

# AN ENDPOINT $(1, \infty)$ BALIAN-LOW THEOREM

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ABSTRACT. It is shown that a  $(1, \infty)$  version of the Balian-Low Theorem holds. If  $g \in L^2(\mathbb{R})$ ,  $\Delta_1(g) < \infty$  and  $\Delta_\infty(\widehat{g}) < \infty$ , then the Gabor system  $\mathcal{G}(g, 1, 1)$  is not a Riesz basis for  $L^2(\mathbb{R})$ . Here,  $\Delta_1(g) = \int |t| |g(t)|^2 dt$  and  $\Delta_\infty(\widehat{g}) = \sup_{N>0} \int |\gamma|^{2N} |\widehat{g}(\gamma)|^2 d\gamma$ .

## 1. INTRODUCTION

Given a square integrable function  $g \in L^2(\mathbb{R})$ , and constants  $a, b > 0$ , the associated *Gabor system*,  $\mathcal{G}(g, a, b) = \{g_{m,n}\}_{m,n \in \mathbb{Z}}$ , is defined by

$$g_{m,n}(t) = e^{2\pi i a m t} g(t - b n).$$

Gabor systems provide effective signal decompositions in a variety of settings ranging from eigenvalue problems to applications in communications engineering. Background on the theory and applications of Gabor systems can be found in [16], [12], [13], [3].

We shall use the Fourier transform defined by  $\widehat{g}(\gamma) = \int g(t) e^{-2\pi i \gamma t} dt$ , where the integral is over  $\mathbb{R}$ . We let  $\mathbf{1}_S(t)$  denote the characteristic function of a set  $S \subseteq \mathbb{R}$ , and let  $S^c$  denote the complement of  $S \subseteq \mathbb{R}$ . Depending on the context,  $|\cdot|$  will denote either the Lebesgue measure of a set, or the modulus of a function or complex number.

The Balian-Low Theorem is a classical manifestation of the uncertainty principle for Gabor systems.

**Theorem 1.1** (Balian-Low). *Let  $g \in L^2(\mathbb{R})$ . If*

$$\int |t|^2 |g(t)|^2 dt < \infty \quad \text{and} \quad \int |\gamma|^2 |\widehat{g}(\gamma)|^2 d\gamma < \infty,$$

*then  $\mathcal{G}(g, 1, 1)$  is not an orthonormal basis for  $L^2(\mathbb{R})$ .*

The Balian-Low Theorem has a long history and some of the original references include [1], [19], [2]. The theorem still holds if “orthonormal basis” is replaced by “Riesz basis”. For this and other generalizations of the Balian-Low Theorem, we refer the reader to the survey articles [6], [9], as well as [4], [5], [7], [8], [10], [14], [17]. The issue of sharpness in the Balian-Low Theorem was investigated in [5], where the following was shown.

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**Theorem 1.2.** *If  $\frac{1}{p} + \frac{1}{q} = 1$ , where  $1 < p, q < \infty$ , and  $d > 2$ , then there exists a function  $g \in L^2(\mathbb{R})$  such that  $\mathcal{G}(g, 1, 1)$  is an orthonormal basis for  $L^2(\mathbb{R})$  and*

$$\int \frac{1 + |t|^p}{\log^d(2 + |t|)} |f(t)|^2 dt < \infty \quad \text{and} \quad \int \frac{1 + |\gamma|^q}{\log^d(2 + |\gamma|)} |\widehat{f}(\gamma)|^2 d\gamma < \infty.$$

When  $(p, q) = (2, 2)$ , this says that the Balian-Low Theorem no longer holds if the weights  $(t^2, \gamma^2)$  are weakened by appropriate logarithmic terms. In view of Theorem 1.2, it is also natural to ask if there exist versions of the Balian-Low Theorem for the general  $(p, q)$  case corresponding to the weights  $(t^p, \gamma^q)$ . The best that is known is the following.

**Theorem 1.3.** *Suppose  $\frac{1}{p} + \frac{1}{q} = 1$  with  $1 < p < \infty$  and let  $\epsilon > 0$ . If*

$$\int |t|^{(p+\epsilon)} |g(t)|^2 dt < \infty \quad \text{and} \quad \int |\gamma|^{(q+\epsilon)} |\widehat{g}(\gamma)|^2 d\gamma < \infty$$

*then  $\mathcal{G}(g, 1, 1)$  is not an orthonormal basis for  $L^2(\mathbb{R})$ .*

The above theorem follows by combining Theorem 4.4 of [11] and Theorem 1 in [15]. The  $\epsilon > 0$  can, of course, be removed in the case  $(p, q) = (2, 2)$ , by the Balian-Low Theorem.

This note shows the existence of a Balian-Low Theorem in the case  $(p, q) = (1, \infty)$ , and thus extends Theorems 1.1 and 1.3. To define what this means, let  $g \in L^2(\mathbb{R})$  and  $1 \leq p < \infty$  and set

$$\Delta_p(g) = \int |t|^p |g(t)|^2 dt \quad \text{and} \quad \Delta_\infty(g) = \sup_{N>0} \int |t|^N |g(t)|^2 dt.$$

With this notation, the classical Balian-Low Theorem says that if  $\Delta_2(g) < \infty$  and  $\Delta_2(\widehat{g}) < \infty$  then  $\mathcal{G}(g, 1, 1)$  is not an orthonormal basis for  $L^2(\mathbb{R})$ .

Our main result of this note is the following theorem.

**Theorem 1.4.** *Let  $g \in L^2(\mathbb{R})$  and suppose that  $\mathcal{G}(g, 1, 1)$  is a Riesz basis for  $L^2(\mathbb{R})$ . Then*

$$\Delta_1(g) = \infty \quad \text{or} \quad \Delta_\infty(\widehat{g}) = \infty.$$

This yields the following  $(1, \infty)$  version of the classical Balian-Low Theorem.

**Corollary 1.5.** *Let  $g \in L^2(\mathbb{R})$  and suppose*

$$\Delta_1(g) < \infty \quad \text{and} \quad \Delta_\infty(\widehat{g}) < \infty.$$

*Then  $\mathcal{G}(g, 1, 1)$  is not an orthonormal basis for  $L^2(\mathbb{R})$ .*

## 2. BACKGROUND

A collection  $\{e_n\}_{n \in \mathbb{Z}} \subseteq L^2(\mathbb{R})$  is a *frame* for  $L^2(\mathbb{R})$  if there exist constants  $0 < A \leq B < \infty$  such that

$$\forall f \in L^2(\mathbb{R}), \quad A \|f\|_{L^2(\mathbb{R})}^2 \leq \sum_{n \in \mathbb{Z}} |\langle f, e_n \rangle|^2 \leq B \|f\|_{L^2(\mathbb{R})}^2.$$

$A$  and  $B$  are the *frame constants* associated to the frame. If  $\{e_n\}_{n \in \mathbb{Z}}$  is a frame for  $L^2(\mathbb{R})$ , but is no longer a frame if any element is removed, then we say that  $\{e_n\}_{n \in \mathbb{Z}}$  is a *Riesz basis* for  $L^2(\mathbb{R})$ . Riesz bases are also known as *exact frames* or *bounded unconditional bases*, e.g., see [3]. The Zak transform is an important tool for studying Riesz bases given by Gabor systems.

Given  $g \in L^2(\mathbb{R})$ , the *Zak transform* is formally defined by

$$\forall (t, \gamma) \in Q \equiv [0, 1)^2, \quad Zg(t, \gamma) = \sum_{n \in \mathbb{Z}} g(t - n) e^{2\pi i n \gamma}.$$

This defines a unitary operator from  $L^2(\mathbb{R})$  to  $L^2(Q)$ . Further background on the Zak transform, as well as the next theorem, can be found in [3], [16].

**Theorem 2.1.** *Let  $g \in L^2(\mathbb{R})$ .  $\mathcal{G}(g, 1, 1)$  is a Riesz basis for  $L^2(\mathbb{R})$  with frame constants  $0 < A \leq B < \infty$  if and only if  $A \leq |Zg(t, \gamma)|^2 \leq B$  for a.e.  $(t, \gamma) \in Q$ .*

A function  $g \in L^2(\mathbb{R})$  is said to be in the *homogeneous Sobolev space* of order  $s > 0$ , denoted  $\dot{H}^s(\mathbb{R})$ , if  $\|g\|_{\dot{H}^s(\mathbb{R})}^2 \equiv \int |\gamma|^{2s} |\widehat{g}(\gamma)|^2 d\gamma < \infty$ . Since the condition  $\Delta_1(g) < \infty$  in Theorem 1.5 is equivalent to  $\widehat{g} \in \dot{H}^{1/2}(\mathbb{R})$ , we shall need some results on  $\dot{H}^{1/2}(\mathbb{R})$ . The following alternate characterization of  $\dot{H}^{1/2}(\mathbb{R})$  will be useful, e.g., [18].

**Theorem 2.2.** *If  $f \in \dot{H}^{1/2}(\mathbb{R})$  then*

$$\|f\|_{\dot{H}^{1/2}(\mathbb{R})}^2 = \frac{1}{4\pi^2} \int \int \frac{|f(x) - f(y)|^2}{|x - y|^2} dx dy.$$

Given  $f \in L^2(\mathbb{R})$ , the *symmetric-decreasing rearrangement*  $f^*$  of  $f$  is defined by

$$f^*(t) = \int_0^\infty \mathbf{1}_{S_x}(t) dx,$$

where  $S_x = (-s_x/2, s_x/2)$  and  $s_x = |\{t : |f(t)| > x\}|$ . An important property of a symmetric-decreasing rearrangement is that it decreases the  $\dot{H}^{1/2}(\mathbb{R})$  norm of functions, [18].

**Theorem 2.3.** *If  $f \in \dot{H}^{1/2}(\mathbb{R})$  then*

$$\|f\|_{\dot{H}^{1/2}(\mathbb{R})} \geq \|f^*\|_{\dot{H}^{1/2}(\mathbb{R})}.$$

This has the following useful corollary, [18].

**Corollary 2.4.** *If  $S \subset \mathbb{R}$  is a measurable set of positive and finite measure then  $\|\mathbf{1}_S\|_{\dot{H}^{1/2}(\mathbb{R})} = \infty$ .*

### 3. PROOF OF THE $(1, \infty)$ BALIAN-LOW THEOREM

The proof of Theorem 1.4 requires the following preliminary technical theorem.

**Theorem 3.1.** *Let  $f$  be a non-negative measurable function supported in the interval  $[-1, 1]$  and suppose that there exist constants  $0 < A \leq B < \infty$  such that*

$$(3.1) \quad A \leq |f(x) \pm f(x-1)| \leq B, \quad \text{a.e. } x \in [-1, 1].$$

*Then  $\|f\|_{\dot{H}^{1/2}(\mathbb{R})} = \infty$ .*

*Proof.* We begin by defining the measurable sets

$$\begin{aligned} S &= \{x \in [0, 1] : f(x-1) \leq f(x)\}, \\ T &= S^c \cap [0, 1] = \{y \in [0, 1] : f(y) < f(y-1)\}, \end{aligned}$$

and note that (3.1) implies

$$(3.2) \quad A \leq f(x) - f(x-1), \quad \text{a.e. } x \in S,$$

$$(3.3) \quad A \leq f(y-1) - f(y), \quad \text{a.e. } y \in T.$$

We break up the proof into two cases depending on whether or not  $S$  is a proper non-trivial subset of  $[0, 1]$ .

*Case I.* We shall first consider the case where

$$(3.4) \quad 0 < |S| < 1,$$

and hence that  $0 < |T| < 1$ .

Define the following capacity type integral over the product set  $S \times T$ .

$$(3.5) \quad I = \int_S \int_T \frac{1}{|x-y|^2} dy dx.$$

Conditions (3.2) and (3.3) allow one to bound  $I$  in terms of the  $\dot{H}^{\frac{1}{2}}(\mathbb{R})$  norm of  $f$  as follows.

$$\begin{aligned} I &\leq \frac{1}{4A^2} \int_S \int_T \frac{|f(x) - f(x-1) + f(y-1) - f(y)|^2}{|x-y|^2} dy dx \\ &\leq \frac{1}{2A^2} \left( \int_S \int_T \frac{|f(x) - f(y)|^2}{|x-y|^2} dy dx + \int_S \int_T \frac{|f(y-1) - f(x-1)|^2}{|x-y|^2} dy dx \right) \\ &\leq \frac{1}{A^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|f(x) - f(y)|^2}{|x-y|^2} dy dx \\ &= \frac{4\pi^2}{A^2} \|f\|_{\dot{H}^{1/2}}^2. \end{aligned}$$

It therefore suffices to show that  $I = \infty$ .

Since by the Lebesgue differentiation theorem almost every point of  $T$  is a point of density, it follows from (3.4) that we may choose  $a \in (0, 1)$  such that  $a$  is a point of density of  $T$  which satisfies either

$$(3.6) \quad 0 < |S \cap [0, a]| < a$$

or

$$(3.7) \quad 0 < |S \cap [a, 1]| < 1 - a.$$

Without loss of generality, we assume (3.6). If (3.7) holds then our arguments proceed analogously; for example in the first subcase below we would symmetrize about  $x = 1$  instead of  $x = 0$ .

To estimate  $I$ , we shall proceed separately depending on whether  $\int_0^a \frac{\mathbf{1}_S(x)}{|x-a|} dx$  is finite or infinite.

*Subcase i.* Suppose  $\int_0^a \frac{\mathbf{1}_S(x)}{|x-a|} dx < \infty$ . It will be convenient to work with the following set

$$\tilde{S} = (S \cup (-S)) \cap [-a, a].$$

By (3.6) we have  $|\tilde{S}| = 2|S \cap [0, a]| \neq 0$ . It follows from Corollary 2.4 and the definition of  $\tilde{S}$  that

$$\begin{aligned} \tilde{I} &\equiv \int_{-a}^a \int_{-\infty}^{\infty} \frac{\mathbf{1}_{\tilde{S}}(x)\mathbf{1}_{(\tilde{S})^c}(y)}{|x-y|^2} dy dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\mathbf{1}_{\tilde{S}}(x)\mathbf{1}_{(\tilde{S})^c}(y)}{|x-y|^2} dy dx \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|\mathbf{1}_{\tilde{S}}(x) - \mathbf{1}_{\tilde{S}}(y)|^2}{|x-y|^2} dx dy \\ &= 4\pi^2 \|\mathbf{1}_{\tilde{S}}\|_{\dot{H}^{1/2}(\mathbb{R})}^2 = \infty. \end{aligned}$$

The symmetric definition of  $\tilde{S}$  implies that

$$(3.8) \quad \tilde{I} = 2 \int_0^a \int_{-\infty}^{\infty} \frac{\mathbf{1}_{\tilde{S}}(x)\mathbf{1}_{(\tilde{S})^c}(y)}{|x-y|^2} dy dx = 2(I_1 + I_2 + I_3),$$

where

$$\begin{aligned} I_1 &\equiv \int_0^a \int_{-\infty}^{-a} \frac{\mathbf{1}_{\tilde{S}}(x)\mathbf{1}_{(\tilde{S})^c}(y)}{|x-y|^2} dy dx \leq \int_0^a \int_{-\infty}^{-a} \frac{1}{|y|^2} dy dx < \infty, \\ I_2 &\equiv \int_0^a \int_a^{\infty} \frac{\mathbf{1}_{\tilde{S}}(x)\mathbf{1}_{(\tilde{S})^c}(y)}{|x-y|^2} dy dx \leq \int_0^a \frac{\mathbf{1}_S(x)}{|x-a|} dx < \infty, \\ I_3 &\equiv \int_0^a \int_{-a}^a \frac{\mathbf{1}_{\tilde{S}}(x)\mathbf{1}_{(\tilde{S})^c}(y)}{|x-y|^2} dy dx. \end{aligned}$$

A simple calculation for  $I_3$  shows that

$$(3.9) \quad I_3 = \int_0^a \int_0^a \frac{\mathbf{1}_{\tilde{S}}(x)\mathbf{1}_{(\tilde{S})^c}(y)}{|x-y|^2} dy dx + \int_0^a \int_0^a \frac{\mathbf{1}_{\tilde{S}}(x)\mathbf{1}_{(\tilde{S})^c}(y)}{|x+y|^2} dy dx \leq 2I,$$

where the inequality for the second term in the middle of (3.9) follows from the fact that  $|x-y| \leq |x+y|$  in the square  $[0, a] \times [0, a]$ .

It follows from (3.8) and (3.9) that

$$\infty = \tilde{I} \leq 2I_1 + 2I_2 + 4I.$$

Since  $I_1$  and  $I_2$  are finite, we have  $I = \infty$ , as desired.

*Subcase ii.* Suppose  $\int_0^a \frac{\mathbf{1}_S(x)}{|x-a|} dx = \infty$ . Define

$$I_D = \int_0^a \int_0^a \frac{\mathbf{1}_S(x)\mathbf{1}_T(y)}{|x-y|^2} \mathbf{1}_D(x, y) dy dx \leq I,$$

where  $D = \{(x, y) \in \mathbb{R}^2 : x < y\}$ . To compute a lower bound for  $I_D$  first note that since  $a$  is a point of density of  $T$ , there exists a sufficiently large constant  $0 < C < \infty$  such that

$$|a-x| \leq C |T \cap [x, a]|, \quad a.e. x \in [0, 1].$$

Therefore for a.e.  $x \in [0, a)$

$$\begin{aligned} \frac{1}{|a-x|} &\leq \frac{C|T \cap [x, a]|}{|a-x|^2} = C |T \cap [x, a]| \cdot \min_{y \in [x, a]} \left\{ \frac{1}{|x-y|^2} \right\} \\ &\leq C \int_x^a \frac{\mathbf{1}_T(y)}{|x-y|^2} dy. \end{aligned}$$

This implies that

$$\infty = \int_0^a \frac{\mathbf{1}_S(x)}{|a-x|} dx \leq C \int_0^a \int_x^a \frac{\mathbf{1}_S(x)\mathbf{1}_T(y)}{|x-y|^2} dy dx = CI_D,$$

and it follows that  $I_D = \infty$ , and hence  $I = \infty$ , as desired.

*Case II.* We conclude by addressing the cases where  $|S| = 0$  or  $|S| = 1$ . Without loss of generality we only consider  $|S| = 1$ , and hence assume that  $S = [0, 1]$  up to a set of measure zero. It follows from (3.2) and the positivity of  $f$  that

$$A \leq f(x), \quad \text{a.e. } x \in [0, 1].$$

This, together with the fact that  $f$  is supported in  $[-1, 1]$ , implies that

$$\begin{aligned} \infty &= \int_1^\infty \int_0^1 \frac{1}{|x-y|^2} dx dy \\ &\leq \frac{1}{A^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|f(x) - f(y)|^2}{|x-y|^2} dx dy \\ &= \frac{4\pi^2}{A^2} \|f\|_{\dot{H}^{1/2}(\mathbb{R})}^2, \end{aligned}$$

as desired. This completes the proof.  $\square$

We are now ready to prove Theorem 1.4.

*Proof of Theorem 1.4.* We proceed by contradiction. Assume that  $g \in L^2(\mathbb{R})$ , that  $\mathcal{G}(g, 1, 1)$  is a Riesz basis for  $L^2(\mathbb{R})$  with frame constants  $0 < A \leq B < \infty$ , and that  $\Delta_1(g) < \infty$  and  $\Delta_\infty(\hat{g}) < C < \infty$ , for some constant  $C$ .

By Theorem 2.1,

$$\sqrt{A} \leq |Zg(x, w)| \leq \sqrt{B} \quad \text{a.e. on } [0, 1]^2.$$

Since  $Z\hat{g}(x, w) = e^{2\pi i x w} Zg(-w, x)$  we have

$$\sqrt{A} \leq |Z\hat{g}(x, w)| \leq \sqrt{B} \quad \text{a.e. on } [0, 1]^2.$$

Next, the assumption  $\int |\gamma|^N |\hat{g}(\gamma)|^2 d\gamma < C$  for all  $N > 0$  implies that

$$\text{supp } \hat{g} \subseteq [-1, 1].$$

Thus, for  $(x, w) \in [0, 1]^2$ , we have

$$Z\hat{g}(x, w) = \sum_{n \in \mathbb{Z}} \hat{g}(x-n) e^{2\pi i n w} = \hat{g}(x) + \hat{g}(x-1) e^{2\pi i w},$$

so that we have

$$(3.10) \quad \sqrt{A} \leq |\hat{g}(x) + \hat{g}(x-1) e^{2\pi i w}| \leq \sqrt{B} \quad \text{for a.e. } (x, w) \in [0, 1]^2.$$

In particular, it follows that

$$\sqrt{A} \leq | |\hat{g}(x)| \pm |\hat{g}(x-1)| | \leq \sqrt{B}, \quad \text{for a.e. } x \in [0, 1].$$

It now follows from Theorem 3.1 that  $|\hat{g}| \notin \dot{H}^{1/2}(\mathbb{R})$ , which implies that  $\hat{g} \notin \dot{H}^{1/2}(\mathbb{R})$ . In other words,  $\Delta_1(g) = \|\hat{g}\|_{\dot{H}^{1/2}(\mathbb{R})}^2 = \infty$ . This contradiction completes the proof.  $\square$

Since orthonormal bases are Riesz bases with frame constants  $A = B = 1$ , Corollary 1.5 follows from Theorem 1.4.

## 4. FURTHER COMMENTS

1. Theorem 1.5 is sharp in the sense investigated in Theorem 1.2, see [5]. In fact, Theorem 1.5 no longer holds if one weakens the  $\Delta_1$  decay hypotheses by a certain logarithmic amount. For example, if  $d > 1$  and  $\widehat{g}(\gamma) = \mathbf{1}_{[0,1]}(\gamma)$  then  $\mathcal{G}(g, 1, 1)$  is an orthonormal basis for  $L^2(\mathbb{R})$ , and

$$\int \frac{|t|}{\log^d(|t|+2)} |g(t)|^2 dt < \infty \quad \text{and} \quad \sup_{N>0} \int |\gamma|^N |\widehat{g}(\gamma)|^2 d\gamma < \infty.$$

2. There are two noteworthy cases in which the proof of Theorem 1.4 can be significantly simplified. If one assumes that  $\mathcal{G}(g, 1, 1)$  is an orthonormal basis for  $L^2(\mathbb{R})$  then the frame constants satisfy  $A = B = 1$  and it follows from (3.10) that  $|\widehat{g}(x)| = \mathbf{1}_R(x)$  for some set  $R \subset \mathbb{R}$  of positive and finite measure. Corollary 2.4 completes the proof in this case. Likewise, if  $\mathcal{G}(g, 1, 1)$  is a Riesz basis for  $L^2(\mathbb{R})$  whose frame bounds  $A$  and  $B$  are sufficiently close to one another, e.g.,  $\sqrt{B} < 3\sqrt{A}$ , then a direct argument involving Theorem 2.2 and Theorem 2.3 completes the proof. The main difficulty in Theorem 1.4 and Theorem 3.1 arises when the frame constants  $A$  and  $B$  are far apart.

3. We conclude by noting that if one weakens the hypotheses in Theorem 1.4 to  $\Delta_\infty(\widehat{g}) < \infty$  and  $\Delta_{1+\epsilon}(g) < \infty$ , for some  $\epsilon > 0$ , then the result is a simple consequence of the *Amalgam Balian-Low Theorem*. The Amalgam Balian-Low Theorem, e.g., [6], states that if  $\mathcal{G}(g, 1, 1)$  is a Riesz basis for  $L^2(\mathbb{R})$  then

$$g \notin W(C_0, l^1) \quad \text{and} \quad \widehat{g} \notin W(C_0, l^1),$$

where

$$W(C_0, l^1) = \{f : f \text{ is continuous and } \sum_{k \in \mathbb{Z}} \|f \mathbf{1}_{[k, k+1)}\|_{L^\infty(\mathbb{R})} < \infty\}.$$

The assumptions  $\Delta_{1+\epsilon}(g) < \infty$  and  $\Delta_\infty(\widehat{g}) < \infty$  imply that  $\widehat{g}$  is continuous and supported in  $[-1, 1]$ , which, in turn, implies that  $\widehat{g} \in W(C_0, l^1)$ .

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