Brief review of basic notions
K-Theory

Topological K–theory extends to C^*–algebras:

- even: equivalence classes of idempotents

  $p^2 = p \in M_\infty(A) := \lim_n M_n(A)$ with addition:

  $[p] + [q] := \left[\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \right]$

  $K_0(A) :=$ Grothendieck group;

- odd: $K_1(A) := \pi_0(\mathrm{GL}_\infty(A))$

- $K_i(A) := \pi_{i-1}(\mathrm{GL}_\infty(A)), \quad i \geq 2,$

- Bott Periodicity: $K_i(A) \cong K_{i+2}(A)$.

K–homology

- Atiyah (1969) K_*-**cycle** on manifold $= (A, \mathfrak{H}, F)$, where $A = C(M)$ acting on $\mathfrak{H} = L^2(M, E)$ by multiplication, and $F : L^2(M, E) \to L^2(M, E)$ is a 0-order pseudodifferential *elliptic* operator:

$$\text{(1)} \qquad F = F^*, \quad F^2 - \text{Id} \text{ compact}$$
$$\text{(2)} \qquad [F, \pi(a)] = F\,\pi(a) - \pi(a)\,F \text{ compact} \quad \forall\, a \in A.$$

- Baum-Douglas (1980): equivalence relation for K-cycles.

- Brown-Douglas-Fillmore (1973), Kasparov (1975): *K_*-theory for (unital) C^*-algebra A*, based on the notion of Fredholm module $(A, \mathfrak{H}, F)$ satisfying (1), (2).

Index Pairing and Cyclic cohomology

- Connes (1981) $F^2 = \text{Id}$, $[F,a] \in \mathcal{L}^p$ even:
  $\forall\, e^2 = e \in \mathcal{M}_\infty(\mathcal{A})\,, \qquad n > p$,

  $$\langle [F],\, [e] \rangle := \text{Index}\left(e\, F^+\, e\right) = (-1)^{\frac{n}{2}}\ \text{Tr}\,(\gamma e\, [F,e]^n)$$

  odd: $\forall\, g \in GL_\infty(\mathcal{A})\,, \quad E = \frac{F+\text{Id}}{2}, \quad n > p$,

  $$\langle [F],\, [g] \rangle := \text{Index}\,(E\, g\, E) = \frac{1}{2^n}\,\text{Tr}\left((g\,[F, g^{-1}])^n\right)$$

- With $da = i[F,a]$ viewed as a *quantized differential*, the formula is reminiscent of Chern-Weil theory.

- In polarized form,

  $$\tau_F(a^0, a^1, \ldots, a^n) := c_n\ \text{Tr}\left(\gamma\, a^0[F,a^1] \ldots [F,a^n]\right),$$

  defines the *Connes-Chern character*

  $$ch^*(\mathfrak{H}, F) \in HC^*(\mathcal{A}).$$

- **Hochschild coboundary** $HH^*(\mathcal{A}) \equiv H^*(\mathcal{A}, \mathcal{A}^*)$

$$b\phi(a^0, a^1, \ldots, a^{n+1}) := \sum_{j=0}^{n} (-1)^j \phi(a^0, \ldots, a^j a^{j+1}, \ldots, a^{n+1}) + (-1)^{n+1} \phi(a^{n+1} a^0, a^1, \ldots, a^n).$$

- **Cyclic cohomology** $HC^n(\mathcal{A})$ = cohomology of cyclic subcomplex satisfying

$$\phi\,(a^0, a^1, ..., a^n) = (-1)^n \phi(a^n, a^0, ..., a^{n-1})\,;$$

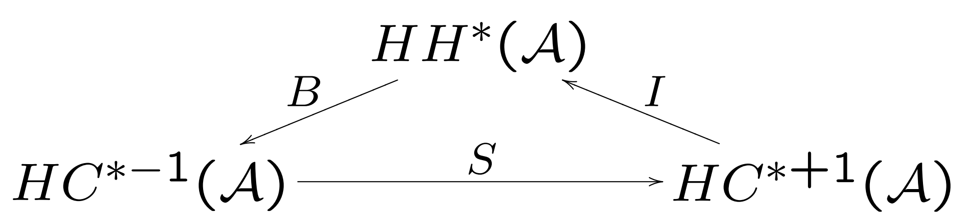
- or $HC^n(\mathcal{A})$ = the cohomology of the total complex for the (b, B)-bicomplex $C^n(\mathcal{A}) = \left(\mathcal{A} \otimes (\mathcal{A}/\mathbb{C})^{\otimes n}\right)^*$, $B^2 = b^2 = 0$, $Bb + bB = 0$,

$$B\phi(a_0, \ldots, a_{n-1}) := \sum_{j=0}^{n-1} (-1)^{(n-1)j} \phi(1, a_j, \ldots, a_{n-1}, a_0, \ldots, a_{j-1}).$$

- **In categorical terms**, $HC^n(\mathcal{A})$ = the cohomology of the Λ-module

$$\{C^n(\mathcal{A}) = \left(\mathcal{A}^{\otimes^{n+1}}\right)^*, \delta_i^n, \sigma_i^n, \tau_n\}_n \geq 0\,,$$
$$\tau_n\phi\,(a^0, a^1, ..., a^n) \;:=\; \phi(a^n, a^0, ..., a^{n-1})$$
$$\Lambda = C \cdot \Delta \qquad \text{cyclic category}$$

- **Exact triangle:**



- **Cohomological Pairing Formula:**

$$K^*(A) \otimes K_*(A) \to HC^*(\mathcal{A}) \otimes HC_*(\mathcal{A})$$
$$\langle [F],\, [c] \rangle \;=\; \langle ch^*(\mathfrak{H}, F), ch_*[c] \rangle.$$

Transgression du caractère de Chern et cohomologie cyclique

[A. Connes + H.M., C.R.A.S. Paris **303** (1986)]

Spectral triple (p-summable): $(\mathcal{A}, \mathfrak{H}, D)$

$\mathcal{A} \subset \mathcal{L}(\mathfrak{H})$ unital $*$ −algebra
$D = D^*$ unbounded ($F =$ Sign D)
D^{-1} or $(D^2 + 1)^{-\frac{1}{2}} \in \mathcal{L}^p(\mathfrak{H})$
$[D, a] := D\,a - a\,D$ bounded $\forall\, a \in \mathcal{A}$.

Auxiliary norm:

$$\|a\|_D := \|a\| + \|[D, a]\|$$

Geometric paradigm = Dirac operator on closed Riemannian spin manifold (M, g):

$$\Big(\mathcal{A} := C^\infty(M),\ \mathfrak{H} := \mathcal{L}^2(\mathcal{S}),\ D = \not{D}\Big).$$

Space of vector potentials $\mathcal{V} := \{A = A^* \mid A = \sum_{i=1}^{r} a_i[D, b_i]\,,\ a_i, b_i \in \mathcal{A}\} \cong \{c \in \mathcal{A} \otimes \mathcal{A} \mid c = \sum_{i=1}^{r} a_i \otimes b_i, \sum_{i=1}^{r} a_i\, b_i = 0, \sum_{i=1}^{r} a_i \otimes b_i = \sum_{i=1}^{r} b_i^* \otimes a_i^*\}$.

- $\forall\, A \in \mathcal{V}$, $D_A = D + A$, D_A is selfadjoint and $\dim \operatorname{Ker} D_A < \infty$;
- If $\operatorname{Ker} D_A = 0$, then D_A is invertible and $D_A^{-1} \in \mathcal{L}^p(\mathfrak{H})$;
- $\mathcal{V}^\times = \{A \in \mathcal{V}\,; \operatorname{Ker} D_A = 0\}$ is open in $\mathcal{V}$.

Group of gauge transformations $\mathcal{U} \equiv \mathcal{U}(\mathcal{A}) := \{u \in \mathcal{A}\,;\ u^*u = uu^* = 1\}$, with affine action

$$\nu_u(A) = u\,[D, u^*] + uAu^*\,;$$
$$D_{\nu_u(A)} = u\, D_A\, u^*, \quad u \in \mathcal{U},\ A \in \mathcal{V}.$$

Hilbertian vector bundle with connection

- Trivial bundle $\mathcal{H} = \mathfrak{H} \times \mathcal{V}$ with trivial flat connection d, $\forall \xi \in C^\infty(\mathcal{V}, \mathfrak{H})$,

$$(d\xi)_A(X) = (\mathbf{X}\,\xi)(A) \equiv \frac{d}{dt}\bigg|_{t=0} \xi(A + tX).$$

- $\mathcal{L}^{\wedge k}(\mathcal{V}, \mathfrak{H})$= Banach space of norm continuous k-multilinear alternating maps;

- $\Omega^\bullet(\mathcal{V}, \mathcal{H})$= smooth maps from $\mathcal{V}$ to $\mathcal{L}^{\wedge \bullet}(\mathcal{V}, \mathfrak{H})$, = graded right module over $\Omega^\bullet(\mathcal{V})$;

- $\forall \omega \in \Omega^r(\mathcal{V}, \mathcal{H})$,

$$d\omega \quad (\mathbf{X}^0, \ldots, \mathbf{X}^r) = \sum_{i=0}^{r} (-1)^i\, \mathbf{X}^i\, \omega(\mathbf{X}^0, \ldots, \widehat{\mathbf{X}}^i, \ldots, \mathbf{X}^r).$$

Superconnection $\nabla := \gamma d + \mathcal{D}$,
where $\mathcal{D} \in \text{End}(\mathcal{H})$ is $\mathcal{D}_A := D_A = D + A$.

- $\mathcal{D}' := [\gamma d, \mathcal{D}] \equiv \gamma d \circ \mathcal{D} + \mathcal{D} \circ \gamma d \in \text{End}\, \Omega^*(\mathcal{V}, \mathcal{H})$

$$(\mathcal{D}'\omega)_A(X^0, \dots, X^r) =$$
$$\gamma \sum_{i=0}^{r} (-1)^i X^i \Big(\omega_A(X^0, \dots, \widehat{X}^i, \dots, X^r) \Big);$$

- with $\nabla_z := \gamma d + z\mathcal{D}$, one has $\forall z \in \mathbb{C}$

$$\nabla_z^2 = z\mathcal{D}' + z^2 \mathcal{D}^2 \in \text{End}\, \Omega^*(\mathcal{V}, \mathcal{H}).$$

Then $e^{-\nabla_z^2} = \bigoplus_{n \geq 0} \Omega_z^{(n)}$, where

$$\Omega_z^{(n)} = z^n \int_{\Delta_n} e^{-s_1 z^2 \mathcal{D}^2} \mathcal{D}' \, e^{(s_1 - s_2) z^2 \mathcal{D}^2} \mathcal{D}'$$
$$\dots e^{(s_{n-1} - s_n) z^2 \mathcal{D}^2} \mathcal{D}' \, e^{(s_n - 1) z^2 \mathcal{D}^2} \, ds,$$

by iterated Duhamel (Expansional) formula.

Quillen's superconnection formalism

Proposition 1. *For* $\mathrm{Re}(z^2) > 0$, *let*

$$\mathrm{Tr}\left(\gamma e^{-\nabla_z^2}\right) = \bigoplus_{n \geq 0} \omega_z^{(n)} \in \Omega^*(\mathcal{V}),$$

$$\omega_{z,A}^{(n)}(X^1, \ldots, X^n) = \mathrm{Tr}\left(\gamma \Omega_{z,A}^{(n)}(X^1, \ldots, X^n)\right).$$

- $\omega_z \in \Omega^{\mathrm{even}}(\mathcal{V})$ *is closed and* $\mathcal{U}$*-invariant;*

- *if* $\mathrm{Re}(z^2) \to \infty$, *then* $\omega_{z,A}^{(n)} \to 0$ *pointwise,* $\forall A \in \mathcal{V}^\times$*;*

- *if* $n \geq p$ *and* $0 < \mathrm{Re}(z^2) \to 0$, *then* $\Omega_{z,A}^{(n)} \to 0$ *pointwise,* $\forall A \in \mathcal{V}$.

Lemma 2. *Let* $A \in \mathcal{V}^\times$. *For any* $s \in [0, 1]$,

$$s \, \| D_A e^{-st^2 D_A^2} \|_{\frac{1}{s}} \leq C_p \, t^{-ps-1} \Big(\mathrm{Tr} \, |D_A|^{-p} \Big)^s,$$

with $$C_p = \sup_{s \in [0,1]} \left(s \left(\tfrac{1}{2e} \left(p + \tfrac{1}{s} \right) \right)^{\frac{1}{2} + \frac{ps}{2}} \right).$$

Transgression: $\frac{d}{dt}\omega_t = d\theta_t$, where

$$\mathrm{Tr}(\gamma e^{-(\gamma dt \mathcal{D} + t\mathcal{D}' + t^2\mathcal{D}^2)} =$$
$$\sum_{n\geq 1} t^{n-1} \int_{\Delta_{n-1}} \mathrm{Tr}(\gamma e^{-s_1(\gamma dt\mathcal{D}+t^2\mathcal{D}^2)}\, \mathcal{D}' \ldots$$
$$e^{-(s_{n-1}-s_{n-2})(\gamma dt\mathcal{D}+t^2\mathcal{D}^2)} \mathcal{D}' e^{-(1-s_{n-1})(\gamma dt\mathcal{D}+t^2\mathcal{D}^2)}\, ds$$
$$= (\mathrm{JLO})\sigma_t + \theta_t\, dt.$$

By Hölder's inequality, Lemma 2 and Proposition 1, $\forall\, A \in \mathcal{V}^\times$,

$$\|\theta_{t,A}^{(n-1)}\| = o\left(t^{n-p-2}\right), \quad \text{as} \quad 0 < t \to 0.$$

Theorem 3. *Let $m > p$ be an odd integer.*

$$\Theta_A^{(m)} = \int_0^\infty \theta_{t,A}^{(m)}\, dt$$

converges $\forall A \in \mathcal{V}^\times$ and defines a closed $\mathcal{U}$-invariant differential form on $\mathcal{V}^\times$.

Explicit expression: $\Theta_A^{(m)}(X^1, \dots, X^m) =$

$$= \sum_{\sigma \in S_m} \mathrm{sgn}(\sigma) \int_0^\infty \left(\int_{\Delta_m} \mathrm{Tr} \left(\gamma D_A e^{-s_0 t^2 D_A^2} X^{\sigma(1)} \right.\right.$$

$$\left.\left. e^{-s_1 t^2 D_A^2} \dots X^{\sigma(m)} \, e^{-s_m t^2 D_A^2} \right) ds \right) t^m \, dt \, .$$

To put in rational form, change variables

$$u_i = (s_i - s_{i-1}) \, t^2 \, ,$$

then use the Mellin transform

$$\left(\sum_{i=1}^{m+1} u_i \right)^{-\frac{m+1}{2}}$$

$$\Gamma \left(\frac{m+1}{2} \right)^{-1} \int_0^\infty s^{\frac{m+1}{2}} \exp \left(-s \sum_{i=1}^{m+1} u_i \right) \frac{ds}{s} \, ,$$

and also take $s = \mu^2$.

Rational expression: $\Theta_A^{(m)}(X^1, \ldots, X^m) =$

$$-\Gamma\left(\frac{m+1}{2}\right)^{-1} \sum_{\sigma \in S_m} \mathrm{sgn}(\sigma) \int_0^\infty \mathrm{Tr}\left(\gamma \frac{D_A}{D_A^2+\mu^2} X^{\sigma(1)} \frac{1}{D_A^2+\mu^2} \cdots X^{\sigma(m)} \frac{1}{D_A^2+\mu^2}\right) \mu^m \, d\mu$$

Inflation: $\forall q \in \mathbb{N}$, replace $(\mathcal{A}, \mathfrak{H}, D)$ by $(\mathcal{A} \otimes M_q(\mathbb{C}), \mathfrak{H} \otimes \mathbb{C}^q, D \otimes 1_q)$; the corresponding form is ${}_q\Theta^{(m)}_{A\otimes 1}(X^1 \otimes c^1, \ldots, X^m \otimes c^m) =$

$$-\Gamma\left(\frac{m+1}{2}\right)^{-1} \sum_\sigma \operatorname{sgn}(\sigma) \operatorname{Tr}\left(c^{\sigma(1)} \ldots c^{\sigma(m)}\right)$$
$$\int_0^\infty \operatorname{Tr}\left(\gamma \frac{D_A}{D_A^2+\mu^2} X^{\sigma(1)} \ldots X^{\sigma(m)} \frac{1}{D_A^2+\mu^2}\right) \mu^m \, d\mu.$$

Restriction of ${}_q\Theta^{(m)}$ to $\mathcal{U}_q := \mathcal{U}\Big(M_q(\mathcal{A})\Big)$ orbit of $A_q = A \otimes 1 =$ invariant form on $\mathcal{U}_q$; the tangent map at $1 \in \mathcal{U}_q$ to the map $u \to \nu_u(A_q) \in \mathcal{V}_q^\times$ is $a = -a^* \in M_q(\mathcal{A}) \to [a, D_{A_q}]$,

$$\phi^{(m)}_{A,q}(a^1 \otimes c^1, \ldots, a^m \otimes c^m) =$$
$$-\Gamma\left(\frac{m+1}{2}\right)^{-1} \sum_\sigma \operatorname{sgn}(\sigma) \operatorname{Tr}\left(c^{\sigma(1)} \ldots c^{\sigma(m)}\right)$$
$$\int_0^\infty \operatorname{Tr}\left(\gamma \frac{D_A}{D_A^2+\mu^2} [D_A, a^{\sigma(1)}] \frac{1}{D_A^2+\mu^2} \ldots \right.$$
$$\left. \ldots [D_A, a^{\sigma(m)}] \, \frac{1}{D_A^2+\mu^2}\right) \mu^m \, d\mu.$$

Dual of Loday-Quillen-Tsygan isomorphism

$\mathcal{U}_\infty(\mathcal{A}) := \varinjlim \mathcal{U}\Big(M_q(\mathcal{A})\Big), \quad U_\infty := \varinjlim \mathcal{U}\Big(M_q(\mathbb{C})\Big)$

$\mathfrak{gl}_{\mathbb{C}}\Big(\mathcal{U}_\infty(\mathcal{A})\Big) \cong M_\infty(\mathcal{A}) := \varinjlim M_q(\mathcal{A})$

$\{\Lambda^\bullet(M_\infty(\mathcal{A})^*)^{U_\infty} := \varprojlim \Lambda^\bullet(M_q(\mathcal{A})^*)^{U_q}, d\}$ = complex of U_∞-invariant and coprimitive continuous alternating multilinear forms on $M_\infty(\mathcal{A})$;

Coprimitive: $\Gamma^* \alpha = \iota_1^* \alpha + \iota_2^* \alpha$

where $\iota_1(a,b) = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, \quad \iota_2(a,b) = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix},$
and $\Gamma(a,b) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \quad a \in M_p(A), b \in M_q(A).$

$\{C_\lambda^\bullet(\mathcal{A}), b\}$= complex of continuous cyclic cochains on $\mathcal{A}$.

Theorem 4. [L-Q, Tsy]* The cochain map $\mathscr{I} : C^\bullet_\lambda(\mathcal{A}) \longrightarrow \Lambda^{\bullet+1}(M_\infty(\mathcal{A})^*)^{U_\infty}$ defined by

$$\mathscr{I}(\phi)(X^0, \ldots, X^m) = (-1)^m \sum_\sigma \mathrm{sgn}(\sigma)(\phi \otimes \mathrm{Tr})\Big(X^{\sigma(0)}, X^{\sigma(1)}, \ldots, X^{\sigma(m)}\Big)$$

is an isomorphism of complexes, with inverse

$$\mathscr{I}^{-1}(\alpha)(a^0, \ldots, a^m) = (-1)^m \alpha(a^0 \otimes E_{01},\, a^1 \otimes E_{12}, \ldots, a^m \otimes E_{m0}),$$

where $E_{ij} \in M_\infty(\mathbb{C})$ are elementary matrices.

Corollary 5. *The restricted forms* $\phi^{(\bullet)}_{A,q}$ *are* U_∞*-invariant and coprimitive. In particular*

$$\mathrm{Tch}_{n+1}(a^0, \ldots, a^n) = \Gamma\left(\frac{n}{2}+1\right)^{-1} \sum_{\lambda \in C_n} \mathrm{sgn}(\lambda) \int_0^\infty \mathrm{Tr}\left(\gamma \frac{D}{D^2+\mu^2}[D, a^{\lambda(0)}]\frac{1}{D^2+\mu^2} \cdots \right.$$
$$\left. \cdots [D, a^{\lambda(n)}]\, \frac{1}{D^2+\mu^2}\right)\mu^n \, d\mu$$

is a cyclic n*-cocycle* $\forall\, n > p-1$.

Theorem 6 (**Chern character**). *Let* $(\mathcal{A}, \mathfrak{H}, D)$ *be a* p*-summable spectral triple (with* D *invertible). For any odd (resp. even) integer* $n > p-1$*, the cyclic cocycle* Tch^{n+1} *represents the Chern character of the spectral triple. In particular* $\mathrm{Tch}^{n+3} = S\,\mathrm{Tch}^{n+1}$*.*

Proof. The construction works under a weaker hypothesis: $\exists p_1, p_2 \geq 1$, $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ such that $D^{-1} \in \mathcal{L}^{p_1}$ and $[D, a] \in \mathcal{L}^{p_2}$ for any $a \in \mathcal{A}$.

Lemma 7. *For* $\alpha \in [0, 1]$*, let* $D_\alpha = D\,|D|^{-\alpha}$*. If* $p' > p$*, then* $[D_\alpha, a] \in \mathcal{L}^{p'/\alpha}$ *for any* $a \in \mathcal{A}$*, and* $\|[D_\alpha, a]\|_{p'/\alpha} \leq C(p, p')\,\|[D, a]\|_\infty \Big(1 + \|D^{-1}\|_p^2\Big)$*.*

This allows to apply the construction of Tch on the total space of the bundle over $\mathcal{V} \times [0, 1]$, yielding a 1-parameter family of cochains τ_α, with $\tau_0 = \mathrm{Tch}$ and $\tau_1 =$ Chern character of $(\mathcal{A}, \mathfrak{H}, \mathrm{Sign}\, D)$. □

Transgression and the Chern character of finitely summable K-cycles

[A. Connes + H.M., CMP **155** (1993)]

Recall $\quad \mathrm{Tr}(\gamma e^{-(\gamma dt\mathcal{D}+t\mathcal{D}'+t^2\mathcal{D}^2)} = \sigma_t + \theta_t\, dt.$

The even part gives rise to **JLO cocycle**:

$$\mathrm{Ch}^k(D)(a_0,\cdots,a_k) = \langle a_0,[D,a_1],\cdots,[D,a_k]\rangle$$

$$\langle A_0,\cdots,A_k\rangle_D = \int_{\Delta_k} \mathrm{Str}\left(A_0\, e^{-s_0 D^2}\cdots A_k\, e^{-s_k D^2}\right) ds$$

$$b\,\mathrm{Ch}^{k-1}(D) + B\,\mathrm{Ch}^{k+1}(D) = 0,$$

giving the Connes-Chern character in *entire cyclic cohomology*, for $(\mathcal{A},\mathcal{H},D)$ satisfying

$$\mathrm{Tr}\, e^{-tD^2} < \infty, \qquad \forall\, t > 0.$$

$$\not{c}\mathsf{h}^k(D,\mathsf{V})(a_0,\cdots,a_k) = \sum_{1\le i\le k}(-1)^{\deg \mathsf{V}}\langle a_0,[D,a_1],\cdots,[D,a_i],\mathsf{V},[D,a_{i+1}],\cdots,[D,a_k]\rangle.$$

Transgression formula :

$$-\frac{d}{dt}\mathsf{Ch}^k(tD) = b\,\not{c}\mathsf{h}^{k-1}(tD,D) + B\,\not{c}\mathsf{h}^{k+1}(tD,D).$$

By integration on $[\varepsilon, t]$,

$$\mathsf{Ch}^k(\varepsilon D) - \mathsf{Ch}^k(tD) = b\int_\varepsilon^t \not{c}\mathsf{h}^{k-1}(sD,D)ds + B\int_\varepsilon^t \not{c}\mathsf{h}^{k+1}(sD,D)ds.$$

From now on $(\mathcal{A},\mathcal{H},D)$ is p-summable.

Lemma 8. *Then for $n > p$, and as $t \to 0^+$,*

$$\begin{aligned} \|\,\mathsf{Ch}^n(tD)\| &= O(t^{n-p}), \\ \|\,\not{c}\mathsf{h}^n(tD,D)\| &= O(t^{n-p-1}). \end{aligned}$$

Thus $\lim_{\varepsilon \searrow 0} \mathrm{Ch}^n(\varepsilon D) = 0$ for all $n > p$,

and

$$-\,\mathrm{Ch}^n(tD) = b\,\mathrm{T}\not{\mathrm{ch}}_t^{n-1}(D) + B\,\mathrm{T}\not{\mathrm{ch}}_t^{n+1}(D),$$

where

$$\mathrm{T}\not{\mathrm{ch}}_t^{n}(D) := \int_0^t \not{\mathrm{ch}}^n(sD, D)\,ds.$$

In particular,

$$b\Big(\mathrm{Ch}^n(tD) + B\,\mathrm{T}\not{\mathrm{ch}}_t^{n+1}(D)\Big) = 0.$$

Retracted cocycle : for $n > p$,

$$\mathrm{ch}_t^n(D) := \sum_{j\geq 0} \mathrm{Ch}^{n-2j}(tD) + B\,\mathrm{T}\not{\mathrm{ch}}_t^{n+1}(D)$$

satisfies $(b + B)\,\mathrm{ch}_t^n(D) = 0$. In addition,

$$\mathrm{ch}_t^{n+2}(D) - \mathrm{ch}_t^n(D) = -(b + B)\,\mathrm{T}\not{\mathrm{ch}}_t^{n+1}(D),$$

hence its periodic cyclic cohomology class is independent of $n > p$ and of $t > 0$.

Lemma 9. *For any $k \geqq 1$ and $a_0, \ldots, a_k \in \mathcal{A}$,*

$$\not{\mathrm{c}}\mathrm{h}^k(tD, D)(a_0, \ldots, a_k) = O(t^{-2}) \qquad \textit{as } t \to \infty\,;$$

if k is odd, one has in fact

$$\not{\mathrm{c}}\mathrm{h}^k(tD, D)(a_0, \ldots, a_k) = O(t^{-3}) \qquad \textit{as } t \to \infty\,.$$

$H =$ the orthogonal projection on $\operatorname{Ker} D$,
$\pi_H(a) = HaH\,, \quad \forall\, a \in \mathcal{A}$, and

$$\varpi_H(a, b) = \pi_H(ab) - \pi_H(a)\,\pi_H(b)\,, \quad a, b \in \mathcal{A}.$$

Lemma 10. *For $k \geqq 1$, and $a_0, \ldots, a_k \in \mathcal{A}$,*

$$\lim_{t \nearrow \infty} \mathrm{Ch}^{2k-1}(tD)(a_0, \ldots, a_k) \,=\, 0,$$

$$\lim_{t \nearrow \infty} \mathrm{Ch}^{2k}(tD)(a_0, \ldots, a_{2k}) \,=$$

$$\frac{(-1)^k}{k!}\,\mathrm{Str}(\pi_H(a_0)\,\varpi_H(a_1, a_2) \ldots \varpi_H(a_{2k-1}, a_{2k}))\,.$$

Theorem 11. *For* $n = 2m$, $0 \leqq k \leqq m$,

$$\mathrm{ch}^n_\infty(D)_{(2k)}(a_0, \ldots, a_{2k}) = \frac{(-1)^k}{k!} \mathrm{Str}(\varpi_H(a_0)\, \varpi_H(a_1, a_2) \ldots \varpi_H(a_{2k-1}, a_{2k});$$

$$\mathrm{ch}^n_\infty(D)^{\mathrm{add}}_{(2m)}(a_0, \ldots, a_{2m}) = \sum_{\lambda \in C_{2m}} \mathrm{sgn}(\lambda) \int_0^\infty \Big(\int_{\Delta_{2k+1}} \mathrm{Str}(D e^{-s_0 t^2 D^2} [D, a^{\lambda(0)}] e^{-(s_1 - s_0) t^2 D^2} \ldots [D, a^{\lambda(2m)}]\, e^{-(1-s_{2m}) t^2 D^2})\, ds \Big) t^{2m+1}\, dt.$$

For $n = 2m+1$, $\mathrm{ch}^n_\infty(D)(a_0, \ldots, a_{2m+1}) =$

$$\sum_{\lambda \in C_n} \mathrm{sgn}(\lambda) \int_0^\infty \Big(\int_{\Delta_{2k+2}} \mathrm{Tr}(D e^{-s_0 t^2 D^2} [D, a^{\lambda(0)}] e^{-(s_1 - s_0) t^2 D^2} \ldots [D, a^{\lambda(2k+1)}]\, e^{-(1-s_{2k+1}) t^2 D^2})\, ds \Big) t^{2k+2}\, dt\,.$$

gives a cocycle cohomologous to

$$(a^0, \ldots, a^n) \mapsto c_n\ \mathrm{Tr}\Big(\gamma F [F, a^0] \ldots [F, a^n] \Big).$$

ΨDO calculus for spectral triples

For $(\mathcal{A}, \mathcal{H}, D)$ spectral triple, let

$$\mathcal{H}^s := \text{Domain } (|D|^s), \quad s \in \mathbb{R}$$
$$\mathcal{H}^\infty = \cap_{s\geq 0}\mathcal{H}^s \quad , \quad \mathcal{H}^{-\infty} = \text{dual of } \mathcal{H}^\infty \ ,$$
$$op^r = \{T \in \mathcal{L}(\mathcal{H}^\infty) \,|\, T(\mathcal{H}^s) \subset \mathcal{H}^{s-r}, \forall s \in \mathbb{R}\}$$

Regularity assumption: $\forall a \in \mathcal{A}$, a and $[D, a]$ are in the domains of all powers of the derivation $\delta = [|D|, \cdot]$.

Lemma 12. *Let* $b = a$ or $[D, a]$, $a \in \mathcal{A}$. *Then*
(1) $b \in op^0$ and $b - |D|\, b\, |D|^{-1} \in op^{-1}$;
(2) $\left[D^2, \left[D^2, \ldots [D^2, b\underbrace{]\left] \cdots \right]}_{n}\right. \in op^n$.

Order filtration: $P \in OP^s$ iff

$$|D|^{-s} P \in \bigcap_{n\geq 0} \text{Dom}\, \delta^n \ .$$

Thus $OP^0 = \bigcap_{n\geq 0} \text{Dom}\, \delta^n$ and $OP^s \subset op^s$.

$\nabla(T) = [D^2, T]$, $\mathcal{D}$ = algebra generated by the operators $\nabla^n(T)$, $T \in \mathcal{A}$ or $[D, \mathcal{A}]$, filtered by the total power of ∇; $\mathcal{D}^n \subset OP^n$;

$$\sigma^{2z} = \Delta^z \cdot \Delta^{-z}, \qquad \Delta = D^2;$$

$$\sigma^2 = 1 + \rho, \qquad \rho(T) = \nabla(T) \cdot \Delta^{-1} .$$

Theorem 13. *Let $P \in \mathcal{D}^n$; for all $N \in \mathbb{N}$ and $z \in \mathbb{C}$, one has*

$$\sigma^{2z}(P) - \sum_{k=0}^{N-1} \frac{z(z-1)\cdots(z-k+1)}{k!} \rho^k(P) \in OP^{n-N}.$$

This allows to develop an "asymptotic calculus" for the algebra of operators $\Psi(\mathcal{A}, \mathcal{H}, D)$ generated by the "differential" operators $P \in \mathcal{D}$, the complex powers $|D|^z$, and the "smoothing" operators $OP^{-\infty}$.

More precisely, $\Psi(\mathcal{A},\mathcal{H},D)$ consists of operators which admit an expansion

$$P \sim b_q|D|^q + b_{q-1}|D|^{q-1} + \cdots \quad , \quad b_q \in \mathcal{B} ,$$

$\mathcal{B}$= algebra generated by $\{\delta^k(a)\,|\,a \in \mathcal{A},\, k \in \mathbb{N}\}$ and the remainder of order N is in OP^{-N}.

The fact that it is an algebra is guaranteed by the following consequence of Theorem 13 :

Corollary 14. *For any* $b \in \mathcal{B}$ *, one has*

$$|D|^\alpha \quad b \quad \sim \sum_{k=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}\, \delta^k(b)\ |D|^{\alpha-k}.$$

The next step is to produce a trace functional analogous to Wodzicki's noncommutative residue on the algebra $\Psi(\mathcal{A},\mathcal{H},D)$.

Noncommutative residue

Dimension Spectrum assumption: $\exists$ a discrete set $\Sigma \subset \mathbb{C}$, such that $\forall b \in \mathcal{B}$, the holomorphic function on $\Re z > p$

$$\zeta_b(z) = \mathrm{Tr}\left(b\,|D|^{-z}\right),$$

extends holomorphically to $\mathbb{C} \setminus \Sigma$.

$⨍_k(P) := \mathrm{Res}_{z=0}\, z^k\, \zeta_P(z))$, $P \in \Psi^N(\mathcal{A}, \mathcal{H}, D)$.

Proposition 15.

$$⨍_k(PQ - QP) = \sum_{n>0} \frac{(-1)^{n-1}}{n!} ⨍_{k+n}\Big(P\; L^n(Q)\Big),$$

where L is the derivation $L = [\log |D|^2, \cdot]$.

Corollary 16. *If all singularities are simple poles, then $⨍ := ⨍_0$ is a trace on $\Psi^N(\mathcal{A}, \mathcal{H}, D)$.*

Using the order filtration one makes sense of

$$e^{-t^2D^2}T \sim \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\, t^{2k}\, \nabla^k(T)\, e^{-t^2D^2}\,.$$

By iteration, one moves the heat operators in the JLO expression

$$\langle a_0, [tD, a_1], \cdots, [tD, a_n]\rangle_{tD} =$$

$$t^n \int_{\Delta_n} \mathrm{Tr}\left(a_0\, e^{-s_0 t^2 D^2}[D, a_1] \cdots [D, a_n]\, e^{-s_n t^2 D^2}\right) ds$$

to the back of the formula, replacing them by terms of the form

$$\mathrm{Tr}\left(a_0\, \nabla^{k_1}[D, a_1]\, \cdots\, \nabla^{k_n}[D, a_n]\, e^{-t^2D^2}\right)$$

Note that

$$\int_{\Delta_n} s_0^{k_1}(s_0 + s_1)^{k_2} \ldots (s_0 + \cdots + s_{n-1})^{k_n} ds = \frac{1}{(k_1 + 1)(k_1 + k_2 + 2) \cdots (k_1 + \cdots + k_n + n)}\,.$$

Via the inverse Mellin transform, the meromorphicity assumption gives expansions

$$t^{2|k|+n}\,\mathrm{Tr}\left(a_0\,\nabla^{k_1}[D,a_1]\cdots\nabla^{k_n}[D,a_n]\,e^{-t^2D^2}\right)$$

$$=\sum c_{m,\ell}\,t^{n-2p_m}\,\log^{\ell}(t^2)+O(1)$$

where p_m corresponds to the poles whose real part $> n$.

Recall retracted cocycle : for $n > p$,

$$\mathrm{ch}_t^n(D) := \sum_{j\geq 0}\mathrm{Ch}^{n-2j}(tD)+B\,\mathrm{T}\!\!\!/\mathrm{ch}_t^{n+1}(D)$$

Taking the constant term gives the residue cocycle in the theorem below*.

*[A. Connes & H. M., *The local index formula in noncommutative geometry*, Geom. Funct. Anal. **5** (1995)]

Theorem 17. *For a spectral triple with discrete dimension spectrum, the following gives a cocycle in the (b, B)-bicomplex of $\mathcal{A}$,*

$$\varphi_n(a^0, \ldots, a^n) = \sum_{\mathbf{k},\ell} c_{n,\mathbf{k},\ell} \int\!\!\!\!-_{\ell} a^0 [D, a^1]^{(k_1)} \ldots \ldots [D, a^n]^{(k_n)} |D|^{-n-2|k|}$$

$$\nabla(T) = [D^2, a],\ T^{(k)} = \nabla^k(T),$$
$$|\mathbf{k}| = k^1 + \ldots + k^n,$$

$$c_{n,\mathbf{k},\ell} = \frac{(-1)^{|k|}\,\Gamma^{(\ell)}\left(|\mathbf{k}| + \frac{n}{2}\right)}{k_1! \ldots k_n! (k_1 + 1) \ldots (k_1 + \cdots + k_n + n)}.$$

In the even case the first component is

$$\varphi_0(a^0) = \mathrm{Res}_{s=0}(\Gamma(s)\,\mathrm{Tr}(\gamma a^0 |D|^{-2s}) + \mathrm{Tr}(\gamma a^0 P_{\mathrm{Ker}\, D}).$$

The class of $\{\varphi_\bullet\}$ gives the Chern character $\mathrm{ch}^*(\mathscr{H}, F) \equiv [\tau_F] \in HC^*(\mathcal{A})$.

Example of Dirac spectral triple

1. all zeta functions associated to the Dirac spectral triple $(C^\infty(M^m),\, L^2(S),\, \not{D})$ are *meromorphic with simple poles*;

2. $\displaystyle ⨍ P \simeq \int_{S^*M} \sigma_{-m}(P). \qquad \forall\, P \in \Psi DO(M^m)$;
   (Wodzicki-Guillemin residue)

3. $\displaystyle ⨍ f^0[\not{D}, f^1]^{(k_1)} \dots [\not{D}, f^n]^{(k_n)}\ |\not{D}|^{-(n+2|k|)} = 0$
   whenever $\quad |\mathbf{k}| > 0$;

4. $\displaystyle ⨍ f^0[\not{D}, f^1] \dots [\not{D}, f^n]\ |\not{D}|^{-n}\ =$
   $$c_n \int_M \det\left(\frac{\nabla^2/4\pi i}{\sinh \nabla^2/4\pi i}\right)^{\frac{1}{2}} \wedge f^0\ df^1 \wedge \dots \wedge df^n;$$

5. under the canonical isomorphism
   $HP^*(C^\infty(M^m)) \cong H_*^{\mathrm{dR}}(M, \mathbb{C})$,
   $$\mathrm{ch}^*(\mathscr{H}, \not{D}) \equiv [(\varphi_n)] \cong [\hat{A}(R)].$$

The twisted case

Vector potentials: $\mathcal{V}_\sigma := \{A = A^* \,|$

$$A = \sum_{i=1}^{r} a_i [D, b_i]_\sigma\,, \;\; a_i, b_i \in \mathcal{A}\} \cong \{\sum_{i=1}^{r} a_i \otimes b_i,$$

$$\sum_{i=1}^{r} a_i\,\sigma(b_i) = 0, \quad \sum_{i=1}^{r} a_i \otimes b_i = \sum_{i=1}^{r} b_i^* \otimes a_i^*\}.$$

Gauge transformations: $\mathcal{U}_\sigma :=$ $\{u \in \mathcal{A}\,;\, \sigma(u)^* u = u\sigma(u)^* = 1\}$, affine action

$$\nu_u(A) = \sigma(u)\,[D, \sigma(u)^*]_\sigma + \sigma(u)\,A\,\sigma(u)^*\,;$$

$$D_{\nu_u(A)} = \sigma(u)\,D_A\,\sigma(u)^*, \; \forall u \in \mathcal{U}_\sigma, \; A \in \mathcal{V}_\sigma.$$

Anti-involution: $\theta(a) = \sigma(a)^*$, $\theta(ia) = -i\theta(a)$,

$$\mathcal{A} = \mathcal{A}_\theta \oplus \mathcal{A}_\theta^-, \quad i\mathcal{A}_\theta^- = \mathcal{A}_\theta.$$

Tangent space $T_1(\mathcal{U}_\sigma) \cong \mathcal{A}_\theta^-$,

$$\forall\, b \in \mathcal{A}_\theta^- \Longrightarrow u_t = e^{tb} \in \mathcal{U}_\sigma,$$

hence $T_1(\mathcal{U}_\sigma) \otimes \mathbb{C} \cong \mathcal{A}$.

Corollary 18 (**Straight cocycles**). *The forms* ${}_\sigma\phi^{(\bullet)}_{A,q}$ *are* U_∞*-invariant and coprimitive. By restriction to* $\mathcal{A} \cong T_1(\mathcal{U}_\sigma)_{\mathbb{C}}$ *, one obtains for any* $n > p-1$ *the cyclic* n*-cocycle*

$$
{}_\sigma\,\mathsf{Tch}_{n+1}(a^0,\dots,a^n) = \Gamma\left(\frac{n}{2}+1\right)^{-1} \sum_{\lambda\in C_n} \mathsf{sgn}(\lambda)
\int_0^\infty \mathsf{Tr}\left(\gamma\frac{D}{D^2+\mu^2}[D,a^{\lambda(0)}]_\sigma\frac{1}{D^2+\mu^2}\cdots \right.
\left.\dots[D,a^{\lambda(n)}]_\sigma\,\frac{1}{D^2+\mu^2}\right)\mu^n\,d\mu =
$$

$$
= \sum_{\lambda\in C_n} \mathsf{sgn}(\lambda) \int_0^\infty \left(\int_{\Delta_{n+1}} \mathsf{Str}\,(De^{-s_0t^2D^2}[D,a^{\lambda(0)}]_\sigma \right.
\left. e^{-(s_1-s_0)t^2D^2}\dots[D,a^{\lambda(n)}]_\sigma\,e^{-(1-s_n)t^2D^2})\,ds\right)t^{n+1}\,dt\,.
$$