)$ *is an increasing sequence of closed linear subspaces* $V_j \subset L^2(\mathbb{R}^d)$ *and an element* $\varphi \in V_0$ *for which the following hold:*

(1) $\overline{\cup_j V_j} = L^2(\mathbb{R}^d)$ *and* $\cap V_j = \{0\}$,
(2) $f \in V_j \Longleftrightarrow Df \in V_{j+1}$, *where* $Df(x) = 2^{d/2} f(2x)$,
(3) $\forall k \in \mathbb{Z}^d, f \in V_0 \Longleftrightarrow \tau_k f \in V_0$,
(4) $\{\tau_k\varphi : k \in \mathbb{Z}^d\}$ *is a frame for* V_0.

- $A'\left(\mathbb{Z}^d\right)$ is the linear space of all sequences of Fourier coefficients of bounded periodic functions. It is called the space of *pseudomeaures* on $\mathbb{Z}^d$.
- $\mathcal{L}^\infty$ is the linear space of measurable functions $f : \mathbb{R}^d \longrightarrow \mathbb{C}$ for which
$$\exists B = B(f) \text{ such that } \sum_{n\in\mathbb{Z}^d} |\tau_n f| \le B \text{ a.e.}$$

2.2. **Preliminaries.** For $\varphi \in L^2(\mathbb{R}^d)$, let $V_0 = \overline{\operatorname{span}}\{\tau_n\varphi : n\in\mathbb{Z}^d\}$ be the closed linear span of the sequence$\{\tau_n\varphi\}_{n\in\mathbb{Z}^d}$. Then it is elementary to prove that $\{\tau_n\varphi\}$ is an exact frame (or a Riesz basis) for its closed linear span if and only if there exist constants A, B, with $0 < A \le B < \infty$ for which
$$A \le \Phi \le B \text{ a.e.}$$

A similar result is true for frames of translates [4, 7, 12, 13, 28]. In order to state this latter result we use the"pullback" notation $[f > 0]$ to designate the set of points x in the domain of f for which f is positive. Then $\{\tau_n\varphi\}_{n\in\mathbb{Z}^d}$ is a frame for its closed linear span if and only if there exist constants A, B, with $0 < A \le B < \infty$ for which
$$A \le \Phi \le B \text{ a.e. on } [\Phi > 0].$$

This result can be generalized in the case of several $\varphi_i s$ in terms of the *Gramian matrix*

$$G_X(\gamma) = \left(\sum_{n\in\mathbb{Z}^d} \widehat{\varphi_k}(\gamma - n)\overline{\widehat{\varphi_j}(\gamma - n)}\right)_{1\le k,j\le N} = \left(\langle \mathcal{X}(\varphi_k)(\gamma), \mathcal{X}(\varphi_j)(\gamma)\rangle\right)_{1\le k,j\le N},$$

where $X = \{\varphi_i\}_{i=1}^N$. Let $m(\gamma), m^+(\gamma)$, and $M(\gamma)$ be the smallest, smallest positive, and largest eigenvalues of $G_X(\gamma)$, respectively. Then $\left\{\tau_n\varphi_i : n \in \mathbb{Z}^d, 1 \le i \le N\right\}$ is a frame for its closed linear span if and only if there exist constants A, B, with $0 < A \le B < \infty$ for which
$$A \le m^+(\gamma) \le M(\gamma) \le B \text{ a.e.}$$
in a particular set called the *spectrum* of $\overline{\operatorname{span}}_{k\in\mathbb{Z}^d}\{\tau_k\varphi : \varphi \in X\}$ [8, 9, 10, 11]. The definition of spectrum will be given in section 3. In the case that the translations of elements of X form a Riesz basis, $m^+(\gamma)$ can be replaced by $m(\gamma)$, and the inequality holds a.e.

3. Shift invariant subspaces

We shall use the shift-invariant approach in section 5 to prove Theorem 8. Here are the main concepts and definitions needed for Theorem 8. For more details about the shift-invariant subspace theory see [8, 9, 10, 11, 17].

If $W \subset L^2\left(\mathbb{R}^d\right)$ and $\gamma \in \widehat{\mathbb{R}}^d$, we set
$$\mathcal{X}(W)(\gamma) = \{(\mathcal{X}(f))(\gamma) : f \in W\},$$
and hence $\mathcal{X}(W)(\gamma) \subset l^2\left(\mathbb{Z}^d\right)$ for almost all γ. If $W \subset L^2\left(\mathbb{R}^d\right)$ is a linear subspace of $L^2\left(\mathbb{R}^d\right)$, then, by the linearlity of $\mathcal{X}$, $\mathcal{X}(W)(\gamma)$ is a linear subspace of $l^2\left(\mathbb{Z}^d\right)$. $Sp_{\mathcal{X},\gamma}(W)$ is defined to be
$$Sp_{\mathcal{X},\gamma}(W) = \overline{\operatorname{span}}\{(\mathcal{X}(f))(\gamma) : f \in W\}.$$

For $W \subset L^2\left(\mathbb{R}^d\right)$, we define $S(W)$ as

$$S(W) = \overline{\operatorname{span}}_{k\in\mathbb{Z}^d}\left\{\tau_k f : f \in W\right\},$$

the *shift-invariant space generated by* W. If W is a finite set, we say that $S = S(W)$ is a *finitely generated shift-invariant space* (FSI). The *length* of a shift-invariant subspace S is defined to be len $S = \min\operatorname{card}\{W : S = S(W)\}$. For S a shift-invariant subspace of $L^2\left(\mathbb{R}^d\right)$, the *spectrum of* S, $\sigma(S)$, is defined by

$$\sigma(S) = \left\{\gamma \in \mathbb{T}^d : \dim Sp_{\mathcal{X},\gamma}(W) > 0\right\},$$

where dim indicates dimension. Note that $\sigma(V_0) = [\Phi > 0] = \left\{\gamma \in \mathbb{T}^d : \Phi(\gamma) > 0\right\}$ for the case of an FMRA. Let H be a Hilbert space and let F be a linear subspace of H. Then $F^\perp$, the *orthogonal complement of* F *on* H, is defined as

$$F^\perp = \{x \in H : \forall f \in F,\ \langle x, f\rangle = 0\}.$$

The continuity of the inner product implies that $F^\perp$ is a closed linear subspace of H. We now state some results which we shall need in section 5.

Theorem 1. *Let S be a finitely generated shift-invariant space and let T be a shift-invariant subspace of S. Then $T^\perp$ is also shift-invariant and, for almost every $\gamma \in \widehat{\mathbb{R}}^d$,*

$$\mathcal{X}(S)(\gamma) = \mathcal{X}(T)(\gamma) \bigoplus \mathcal{X}\left(T^\perp\right)(\gamma).$$

Theorem 2. *Given any FSI S, there is a finite subset $W \subset L^2\left(\mathbb{R}^d\right)$, for which the multi-integer translates of W are a frame for S.*

Theorem 3. *For a shift-invariant subspace $S \subset L^2\left(\mathbb{R}^d\right)$, we have*

$$\text{len } S = \textit{ess-sup}\left\{\dim \mathcal{X}(S)(\gamma), \gamma \in \mathbb{T}^d\right\}.$$

The map $D_S(\gamma) = \dim \mathcal{X}(S)(\gamma)$ is the dimension function *of the subspace S.*

These theorems together with their proofs can be found in [8], [28], and [9], respectively.

4. FMRA Frames

The following results are well known for the case $d = 1$, see [6, 7]. The equations in these results are the key for the construction of FMRA frames.

Theorem 4. *Let (V_j, φ) be an FMRA, let $\omega = \{\psi_1, ..., \psi_m\} \subset W_0$, the orthogonal complement of V_0 in V_1, and set $\psi_0 = \varphi$.*

(1) *If $\cup_{k\in\mathbb{Z}^d}\tau_k\omega = \left\{\tau_k\psi_p : 1 \le p \le m; k \in \mathbb{Z}^d\right\}$ defines W_0, i.e., $\overline{span}\left(\cup_{k\in\mathbb{Z}^d}\tau_k\omega\right) = W_0$, then there are $g_0, ..., g_m \in l^2\left(\mathbb{Z}^d\right)$ such that*

$$\forall n\in\mathbb{Z}^d,\ \varphi(2x - n) = \sum_{p=0}^{m}\sum_{k\in\mathbb{Z}^d} g_p(2k - n)\,\psi_p(x - k)\ \text{ in } L^2(\mathbb{R}^d). \tag{4.1}$$

(2) *If there are $g_0, ..., g_m \in A'\left(\mathbb{Z}^d\right)$ such that (4.1) is valid, and if $\|\mathcal{X}(\psi_p)(\gamma)\|^2_{l^2(\mathbb{Z}^d)}$ is essentially bounded for each $1 \le p \le m$, i.e., each $\widehat{\psi_p}^2 \in \mathcal{L}^\infty$, then $\cup_{k\in\mathbb{Z}^d}\tau_k\omega$ is a frame for W_0.*

Proof. Any $f \in V_1$ can be written uniquely as $f_0 + k_0$, with $f_0 \in V_0$ and $k_0 \in W_0$. For each $m \in \mathbb{Z}^d$ and for each $u \in \{0,1\}^d$, $\varphi(2x - 2m - u)$ is an element of V_1. Since $\cup_{k\in\mathbb{Z}^d}\tau_k\omega = \{\tau_k\psi_p : 1 \le p \le m;\ k \in \mathbb{Z}^d\}$ generates W_0, and since $\cup_{k\in\mathbb{Z}^d}\tau_k\varphi$ is a frame for V_0, there exists a set $\left\{g_{i,u} \in l^2\left(\mathbb{Z}^d\right) : 0 \le i \le m; u \in \{0,1\}^d\right\}$ such that for each $m \ge 0$ and each $u \in \{0,1\}^d$, we have

$$\varphi(2x - 2m - u) = \sum_{p=0}^{m}\sum_{k\in\mathbb{Z}^d} g_{p,u}(k - m)\,\psi_p(x - k).$$

Now, define $\{g_p\}_{p=0}^m$ by means of the formula

$$g_p(2k + u) = g_{p,u}(k),\ \ 0 \le p \le m,\ \ u \in \{0,1\}^d,\ \ k \in \mathbb{Z}^d.$$

If $n = 2l + u$, $n, l \in \mathbb{Z}^d, u \in \{0,1\}^d$, then

$$\begin{aligned}\varphi(2x - n) = \varphi(2x - 2l - u) &= \sum_{i=0}^{m}\sum_{k\in\mathbb{Z}^d} g_{p,u}(k - l)\,\psi_p(x - k)\\ &= \sum_{p=0}^{m}\sum_{k\in\mathbb{Z}^d} g_p(2k - 2l - u)\,\psi_p(x - k)\\ &= \sum_{p=0}^{m}\sum_{k\in\mathbb{Z}^d} g_p(2k - n)\,\psi_p(x - k)\ \text{ in } L^2(\mathbb{R}^d).\end{aligned}$$

Thus, the proof of (1) is complete.

Next, assume that the hypotheses of part (2) hold. For each $f \in W_0 \subset V_1$, there is $\{c(n)\}_{n\in\mathbb{Z}^d} \in l^2\left(\mathbb{Z}^d\right)$ such that

$$\begin{aligned}f(x) &= \sum_{n\in\mathbb{Z}^d} c(n)\,\varphi(2x - n)\\ &= \sum_{u\in\{0,1\}^d}\sum_{k\in\mathbb{Z}^d} c(2k + u)\,\varphi(2x - 2k - u).\end{aligned}$$

The previous equation is equivalent to

$$\begin{aligned}\widehat{f}(\gamma) &= \sum_{k\in\mathbb{Z}^d} 2^{-d}\sum_{u\in\{0,1\}^d} c(2k + u)\,\widehat{\varphi}\left(\tfrac{\gamma}{2}\right)e^{-2\pi ik\cdot\gamma}e^{-\pi iu\cdot\gamma}\\ &= \sum_{u\in\{0,1\}^d} 2^{-d}\left[\sum_{k\in\mathbb{Z}^d} c(2k + u)\,e^{-2\pi ik\cdot\gamma}\right]e^{-\pi iu\cdot\gamma}\widehat{\varphi}\left(\tfrac{\gamma}{2}\right)\\ &= \sum_{u\in\{0,1\}^d} e^{-\pi iu\cdot\gamma}\widehat{\varphi}\left(\tfrac{\gamma}{2}\right)C_u(\gamma)\ \text{ in } L^2(\widehat{\mathbb{R}}^d),\end{aligned} \tag{4.2}$$

where $C_u(\gamma) = 2^{-d}\sum_{k\in\mathbb{Z}^d} c(2k + u)\,e^{-2\pi ik\cdot\gamma} \in L^2(\mathbb{T}^d)$, and where the convergence in $L^2(\widehat{\mathbb{R}}^d)$ is in terms of the partial sums of the C_u by the Parseval-Plancherel theorem. If we take the Fourier transform of (4.1), we obtain for $n = 2l + u$, $u \in \{0,1\}^d$, and $l \in \mathbb{Z}^d$, that

$$\varphi(2x - 2l - u) = \sum_{p=0}^{m}\sum_{k\in\mathbb{Z}^d} g_p(2(k - l) - u)\,\psi_p(x - k)$$

if and only if

$$2^{-d}\widehat{\varphi}\left(\tfrac{\gamma}{2}\right)e^{-2\pi il\cdot\gamma}e^{-\pi iu\cdot\gamma}=\sum_{p=0}^{m}\left[\sum_{k\in\mathbb{Z}^d}g_p\left(2\left(k-l\right)-u\right)e^{-2\pi i\mathbf{k}\cdot\gamma}\right]\widehat{\psi_p}\left(\gamma\right).$$

Hence,

$$\begin{aligned}\widehat{\varphi}\left(\tfrac{\gamma}{2}\right)e^{-\pi iu\cdot\gamma}&=2^d\sum_{p=0}^{m}\left[\sum_{k\in\mathbb{Z}^d}g_p\left(2\left(k-l\right)-u\right)e^{-2\pi i(k-l)\cdot\gamma}\right]\widehat{\psi_p}\left(\gamma\right)\\&=\sum_{p=0}^{m}G_{p,u}\left(\gamma\right)\widehat{\psi_p}\left(\gamma\right),\end{aligned}$$

where $G_{p,u}\left(\gamma\right)=2^d\sum_{k\in\mathbb{Z}^d}g_p\left(2k-u\right)e^{-2\pi ik\cdot\gamma}\in L^2(\mathbb{T}^d)$, for all p,u, and the convergence in $L^2(\widehat{\mathbb{R}}^d)$ is in terms of the partial sums of the $G_{p,u}s$. Substituting into equation (4.2), we have

$$\begin{aligned}\widehat{f}\left(\gamma\right)&=\sum_{u\in\{0,1\}^d}\left(e^{-\pi iu\cdot\gamma}\widehat{\varphi}\left(\tfrac{\gamma}{2}\right)\right)C_u\left(\gamma\right)\\&=\sum_{u\in\{0,1\}^d}\left(\sum_{j=0}^{m}G_{p,u}\left(\gamma\right)\widehat{\psi_j}\left(\gamma\right)\right)C_u\left(\gamma\right)\\&=\sum_{p=0}^{m}\left(\sum_{u\in\{0,1\}^d}G_{p,u}\left(\gamma\right)C_u\left(\gamma\right)\right)\widehat{\psi_j}\left(\gamma\right)\\&=\sum_{p=0}^{m}E_p\left(\gamma\right)\widehat{\psi_p}\left(\gamma\right),\end{aligned}$$

where $E_p\left(\gamma\right)=\sum_{u\in\{0,1\}^d}G_{p,u}\left(\gamma\right)C_u\left(\gamma\right)$. Using the hypothesis that $g_p\in A'\left(\mathbb{Z}^d\right),0\le p\le m$, we see that $E_p\in L^2(\mathbb{T}^d)$. Now, for $N\in\mathbb{N}$, we define

$$\widehat{f}_N\left(\gamma\right)=\sum_{p=0}^{m}\left[\sum_{|n|\le N}E_p^{\vee}\left(n\right)e^{-2\pi in\cdot\gamma}\right]\widehat{\psi_p}\left(\gamma\right)=\sum_{p=0}^{m}E_p^N\left(\gamma\right)\widehat{\psi_p}\left(\gamma\right),$$

where $E_p^N\left(\gamma\right)=\sum_{|n|\le N}E_p^{\vee}\left(n\right)e^{-2\pi in\cdot\gamma}$ (the N-th symmetric partial sum of $E_p\left(\gamma\right)$), $n=\left(n_1,...,n_d\right)\in\mathbb{Z}^d$, and $|n|=\sum_{i=1}^{d}|n_i|$. Clearly, $\widehat{f}_N\in L^2(\widehat{\mathbb{R}}^d)$ and

$$\lim_{N\to\infty}||\widehat{f_N}-\widehat{f}||_{L^2(\widehat{\mathbb{R}}^d)}=0,$$

since each $\widehat{\psi_p}^2 \in \mathcal{L}^\infty$ and since

$$\begin{aligned}||\widehat{f_N} - \widehat{f}||_{L^2(\widehat{\mathbb{R}}^d)} &\leq \sum_{p=0}^{m} ||\widehat{\psi_p}\left(E_p - E_p^N\right)||_{L^2(\widehat{\mathbb{R}}^d)} \\ &= \sum_{p=0}^{m}\sum_{n\in\mathbb{Z}^d}\left(\int_{[0,1]^d} \tau_n|\widehat{\psi_p}|^2\left|E_p - E_p^N\right|^2\right)^{\frac{1}{2}} \\ &= \sum_{p=0}^{m} ||\sqrt{\Phi_p}\left(E_p - E_p^N\right)||_{L^2(\mathbb{T}^d)} \\ &\leq \sum_{p=0}^{m} ||\sqrt{\Phi_p}||_{L^\infty(\mathbb{T}^d)}||\left(E_p - E_p^N\right)||_{L^2(\mathbb{T}^d)},\end{aligned}$$

where $\Phi_p = P\left(|\widehat{\psi_p}|^2\right)$, $0 \leq p \leq m$. Hence, the Parseval-Plancherel theorem implies that

$$f = \sum_{p=0}^{m}\sum_{n\in\mathbb{Z}^d} E_p^\vee(n)\,\tau_n\psi_p \text{ in } L^2(\mathbb{R}^d).$$

Because $f \in W_0$, $\sum_{n\in\mathbb{Z}^d} E_0^\vee(n)\,\tau_n\psi_0 = \sum_{n\in\mathbb{Z}^d} E_0^\vee(n)\,\tau_n\varphi = 0$. Hence,

$$f = \sum_{p=1}^{m}\sum_{n\in\mathbb{Z}^d} E_p^\vee(n)\,\tau_n\psi_p \text{ in } L^2(\mathbb{R}^d),$$

where $\left\{E_p^\vee(n)\right\}_{n\in\mathbb{Z}^d} \in l^2\left(\mathbb{Z}^d\right)$, for $p \in \{1,...,m\}$. So far we have proved that $\cup_{k\in\mathbb{Z}^d}\tau_k\omega$ generates W_0. This mean that the linear operator $T_{W_0}^* : l^2\left(\mathbb{Z}^d\right) \to W_0$, $T_\omega^*(\{c_k\}) = \sum_{p=1}^m\sum_k c_k\tau_k\psi_p$, is a surjection onto W_0. Its adjoint, T_{W_0}, is bounded since $\widehat{\psi_p}^2 \in \mathcal{L}^\infty$, $1 \leq p \leq m$. This implies that $T_{W_0}^*$ is also bounded. The open mapping theorem guarantees that $T_{W_0}^*$ is also bounded below on $\mathcal{N}\left(T_{W_0}^*\right)^\perp$, since the restriction of this map to $\mathcal{N}\left(T_{W_0}^*\right)^\perp$ is an invertible operator. It follows that $\cup_{k\in\mathbb{Z}^d}\tau_k\omega$ is a frame for W_0 (see Proposition 3.4 in [7]). $\square$

Theorem 5. *Let $H_0 \in L^\infty\left(\mathbb{T}^d\right)$, and let (V_j,φ) be an FMRA, where H_0 and φ satisfy*

$$\widehat{\varphi}(\gamma) = H_0\left(\frac{\gamma}{2}\right)\widehat{\varphi}\left(\frac{\gamma}{2}\right) \; a.e. \text{ in } \; L^2(\widehat{\mathbb{R}}^d). \tag{4.3}$$

Define W_0 as the orthogonal complement of V_0 in V_1. Further, for $\{h_1[n]\} \in l^2\left(\mathbb{Z}^d\right)$, let $\widehat{h_1} = H_1 \in L^2\left(\mathbb{T}^d\right)$, and let $\psi \in V_1$ be defined as

$$\widehat{\psi}(\gamma) = H_1\left(\frac{\gamma}{2}\right)\widehat{\varphi}\left(\frac{\gamma}{2}\right) \quad a.e. \text{ in } \; L^2(\widehat{\mathbb{R}}^d). \tag{4.4}$$

Then $\psi \in W_0$ if and only if

$$\sum_{u\in\{0,1\}^d}\tau_{\frac{u}{2}}\left(H_1\overline{H_0}\Phi\right) = 0 \;\; a.e. \text{ in } L^2(\mathbb{T}^d). \tag{4.5}$$

Proof. Set $e_k(\gamma) = e^{-2\pi i k\cdot\gamma}$. Then $\psi \in V_0^{\perp}$ *(in V_1)*

$$\begin{aligned}
&\Longleftrightarrow \forall k \in \mathbb{Z}^d, \left\langle \widehat{\psi}, e_k\widehat{\varphi} \right\rangle = \langle \psi, \tau_k \varphi\rangle = 0 \\
&\Longleftrightarrow \forall k \in \mathbb{Z}^d, \left\langle H_1\left(\tfrac{\cdot}{2}\right)\widehat{\varphi}\left(\tfrac{\cdot}{2}\right), e_k H_0\left(\tfrac{\cdot}{2}\right)\widehat{\varphi}\left(\tfrac{\cdot}{2}\right)\right\rangle = 0 \\
&\Longleftrightarrow \forall k \in \mathbb{Z}^d, \langle H_1\widehat{\varphi}, e_{2k}H_0\widehat{\varphi}\rangle = 0 \\
&\Longleftrightarrow \forall k \in \mathbb{Z}^d, \int_{\mathbb{R}^d} H_1\overline{H_0}|\widehat{\varphi}|^2 e_{2k} = 0 \\
&\Longleftrightarrow \forall k \in \mathbb{Z}^d, \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d} H_1\overline{H_0}\Phi e_{2k} = 0 \\
&\Longleftrightarrow \forall k \in \mathbb{Z}^d, \sum_{u\in\{0,1\}^d} \int_{\left[0,\frac{1}{2}\right]^d} \tau_{\frac{u}{2}}\left(H_1\overline{H_0}\Phi\right)e_{2k} = 0 \\
&\Longleftrightarrow \quad \int_{\left[0,\frac{1}{2}\right]^d} \sum_{u\in\{0,1\}^d} \tau_{\frac{u}{2}}\left(H_1\overline{H_0}\Phi\right)e_{2k} = 0.
\end{aligned}$$

The result follows by the L^1- uniqueness theorem for Fourier series. □

Remark 2. *After periodizing the modulus squared of (4.3), equation (4.3) is equivalent to the following equation:*

$$\Phi = \sum_{u\in\{0,1\}^d} \tau_{-\frac{u}{2}}\left[\left(|H_0|^2\,\Phi\right)\left(\frac{\cdot}{2}\right)\right]. \tag{4.6}$$

This equation will be needed for a remark after Theorem 8 of section 5.

Theorem 6. *Let (V_j,φ) be an FMRA, let $H_0 \in L^{\infty}(\mathbb{T}^d)$ satisfy $\widehat{\varphi}(\gamma) = H_0\left(\frac{\gamma}{2}\right)\widehat{\varphi}\left(\frac{\gamma}{2}\right)$ in $L^2(\widehat{\mathbb{R}}^d)$, and let $\omega = \{\psi_1, ..., \psi_m\} \subset W_0$ and $H_p \in L^2(\mathbb{T}^d)$, $1 \le p \le m$, satisfy $\widehat{\psi_p}(\gamma) = H_p\left(\frac{\gamma}{2}\right)\widehat{\varphi}\left(\frac{\gamma}{2}\right)$ in $L^2(\widehat{\mathbb{R}}^d)$. Suppose that $\widehat{g_p} = G_p \in L^{\infty}(\mathbb{T}^d)$, $0 \le p \le m$. Then (4.1) holds if and only if*

$$\forall u \in \{0,1\}^d,\ \sum_{p=0}^{m} \tau_{-\frac{1}{2}u}(H_p\Phi)G_p = \Phi\delta(0,u). \tag{4.7}$$

Here, $\delta(0,u)$ is the Kronecker delta.

Proof. Equation (4.1) is equivalent to the following equation:

$$\begin{aligned}
\forall\ n \in \mathbb{Z}^d,\ \widehat{\varphi}(\gamma) &= 2^d \sum_{p=0}^{m}\sum_{k\in\mathbb{Z}^d} g_p(2k-n)\,e^{-2\pi i(2k-n)\cdot\gamma}\widehat{\psi_p}(2\gamma) \\
&= 2^d \sum_{p=0}^{m}\sum_{k\in\mathbb{Z}^d} g_p(2k-n)\,e^{-2\pi i(2k-n)\cdot\gamma}H_p(\gamma)\,\widehat{\varphi}(\gamma).
\end{aligned} \tag{4.8}$$

Adding these equations over all $v \in \{0,1\}^d$ we obtain

$$\begin{aligned} 2^d\widehat{\varphi}(\gamma) &= 2^d \sum_{p=0}^{m} \sum_{v\in\{0,1\}^d} \sum_{k\in\mathbb{Z}^d} g_p\left(2k-v\right) e^{-2\pi i (2k-v)\cdot\gamma} H_p\left(\gamma\right) \widehat{\varphi}(\gamma) \\ &= 2^d \sum_{p=0}^{m} \sum_{q\in\mathbb{Z}^d} g_p\left(q\right) e^{-2\pi i q\cdot\gamma} H_p\left(\gamma\right) \widehat{\varphi}(\gamma) \\ &= 2^d \sum_{p=0}^{m} G_p\left(\gamma\right) H_p(\gamma) \widehat{\varphi}(\gamma). \end{aligned}$$

Here, $G_p\left(\gamma\right) = \sum_{q\in\mathbb{Z}^d} g_p\left(q\right) e^{-2\pi i q\cdot\gamma}$, $0 \le p \le m$. The second equality follows from the fact that any $q \in \mathbb{Z}^d$ can be written as $2k - v$, where $k \in \mathbb{Z}^d$ and $v \in \{0,1\}^d$ are uniquely determined. Now, let $v \in \{0,1\}^d$ and $u \in \{0,1\}^d \setminus \{0\}$ be fixed. Multiplying (4.8) by $e^{-\pi i v\cdot u}$, we have

$$\begin{aligned} \widehat{\varphi}(\gamma)e^{-\pi i v\cdot u} &= 2^d \sum_{p=0}^{m} \sum_{k\in\mathbb{Z}^d} g_p\left(2k-v\right) e^{-2\pi i (2k-v)\cdot\gamma} e^{-\pi i v\cdot u} H_p\left(\gamma\right) \widehat{\varphi}(\gamma) \\ &= 2^d \sum_{p=0}^{m} \sum_{k\in\mathbb{Z}^d} g_p\left(2k-v\right) e^{-2\pi i (2k-v)\cdot\left(\gamma-\frac{u}{2}\right)} H_p\left(\gamma\right) \widehat{\varphi}(\gamma). \end{aligned}$$

On the other hand, $\sum_{v\in\{0,1\}^d} e^{-\pi i v\cdot u} = 0$ for a fixed $u \in \{0,1\}^d \setminus \{0\}$ since $e^{-\pi i v\cdot u} = (-1)^{v\cdot u}$. In fact, $\sum_{v\in\{0,1\}^d} e^{-\pi i v\cdot u} = \sum_{v\in\{0,1\}^d} (-1)^{v\cdot u}$; and so, for a fixed $u \in \{0,1\}^d \setminus \{0\}$, half of the $\{v \cdot u\}_{v\in\{0,1\}^d}$ are even and half are odd, and so the original sum of exponentials is zero. Hence,

$$\begin{aligned} 0 &= \Big(\sum_{v\in\{0,1\}^d} e^{-\pi i v\cdot u}\Big)\widehat{\varphi}(\gamma) \\ &= 2^d \sum_{p=0}^{m} \sum_{v\in\{0,1\}^d} \sum_{k\in\mathbb{Z}^d} g_p\left(2k-v\right) e^{-2\pi i (2k-v)\cdot(\gamma-\frac{u}{2})} H_p\left(\gamma\right) \widehat{\varphi}(\gamma) \\ &= 2^d \sum_{p=0}^{m} \sum_{q\in\mathbb{Z}^d} g_p\left(q\right) e^{-2\pi i q\cdot(\gamma-\frac{u}{2})} H_p\left(\gamma\right) \widehat{\varphi}(\gamma) \\ &= 2^d \sum_{p=0}^{m} G_p(\gamma - \frac{u}{2}) H_p(\gamma)\widehat{\varphi}(\gamma) \\ &= 2^d \sum_{p=0}^{m} G_p\left(\xi\right) H_p(\xi + \frac{u}{2})\widehat{\varphi}(\xi + \frac{u}{2}). \end{aligned}$$

Summarizing the previous calculations, we have verified that

$$\widehat{\phi}(\xi) = \sum_{p=0}^{m} G_p\left(\xi\right) H_p(\xi)\widehat{\varphi}(\xi), \text{ a.e. } \xi \in \widehat{\mathbb{R}}^d \tag{4.9}$$

and

$$0 = \sum_{p=0}^{m} G_p\left(\xi\right) H_p(\xi + \frac{u}{2})\widehat{\varphi}(\xi + \frac{u}{2}), \text{ a.e. } \xi \in \widehat{\mathbb{R}}^d \tag{4.10}$$

are equivalent to (4.1).

To prove the theorem, assume (4.1), i.e., assume (4.9) and (4.10). Multiplying (4.9) and (4.10) by $\overline{\widehat{\varphi}(\xi)}$ and $\overline{\widehat{\varphi}(\xi+\frac{u}{2})}$, respectively, we obtain

$$|\widehat{\phi}(\xi)|^2 = \sum_{p=0}^{m} G_p\,(\xi)\,H_{pj}(\xi)|\widehat{\varphi}(\xi)|^2 \text{ a.e. } \xi \in \widehat{\mathbb{R}}^d$$

and

$$0 = \sum_{p=0}^{m} G_p\,(\xi)\,H_p(\xi+\frac{u}{2})|\widehat{\varphi}(\xi+\frac{u}{2})|^2, \text{ a.e. } \xi \in \widehat{\mathbb{R}}^d.$$

By the periodicity of the $G_p s$ and the $H_p s$, the change of variable $\xi \to \xi+k$ produces

$$|\widehat{\varphi}(\xi+k)|^2 = \sum_{p=0}^{m} G_p\,(\xi)\,H_p(\xi)|\widehat{\varphi}(\xi+k)|^2, \text{ a.e. } \xi \in \widehat{\mathbb{R}}^d$$

and

$$0 = \sum_{p=0}^{m} G_p\,(\xi)\,H_p(\xi+\frac{u}{2})|\widehat{\varphi}(\xi+k+\frac{u}{2})|^2, \text{ a.e. } \xi \in \widehat{\mathbb{R}}^d.$$

Summing over all k in $\mathbb{Z}^d$, we have

$$\Phi(\xi) = \sum_{p=0}^{m} G_p\,(\xi)\,H_p(\xi)\Phi(\xi), \text{ a.e. } \xi \in \widehat{\mathbb{R}}^d$$

and

$$0 = \sum_{p=0}^{m} G_p\,(\xi)\,H_p(\xi+\frac{u}{2})\Phi(\xi+\frac{u}{2}), \text{ a.e. } \xi \in \widehat{\mathbb{R}}^d,$$

i.e.,

$$\sum_{p=0}^{m} \tau_{-\frac{1}{2}u}(H_p\Phi)G_p = \Phi\delta\,(0,u)\,, \text{ a.e. on } \mathbb{T}^d.$$

This proves the“only if” part.

Conversely, if equation (4.7) has a solution $G_p \in L^\infty(\mathbb{T}^d)$, $0 \le p \le m$, then

$$1 = \sum_{p=0}^{m} G_p\,(\xi)\,H_p(\xi), \text{ a.e. } \xi \in [\Phi > 0]$$

and

$$0 = \sum_{p=0}^{m} G_p\,(\xi)\,H_p(\xi+\frac{u}{2}), \text{ a.e. } \xi \in \left[\tau_{-\frac{u}{2}}\Phi > 0\right].$$

Clearly, $\widehat{\varphi}(\xi) \neq 0$ implies that $\Phi(\xi) \neq 0$; and therefore $[\widehat{\varphi} \neq 0] \subseteq [\Phi > 0]$ and $\left[\tau_{-\frac{u}{2}}\varphi \neq 0\right] \subseteq \left[\tau_{-\frac{u}{2}}\Phi > 0\right]$. This implies that the previous two equations hold a.e. on $[\Phi{>}0]$ and on $\left[\tau_{-\frac{u}{2}}\Phi > 0\right]$, respectively. This yields (4.9) and (4.10), and, hence, the proof is complete. □

Combining Theorems 4 and 6 we obtain the following result:

Theorem 7. *Let (V_j,φ) be an FMRA, let $\omega = \{\psi_1,...,\psi_m\} \subset W_0$, and let H_p, $0 \le p \le m$, be as above. Assume that $\widehat{\psi_p}^{\,2} \in \mathcal{L}^\infty$, $1 \le p \le m$. If there are $\widehat{g}_p = G_p \in L^\infty(\mathbb{T}^d)$, $0 \le p \le m$, such that (4.7) holds, then $\cup_{k\in\mathbb{Z}^d}\tau_k\omega$ is a frame for W_0.*

Once we have a frame for W_0, standard methods can be used to construct an FMRA frame for all of $L^2(\widehat{\mathbb{R}}^d)$, e.g., [4, 5, 6, 7, 13, 15, 21, 22, 23, 28].

5. Measure-theoretic criterion for wavelets and a Mallat-Meyer type algorithm

The main idea of FMRAs is to apply the ideas of classical multiresolution analysis to contruct wavelet frames by means of a generalized Mallat-Meyer algorithm. Theorem 8 is a characterization of when such a construction is possible. As will be seen, this construction is not always guaranteed, but depends solely on the measure properties of a certain set which is intimately related to the spectrum of V_0. On the other hand, for the one dimensional case, H. O. Kim et al. construct two wavelets generating $L^2(\mathbb{R})$ independently of the measure of the aforementioned set [22].

In [6], Benedetto and Li applied the theory of FMRAs to the analysis of narrow band signals. Then, in [7], Benedetto and Treiber presented the main results of the theory of FMRAs from a functional analytic perspective. The proof of the main result in [7] gives a recipe for constructing wavelet frames when a natural measure theoretic criterion is satisfied, see Theorem 8. In this case, the construction in [7] can be extended to $\mathbb{R}^d$, for $d > 1$, by tensor products. We shall make this construction in section 6.

Theorem 8. *Suppose (V_j, φ) is an FMRA of $L^2(\mathbb{R}^d)$, and let $H_0 \in L^\infty(\mathbb{T}^d)$ have the property that $\widehat{\varphi}(2\gamma) = H_0(\gamma)\widehat{\varphi}(\gamma)$ a.e. Set*

$$\Gamma = \left\{\gamma \in \mathbb{T}^d : \Phi(2\gamma) = 0,\ \Phi\left(\gamma + \frac{u}{2}\right) > 0,\ u \in \{0,1\}^d\right\}.$$

Then, there is a set of wavelet functions $\omega = \{\psi_1, ..., \psi_m\} \subset W_0$, $m \leq 2^d - 1$, for which the translations of ω are a frame for W_0 if and only if $|\Gamma| = 0$.

Proof. (H. O. Kim, R. Y. Kim, J. K. Lim)

(a) We shall show that len $V_1 \leq 2^d$. We know that $V_0 = \overline{\operatorname{span}}\left\{\tau_k\varphi : k \in \mathbb{Z}^d\right\}$. Now, $V_1 = DV_0$ which implies that $V_1 = \overline{\operatorname{span}}\left\{D\tau_k\varphi : k \in \mathbb{Z}^d\right\}$. The relations $\forall k \in \mathbb{Z}^d$, $D\tau_{2k} = \tau_k D$ give us

$$D\tau_{2k+u}\varphi = D\tau_{2k}\tau_u\varphi = \tau_k D\tau_u\varphi = \tau_k\varphi_u,\ k \in \mathbb{Z}^d,\, u \in \{0,1\}^d.$$

Here we are using the fact that every $n \in \mathbb{Z}^d$ can be written uniquely in the form $2k + u$, with $k \in \mathbb{Z}^d$ and $u \in \{0,1\}^d$. Also, $\varphi_u = D\tau_u\varphi$. Therefore,

$$V_1 = \overline{\operatorname{span}}\left\{\tau_k\varphi_u : k \in \mathbb{Z}^d,\ u \in \{0,1\}^d\right\}.$$

since card $\{0,1\}^d$ is 2^d, len $V_1 \leq 2^d$.

(b) We shall show that len $V_0 \leq 1$. From $\varphi(2\gamma) = H_0(\gamma)\varphi(\gamma)$, we obtain

$$\mathcal{X}(V_0)(\gamma) = \operatorname{span}\left\{ \left(H_0\left(\frac{1}{2}(\gamma - k)\right)\widehat{\varphi}\left(\frac{1}{2}(\gamma - k)\right)\right)_{k\in\mathbb{Z}^d}\right\}$$

which implies that $\mathcal{X}(V_0)(\gamma)$ is at most one dimensional :

(c) The $-k$ th component of $\mathcal{X}(\varphi_u)(\gamma)$ is given by $2^{-\frac{d}{2}}e^{-2\pi i u\cdot\frac{1}{2}(\gamma-k)}\widehat{\varphi}\left(\frac{1}{2}(\gamma - k)\right)$ so that

$$\mathcal{X}(V_1)(\gamma) = \operatorname{span}\left\{ \left(e^{-2\pi i u\cdot\frac{1}{2}(\gamma-k)}\widehat{\varphi}\left(\frac{1}{2}(\gamma - k)\right)\right)_{k\in\mathbb{Z}^d} : u \in \{0,1\}^d\right\}.$$

If we compute,

$$\begin{aligned}(\mathcal{X}(\varphi_u)(\gamma))_{-k} &= \widehat{\varphi}_u(\gamma-k) = \widehat{D\tau_u\varphi}(\gamma-k) = \int 2^{\frac{d}{2}}\varphi(2x-u)\,e^{-2\pi i x\cdot(\gamma-k)}dx \\ &= \int 2^{-\frac{d}{2}}\varphi(y)\,e^{-2\pi i\frac{1}{2}(y+u)\cdot(\gamma-k)}dy = \int 2^{-\frac{d}{2}}\varphi(y)\,e^{-2\pi i\frac{1}{2}u\cdot(\gamma-k)}e^{-2\pi i\frac{1}{2}y\cdot(\gamma-k)}dy \\ &= 2^{-\frac{d}{2}}e^{-2\pi i\frac{1}{2}u\cdot(\gamma-k)}\int\varphi(y)\,e^{-2\pi i y\cdot\frac{1}{2}(\gamma-k)}dy = 2^{-\frac{d}{2}}e^{-2\pi i u\cdot\frac{1}{2}(\gamma-k)}\widehat{\varphi}\left(\frac{1}{2}(\gamma-k)\right).\end{aligned}$$

(d) We now compute $\sigma(V_1)$. For $\mathbf{c}\in l^2(\mathbb{Z}^d)$ define

$$\mathbf{c}_u(k) = \begin{cases}\mathbf{c}(k) & \text{if} \quad k = 2m+u,\ m\in\mathbb{Z}^d, \\ 0 & \text{otherwise.}\end{cases}$$

Two multi-integers m,n are congruent mod $\{0,1\}^d$ if they have the same multi-remainder $u\in\{0,1\}^d$, i.e., if $m=2k_1+u$ and $n=2k_2+u$ where $k_1,k_2\in\mathbb{Z}^d$ and $u\in\{0,1\}^d$. It follows that $\mathbf{c}=\sum_{u\in\{0,1\}^d}\mathbf{c}_u$ and

$$\langle\mathbf{c}_u,\mathbf{c}_v\rangle = \delta(u,v)\,\|\mathbf{c}_u\|^2,$$

where δ is the Kronecker delta. Hence,

$$\begin{aligned}\left(e^{-2\pi i u\cdot\frac{1}{2}(\gamma-k)}\widehat{\varphi}\left(\frac{1}{2}(\gamma-k)\right)\right)_{k\in\mathbb{Z}^d} &= \left(e^{\pi i u\cdot k}e^{-\pi i u\cdot\gamma}\widehat{\varphi}\left(\frac{1}{2}(\gamma-k)\right)\right)_{k\in\mathbb{Z}^d} \\ &= e^{-\pi i u\cdot\gamma}\sum_{v\in\{0,1\}^d}e^{\pi i u\cdot v}\mathbf{c}_{v,\gamma},\end{aligned}$$

where $\mathbf{c}_\gamma(k)=\widehat{\varphi}\left(\frac{1}{2}(\gamma-k)\right)$ and $\mathbf{c}_{v,\gamma}$ is defined as above (only the v-congruent entries survive). Therefore, $\mathcal{X}(V_1)(\gamma)=\operatorname{span}\left\{\mathbf{c}_{u,\gamma} : u\in\{0,1\}^d\right\}$ and

$$\sigma(V_1) = \left\{\gamma\in\mathbb{T}^d : \dim\mathcal{X}(V_1)(\gamma)>0\right\} = \left\{\gamma\in\mathbb{T}^d : \text{ at least one } \mathbf{c}_{u,\gamma}\neq 0\right\}.$$

On the other hand $\mathbf{c}_{u,\gamma}\neq 0$ implies that $\|\mathbf{c}_\gamma\|^2 = \sum_{u\in\{0,1\}^d}\|\mathbf{c}_{u,\gamma}\|^2 > 0$ by the Pythagorean theorem. We compute

$$\begin{aligned}\sigma(V_1) &= \left\{\gamma\in\mathbb{T}^d : \|\mathbf{c}_\gamma\|^2>0\right\} \\ &= \left\{\gamma\in\mathbb{T}^d : \sum_{k\in\mathbb{Z}^d}\left|\widehat{\varphi}\left(\frac{\gamma}{2}-\frac{2k+u}{2}\right)\right|^2>0,\ \text{for some } u\right\} \\ &= \left\{\gamma\in\mathbb{T}^d : \sum_{k\in\mathbb{Z}^d}\left|\widehat{\varphi}\left(\frac{\gamma}{2}-\frac{u}{2}+k\right)\right|^2>0,\ \text{for some } u\right\} \\ &= \left\{\gamma\in\mathbb{T}^d : \Phi\left(\frac{\gamma}{2}-\frac{u}{2}\right)>0,\ \text{for some } u\right\} \\ &= \left\{\gamma\in\mathbb{T}^d : \Phi\left(\frac{\gamma}{2}+\frac{u}{2}\right)>0,\ \text{for some } u\right\}.\end{aligned}$$

If we now define $H_{0,u}=H_0\left(\frac{\gamma}{2}-\frac{u}{2}\right)$, then,

$$\left(H_0\left(\frac{1}{2}(\gamma-k)\right)\widehat{\varphi}\left(\frac{1}{2}(\gamma-k)\right)\right)_{k\in\mathbb{Z}^d} = \sum_{u\in\{0,1\}^d}H_{0,u}\mathbf{c}_{u,\gamma}$$

because H_0 is 1-periodic in each variable. Hence,

$$\mathcal{X}(V_0)(\gamma) = \operatorname{span}\left\{ \sum_{u\in\{0,1\}^d} H_{0,u}\mathbf{c}_{u,\gamma} \right\} \subset \operatorname{span}\left\{\mathbf{c}_{u,\gamma} : u \in \{0,1\}^d\right\} = \mathcal{X}(V_1)(\gamma).$$

(*e*) $\Gamma = \frac{1}{2}\left\{\gamma \in \mathbb{T}^d : \dim \mathcal{X}(V_1)(\gamma) = 2^d \text{ and } \forall u \in \{0,1\}^d,\ H_{0,u} = 0\right\}$. If we define $\Gamma_j := \left\{\gamma \in \mathbb{T}^d : D_{V_1}(\gamma) = \dim \mathcal{X}(V_1)(\gamma) = j.\right\}, 1 \le j \le 2^d$, then,

$$\sigma(V_1) = \textstyle\bigcup_j \Gamma_j,$$

a disjoint union. If $\gamma \in \Gamma_{2^d}$ and $H_{0,u} = 0$ for every $u \in \{0,1\}^d$, then we have that $\mathcal{X}(V_0)(\gamma) = \{0\}$ and $\mathcal{X}(W_0)(\gamma) = \mathcal{X}(V_1)(\gamma)$. Hence, $D_{W_0}(\gamma) = \dim \mathcal{X}(W_0)(\gamma) = 2^d$, since $D_{V_1}(\gamma) = \dim \mathcal{X}(V_1)(\gamma) = 2^d$ $(\gamma \in \Gamma_{2^d})$. Now,

$$\begin{aligned}
\Theta &= \left\{\gamma \in \Gamma_{2^d} : \forall u \in \{0,1\}^d,\ H_{0,u} = 0\right\} \\
&= \left\{\gamma \in \Gamma_{2^d} : \forall u \in \{0,1\}^d,\ H_0\left(\frac{\gamma}{2} - \frac{u}{2}\right) = 0\right\} \\
&= \left\{\gamma \in \Gamma_{2^d} : \forall u \in \{0,1\}^d,\ H_0\left(\frac{\gamma}{2} + \frac{u}{2}\right) = 0\right\} \\
&= \left\{\gamma \in \mathbb{T}^d : \forall u \in \{0,1\}^d,\ \Phi(\gamma) = 0, \Phi\left(\frac{\gamma}{2} + \frac{u}{2}\right) > 0\right\} \\
&= \left\{2\lambda \in \mathbb{T}^d : \forall u \in \{0,1\}^d,\ \Phi(2\lambda) = 0, \Phi\left(\lambda + \frac{u}{2}\right) > 0\right\} = 2\Gamma.
\end{aligned}$$

Hence, $|\Theta| = 2^d|\Gamma|$, which implies that $|\Theta| > 0$ if and only if $|\Gamma| > 0$. It is now clear that if $|\Gamma| > 0$, then $D_{W_0}(\gamma) = 2^d$ in a subset of $\mathbb{T}^d$ with positive measure. Further, because of the way Θ is defined, *at least* 2^d *wavelets are necessary*, since in this case, len $W_0 = 2^d$. Now, applyng Theorems 1, 2, and 3 concerning shift-invariant subspaces, we obtain the result. □

Remark 3. *The equation (4.6),*

$$\Phi(2\gamma) = \sum_{u\in\{0,1\}^d} |H_0|^2\left(\gamma + \frac{u}{2}\right)\Phi\left(\gamma + \frac{u}{2}\right),$$

is true a.e. The left side of this equation is zero in Γ, *and* $\Phi\left(\gamma + \frac{u}{2}\right) > 0$ *in* Γ. *Hence,* $H_0\left(\gamma + \frac{u}{2}\right) = 0$, *i.e.,*

$$\Gamma \subset \left\{\gamma \in \mathbb{T}^d : \forall u \in \{0,1\}^d,\ H_0\left(\gamma + \frac{u}{2}\right) = 0\right\}.$$

In other words, Γ *can be seen as a subset of the zero set of the low pass filter* H_0 *with the additional geometric condition that its elements* γ *also have the property that* $\gamma + \frac{u}{2}$ *is a zero of* H_0. *This observation can also be obtained from the proof of the previous theorem (see part (e) in the proof of Theorem 8).*

Because of the conclusions of Theorem 7 and Theorem 8, it seems reasonable to point out the distinction between these two results. Theorem 7 provides sufficient conditions in terms of equations (4.1) and (4.7) developed in section 4, that a finite sequence of elements is a frame of W_0. Theorem 8 provides necessary and sufficient conditions for the existence of such generators by a measure theoretic criterion. Theorem 8 is an existence theorem and and Theorem 7 can be view as part of a recipe to give explicitly the generators.

6. Tensors

The following calculation allows one to construct a frame of translates in higher dimensions by means of tensor products. It is a special case of a result found in [18].

Lemma 1. *Let $\{\tau_k\varphi\}_{k\in\mathbb{Z}}$ and $\{\tau_k\varphi'\}_{k\in\mathbb{Z}}$ be two frames for $L^2(\mathbb{R})$ with frame bounds $0<A\le B<\infty$ and $0<A'\le B'<\infty$ respectively, then $\{\tau_m(\varphi\otimes\varphi')\}_{m\in\mathbb{Z}^2}$ is a frame for $L^2(\mathbb{R}^2)$ with frame bounds AA' and BB'.*

Proof. Consider a function of the form $f=\sum_{i=1}^n \alpha_i \mathbf{1}_{A_i\times B_i}$, where the $A_i\times B_i$ are disjoint measurable rectangles. The set of functions of this form is dense in $L^2(\mathbb{R}^2)$. Now, lets compute:

$$\sum_{m\in\mathbb{Z}^2}\left|\langle f,\tau_m(\varphi\otimes\varphi')\rangle\right|^2=$$

$$\sum_{k\in\mathbb{Z}^2}\left|\iint_{\mathbb{R}^2} f(x,y)\,\tau_k(\varphi\otimes\varphi')(x,y)\,dxdy\right|^2=$$

$$\sum_{\mathbf{k}\in\mathbb{Z}^2}\left|\iint_{\mathbb{R}^2}\sum_{i=1}^n \alpha_i\mathbf{1}_{A_i\times B_i}(x,y)\,\tau_m(\varphi\otimes\varphi')(x,y)\,dxdy\right|^2=$$

$$\sum_{m\in\mathbb{Z}^2}\sum_{i=1}^n\left|\iint_{A_i\times B_i}\alpha_i\mathbf{1}_{A_i\times B_i}(x,y)\,\tau_m(\varphi\otimes\varphi')(x,y)\,dxdy\right|^2=$$

$$\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}}\sum_{i=1}^n\left|\iint_{A_i\times B_i}\alpha_i\mathbf{1}_{A_i}(x)\,\mathbf{1}_{B_i}(y)\,\varphi(x-k_1)\,\varphi'(y-k_2)\,dxdy\right|^2=$$

$$\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}}\sum_{i=1}^n|\alpha_i|^2\left|\iint_{A_i\times B_i}\mathbf{1}_{A_i}(x)\,\mathbf{1}_{B_i}(y)\,\varphi(x-k_1)\,\varphi'(y-k_2)\,dxdy\right|^2=$$

$$\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}}\sum_{i=1}^n|\alpha_i|^2\left|\int_{B_i}\mathbf{1}_{B_i}(y)\,\varphi'(y-k_2)\,dy\int_{A_i}\mathbf{1}_{A_i}(x)\,\varphi(x-k_1)\,dx\right|^2=$$

$$\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}}\sum_{i=1}^n|\alpha_i|^2\left|\int_{B_i}\mathbf{1}_{B_i}(y)\,\varphi'(y-k_2)\,dy\right|^2\left|\int_{A_i}\mathbf{1}_{A_i}(x)\,\varphi(x-k_1)\,dx\right|^2=$$

$$\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}}\sum_{i=1}^n|\alpha_i|^2\left|\langle\mathbf{1}_{B_i},\tau_{k_2}\varphi'\rangle\right|^2\left|\langle\mathbf{1}_{A_i},\tau_{k_1}\varphi\rangle\right|^2=$$

$$\sum_{i=1}^n|\alpha_i|^2\sum_{k_2\in\mathbb{Z}}\left|\langle\mathbf{1}_{B_i},\tau_{k_2}\varphi'\rangle\right|^2\sum_{k_1\in\mathbb{Z}}\left|\langle\mathbf{1}_{A_i},\tau_{k_1}\varphi\rangle\right|^2$$

Now, $A'|B_i|\le\sum_{k_2\in\mathbb{Z}}\left|\langle\mathbf{1}_{B_i},\tau_{k_2}\varphi'\rangle\right|^2\le B'|B_i|$ and $A|A_i|\le\sum_{k_1\in\mathbb{Z}}\left|\langle\mathbf{1}_{B_i},\tau_{k_2}\varphi\rangle\right|^2\le B|A_i|$; hence,

$$AA'\sum_{i=1}^n|\alpha_i|^2|A_i||B_i|\le\sum_{i=1}^n|\alpha_i|^2\left[\sum_{k_2\in\mathbb{Z}}\left|\langle\mathbf{1}_{B_i},\tau_{k_2}\varphi'\rangle\right|^2\sum_{k_1\in\mathbb{Z}}\left|\langle\mathbf{1}_{A_i},\tau_{k_1}\varphi\rangle\right|^2\right]\le BB'\sum_{i=1}^n|\alpha_i|^2|A_i||B_i|$$

that is,

$$AA'\|f\|^2\le\sum_{m\in\mathbb{Z}^2}\left|\langle f,\tau_m(\varphi\otimes\varphi')\rangle\right|^2\le BB'\|f\|^2.$$

Hence, the result follows from the fact that these inequalities are satisfied on a dense subset of $L^2\left(\mathbb{R}^2\right)$. By induction, we can extend this argument to $\mathbb{R}^d$, $d \geq 2$. $\square$

6.1. **The Algorithm.** Following Lemma 1, we shall construct FMRA wavelets by tensor products in the same way it is done for the classical MRA case. First, assume that (V_j, φ) is an FMRA of $L^2(\mathbb{R})$. Assume the set Γ defined by

$$\Gamma = \left\{\gamma \in \mathbb{T} : \Phi(2\gamma) = 0,\ \Phi(\gamma) > 0,\ \Phi\left(\gamma + \frac{1}{2}\right) > 0\right\}$$

has measure zero, and the wavelet ψ is given by

$$\widehat{\psi}(2\gamma) = H_1(\gamma)\,\widehat{\varphi}(\gamma),$$

where $H_1(\gamma)$ is defined by

$$H_1(\gamma) = \begin{cases} e^{-2\pi i \gamma}\overline{H_0}\left(\gamma + \frac{1}{2}\right)\Phi\left(\gamma + \frac{1}{2}\right) & \text{if} \quad \gamma \in \Delta_2, \\ 1 & \text{if} \quad \gamma \in \Delta_3 \text{ and } H_0(\gamma) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The sets Δ_j, $j = 1, 2, 3, 4$, are

$$\Delta_1 = \left\{\gamma \in \mathbb{T} : \Phi(\gamma) = 0,\ \Phi\left(\gamma + \frac{1}{2}\right) = 0\right\},$$
$$\Delta_2 = \left\{\gamma \in \mathbb{T} : \Phi(\gamma) > 0 \text{ and } \Phi\left(\gamma + \frac{1}{2}\right) > 0\right\},$$
$$\Delta_3 = \left\{\gamma \in \mathbb{T} : \Phi(\gamma) > 0 \text{ and } \Phi\left(\gamma + \frac{1}{2}\right) = 0\right\},$$
$$\Delta_4 = \left\{\gamma \in \mathbb{T} : \Phi(\gamma) = 0 \text{ and } \Phi\left(\gamma + \frac{1}{2}\right) > 0\right\},$$

and they form a partition of $\mathbb{T}$, see [7, 13]. We now define the d-fold tensor product $V_1^{(d)} = \bigotimes_d V_1$ associated with the given FMRA of $L^2(\mathbb{R})$. Recall that

$$V_1 = V_0 \bigoplus W_0 \subset L^2(\mathbb{R}).$$

Hence, if we set $X_0 = V_0$, $X_1 = W_0$, and

$$X_\nu = X_{v_1} \bigotimes X_{v_2} \bigotimes \ldots \bigotimes X_{\nu_1},$$

for $\nu = (\nu_1, \ldots, \nu_d) \in \{0, 1\}^d$, we have

$$\bigotimes_d V_1 = \bigoplus_{\nu \in \{0,1\}^d} X_\nu.$$

With the convention that $\psi_0 = \varphi$, denote

$$\psi_\nu(x_1, \ldots, x_d) = \psi_{\nu_1}(x_1)\,\psi_{\nu_2}(x_2) \ldots \psi_{\nu_d}(x_d).$$

By Lemma 1, $\{\tau_k \psi_\nu\}_{k \in \mathbb{Z}^d}$ is a frame for X_ν, so that $\{\tau_k \psi_\nu\}_{k \in \mathbb{Z}^d, \nu \in \{0,1\}^d}$ is a frame for $V_1^{(d)}$.

In order to write these wavelets as a Mallat-Meyer algorithm, we compute $\Phi^{(d)}$ obtained by the periodization of the square of the modulus of $\widehat{\varphi}(\gamma_1)\ldots\widehat{\varphi}(\gamma_d)$:

$$\begin{aligned}\Phi^{(d)}(\gamma_1,\ldots,\gamma_d) &= \sum_{(k_1,\ldots,k_d)\in\mathbb{Z}^d} \left|\widehat{\varphi}(\gamma_1+k_1)\ldots\widehat{\varphi}(\gamma_d+k_d)\right|^2 \\ &= \sum_{(k_1,\ldots,k_d)\in\mathbb{Z}^d} \left|\widehat{\varphi}(\gamma_1+k_1)\right|^2\ldots\left|\widehat{\varphi}(\gamma_d+k_d)\right|^2 \\ &= \sum_{k_1\in\mathbb{Z}}\ldots\sum_{k_d\in\mathbb{Z}} \left|\widehat{\varphi}(\gamma_1+k_1)\right|^2\ldots\left|\widehat{\varphi}(\gamma_d+k_d)\right|^2 \\ &= \left[\sum_{k_1\in\mathbb{Z}} \left|\widehat{\varphi}(\gamma_1+k_1)\right|^2\right]\ldots\left[\sum_{k_d\in\mathbb{Z}} \left|\widehat{\varphi}(\gamma_d+k_d)\right|^2\right] \\ &= \Phi(\gamma_1)\ldots\Phi(\gamma_d).\end{aligned}$$

We then define Φ_ν by $\Phi_\nu(\gamma_1,\ldots,\gamma_d) = \Phi_{\nu_1}(\gamma_1)\ldots\Phi_{\nu_d}(\gamma_d)$, where

$$\Phi_{\nu_j}(\gamma_j) = \begin{cases} 1 & \text{if} \quad \nu_j = 0 \\ \Phi(\gamma_j) & \text{if} \quad \nu_j = 1, \end{cases}$$

and H^ν by $H^\nu(\gamma_1,\ldots,\gamma_d) = H^{\nu_1}(\gamma_1)\ldots H^{\nu_d}(\gamma_d)$, where

$$H^{\nu_j}(\gamma_j) = \begin{cases} \overline{H_0}(\gamma_j) & \text{if} \quad \nu_j = 1 \\ H_0(\gamma_j) & \text{if} \quad \nu_j = 0. \end{cases}$$

The sets $\Delta_j^{(d)}$, $j = 1,2,3,4$, for the tensor product, take the form

$$\begin{aligned}\Delta_1^{(d)} &= \left\{\gamma\in\mathbb{T}^d : \forall u\in\{0,1\}^d,\ \Phi_u\left(\gamma+\frac{u}{2}\right)=0\right\}, \\ \Delta_2^{(d)} &= \left\{\gamma\in\mathbb{T}^d : \Phi^{(d)}(\gamma)>0 \text{ and } \exists u\in\{0,1\}^d\setminus\{0\},\ \Phi_u\left(\gamma+\frac{u}{2}\right)>0\right\}, \\ \Delta_3^{(d)} &= \left\{\gamma\in\mathbb{T}^d : \Phi^{(d)}(\gamma)>0 \text{ and } \Phi\left(\gamma_1+\frac{1}{2}\right)=\ldots=\Phi\left(\gamma_d+\frac{1}{2}\right)=0\right\}, \\ \Delta_4^{(d)} &= \left\{\gamma\in\mathbb{T}^d : \Phi^{(d)}(\gamma)=0 \text{ and } \exists u\in\{0,1\}^d\setminus\{0\} \text{ with } \Phi_u\left(\gamma+\frac{u}{2}\right)>0\right\}.\end{aligned}$$

Moreover, the filters $H_\nu(\gamma_1,\ldots,\gamma_d) = H_{\nu_1}(\gamma_1)\ldots H_{\nu_d}(\gamma_d)$, for $\nu\neq(0,\ldots,0)$ are given by

$$H_\nu(\gamma) = \begin{cases} e^{-2\pi i\gamma\cdot\nu} H^\nu\left(\gamma+\frac{\nu}{2}\right)\Phi_\nu\left(\gamma+\frac{\nu}{2}\right) & \text{if} \quad \gamma\in\Delta_2, \\ \prod\limits_{\nu_p=0} H_0(\gamma_p) & \text{if} \quad \gamma\in\Delta_3,\ H^{\nu_j}(\gamma_j)=0, \nu_j=1, \\ 0 & \text{otherwise,} \end{cases}$$

where, by convention, the product $\prod_{\nu_p=0} H_0(\gamma_p) = 1$ in the case none of the ν_p is 0, i.e., $\nu = (1,\ldots,1)$. H_0 is the low pass filter on $\mathbb{T}$. Thus, the FMRA wavelets, ψ_ν, $\nu\neq(0,\ldots,0)$, defined above can now be formulated as in the Mallat-Meyer algorithm as follows:

$$\widehat{\psi_\nu}(2\gamma) = H_\nu(\gamma)\,\widehat{\varphi^{(d)}}(\gamma),$$

where $\varphi^{(d)}(x_1,\ldots,x_d) = \varphi(x_1)\ldots\varphi(x_d)$.

Remark 4. *This argument can be generalized to the case* $(V_j(1),\varphi_1),\ldots,(V_j(d),\varphi_d)$ *of* d *distinct FMRAs of* $L^2(\mathbb{R})$*. The idea is the same, but beginning with*

$$\Phi^{(d)}(\gamma_1,\ldots,\gamma_d) = \Phi_1(\gamma_1)\ldots\Phi_d(\gamma_d).$$

However, the notation becomes cumbersome. If one of the sets

$$\Gamma_j = \left\{\gamma \in \mathbb{T} : \Phi_j(2\gamma) = 0,\ \Phi_j(\gamma) > 0,\ \Phi_j\left(\gamma + \frac{1}{2}\right) > 0\right\}$$

has measure zero, then Theorem 8 says that a set of FMRA wavelets for $L^2(\mathbb{R}^d)$, with cardinality less than or equal $2^d - 1$, exists. Moreover, if Γ_k, $k \neq j$, has positive measure, then it is impossible to construct via tensor products a set of wavelet generators with cardinality less than or equal $2^d - 1$, since we need at least two wavelet generators for $W_0(k) = V_0(k)^{\perp} \cap V_1(k)$ by Theorem 8.

Example 1. *Let $\widehat{\varphi}(\gamma) = \mathbf{1}_{\left[-\frac{1}{4},\frac{1}{4}\right)}(\gamma)$. Then*

$H_0(\gamma) = \mathbf{1}_{\left[-\frac{1}{8},\frac{1}{8}\right)}(\gamma)$, $\Phi(\gamma) = \mathbf{1}_{\left[-\frac{1}{4},\frac{1}{4}\right)}(\gamma)$, *and* $H_1(\gamma) = \mathbf{1}_{\left[-\frac{1}{4},\frac{1}{4}\right)}(\gamma) - \mathbf{1}_{\left[-\frac{1}{8},\frac{1}{8}\right)}(\gamma)$.

This gives

$$\widehat{\psi} = \mathbf{1}_{\left[-\frac{1}{2},\frac{1}{2}\right)}(\gamma) - \mathbf{1}_{\left[-\frac{1}{4},\frac{1}{4}\right)}(\gamma).$$

Example 2. *Example 1 can be extended to any dimension. Let $Q = \left[-\frac{1}{4},\frac{1}{4}\right)^d$, the cube of volume $\left(\frac{1}{2}\right)^d$ center at the origin, and define*

$$\widehat{\varphi}(\gamma) = \mathbf{1}_Q(\gamma).$$

Then

$$H_0(\gamma) = \mathbf{1}_{\frac{1}{2}Q}(\gamma),\ H_1(\gamma) = \mathbf{1}_Q(\gamma) - \mathbf{1}_{\frac{1}{2}Q}(\gamma),$$

and

$$\widehat{\psi}(\gamma) = \mathbf{1}_{2Q}(\gamma) - \mathbf{1}_Q(\gamma).$$

Note that in this case Δ_2 has measure zero. In particular, if the support of $\widehat{\varphi}$ lies inside the cube centered at the origin and with volume $\left(\frac{1}{2}\right)^d$, the algorithm produces one wavelet. This is because the translations by the vectors $\frac{u}{2}$ of this cube intersect the others by either a vertex, an edge, or a face. All of these latter sets have measure zero. For perspective, in the classical MRA theory the required number of wavelet functions is $2^d - 1$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK, MARYLAND, 20742
E-mail address: jjb@math.umd.edu
URL: http://www.math.umd.edu/~jjb/

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK, MARYLAND, 20742
E-mail address: jrr@math.umd.edu