Notes of course at IHP: Applications of noncommutative topology in geometry and string theory

Jonathan Rosenberg

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The purpose of this course will be to give an introduction to ways in which noncommutative topology can be applied to geometry and topology in the usual sense (as studied by topologists, for example) and to string theory:

1. What is noncommutative topology, what is noncommutative geometry, and what's the difference between them? What are some of the techniques available for studying them?

2. Applications of noncommutative topology, especially C^*-algebras of groupoids, to study of group actions on manifolds, geometry of foliations, stratified spaces, and singular spaces.

3. An introduction to twisted K-theory [9], why it appears in physics, and what it has to do with noncommutative topology.

4. Some applications of noncommutative topology and noncommutative geometry that have appeared in the recent physics literature.

Lecture I. Noncommutative topology and geometry

“Et je ne craindrai pas d’introduire ces termes d’arithmétique en la géométrie, afin de me rendre plus intelligible.”

(Descartes, *La Géométrie*, 1638)

Ever since Descartes, it has been standard to study geometry of a space through the algebraic properties of functions on that space. *Noncommutative topology and noncommutative geometry* involve applying this principle when the algebra of functions is noncommutative.

While noncommutative algebraic geometry, where the algebras of functions are typically (left) Noetherian, is a perfectly legitimate subject of current interest, the focus here will be on C^*-algebras and their dense subalgebras, instead.

The purpose of studying noncommutative geometry is quite consistent with a philosophy explained by Émile Borel about a hundred years ago (in his *Introduction géométrique à quelques théories physiques*):

PRÉFACE.

La Science mathématique tout entière doit son origine et la plupart de ses progrès à l'observation et à l'expérience; cette origine ne doit pas être méconnue; séparer par un fossé les Mathématiques et la réalité est une grave erreur pédagogique, dont il semble qu'on soit insuffisamment revenu dans notre enseignement secondaire, malgré les efforts des inspirateurs des programmes de 1902. Il ne faut pas cependant oublier que le but propre de la discipline mathématique est d'abstraire les éléments communs aux réalités diverses, de manière à créer des théories dont le champ d'application soit aussi large que possible; ce point de vue ne s'oppose pas au précédent, mais au contraire le complète.

Definition 1 *A C^*-algebra is a Banach algebra A over $\mathbb{C}$, with a conjugate-linear involution $*$, satisfying the requirement that $\|a^*a\| = \|a\|^2$ for all $a \in A$.*

We recall two important classical theorems:

Theorem 2 (Gelfand-Naimark) *A Banach $*$-algebra is a C^*-algebra if and only if it is isometrically $*$-isomorphic to a $*$-closed, norm-closed, algebra of bounded operators on some Hilbert space.*

Theorem 3 (Gelfand) *$X \mapsto C_0(X)$ sets up a contravariant equivalence of categories, from locally compact Hausdorff spaces and proper maps to commutative C^*-algebras and $*$-preserving homomorphisms.*

Thus it makes sense to view noncommutative C^*-algebras as being algebras of functions on noncommutative spaces. Certain examples are particularly important:

- $C^*(G)$, the *group C^*-algebra* of a locally compact group G, the largest C^*-completion of the convolution algebra $L^1(G)$. This should be viewed as the algebra of functions on the noncommutative space $\widehat{G}$, the unitary dual of G. (Indeed, when G is locally compact abelian, $\widehat{G}$ is also a group, the Pontryagin dual, and $C^*(G) \cong C_0(\widehat{G})$ via the Fourier transform.)

- The noncommutative torus A_Θ defined by a skew-symmetric $n \times n$ matrix Θ. This is the universal C^*-algebra with n unitary generators u_j satisfying the commutation relation

$$u_j u_k = \exp(2\pi i \Theta_{jk}) u_k u_j.$$

In other words, A_Θ is a completion of an algebra of noncommutative (Laurent) polynomials in n variables, which because of the commutation relation can always be ordered as

$$\sum_{r_j \in \mathbb{Z}} c_{r_1 \cdots r_n} u_1^{r_1} \cdots u_n^{r_n}.$$

When $n = 2$ and

$$\Theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix},$$

we write A_θ for A_Θ and call it a rotation algebra. This algebra is simple when θ is irrational.

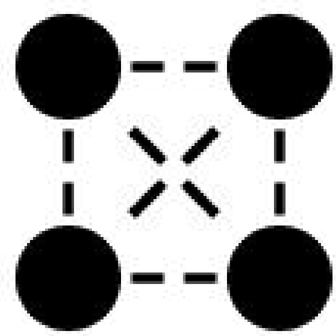
The noncommutative tori are closely related to the first example of group C^*-algebras, since the 2-cocycle $\omega\colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{T}$ defined by $e^{2\pi i \Theta}$ gives rise to a nilpotent group G fitting into an extension

$$1 \to \mathbb{T} \to G \to \mathbb{Z}^n \to 1,$$

and A_Θ is a certain canonical quotient $C^*(\mathbb{Z}^n, \omega)$ of $C^*(G)$.

- We can also consider $C^*(G)$, the *groupoid C^*-algebra* of a locally compact *groupoid* G (with a Haar system), in the sense of Renault [12]. G may be thought of as the set of *morphisms* in a small category in which all morphisms are invertible. There are two maps from G to the set $G^{(0)}$ of *objects* of G, the *range* r and *source* s.

  Example: an equivalence relation R. In this case there is one and only one morphism between any two objects, i.e., $r \times s$ sets up a bijection $G \cong R \subset G^{(0)} \times G^{(0)}$. Such systems arise from spinning particles in quantum mechanics. A particle of spin $(n-1)/2$ has n states, e.g., when $n = 2$, "spin up" and "spin down," with transitions allowed between any of them. The algebra of observables is $M_n(\mathbb{C})$.



State space of a spinning particle, $n = 4$ (or spin 3/2). The noncommutative algebra of "functions" is $M_n(\mathbb{C})$.

- Another important example of a groupoid is a locally compact transformation group $\mathcal{G} = G \times X$, where X is a locally compact space, G is a locally compact group, and G acts on X. In this case, $\mathcal{G}^{(0)} = X$ and the range and source maps are

$$r(g,x) = g \cdot x, \quad s(g,x) = x.$$

The multiplication is

$$(g, h \cdot x) \cdot (h, x) = (gh, x).$$

The associated noncommutative algebra is the crossed product C^*-algebra $C^*(\mathcal{G}) = G \ltimes C_0(X)$, the completion of the convolution algebra $L^1(G \times X)$. When $X = \mathrm{pt}$, this is just $C^*(G)$.

- An example of a noncommutative space that is "not too noncommutative" has as its associated algebra a continuous-trace algebra. These algebras were studied by Fell and Dixmier-Douady [7]; they correspond to *Azumaya algebras* in ring theory. Algebras A of continuous trace are characterized by *Fell's condition*: the dual space $\widehat{A}$ is Hausdorff, and for each $x_0 \in \widehat{A}$, there is an element x which is a local rank-one projection for all x in a neighborhood of x_0.

When X is second-countable and locally compact and $\delta \in H^3(X, \mathbb{Z})$, the stable continuous-trace algebra with Dixmier-Douady invariant δ, $CT(X, \delta)$, is locally isomorphic to $C_0(X, \mathcal{K})$, $\mathcal{K}$ the algebra of compact operators on a separable infinite-dimensional Hilbert space $\mathcal{H}$. More precisely, we define $CT(X, \delta)$ as $\Gamma_0(X, \mathcal{A}_\delta)$, where $\mathcal{A}_\delta$ is the locally trivial bundle of algebras with structure group $\mathrm{Aut}\,\mathcal{K} \cong PU(\mathcal{H}) \cong K(\mathbb{Z}, 2)$ and bundle invariant

$$\begin{aligned} \delta \in [X, BPU(\mathcal{H})] \\ &= [X, K(\mathbb{Z}, 3)] = H^3(X, \mathbb{Z}). \end{aligned}$$

Equivalence Relations

There are several natural equivalence relations on noncommutative spaces:

- Isomorphism. For us this will always mean $*$-isomorphism of C^*-algebras.

- Stable isomorphism. For separable C^*-algebras, this means $*$-isomorphism after tensoring both algebras with $\mathcal{K}$. By a theorem of Brown-Green-Rieffel [2], it coincides with Morita equivalence.

- (Strong) Morita Equivalence. Two C^*-algebras A and B are called (strongly) Morita equivalent if there is an *equivalence bimodule* ${}_AX_B$ such that $\otimes_A X$ and $X \otimes_B$ give equivalences of categories of (Hilbert space) representations of A and B. This notion is due to Rieffel [13].

For separable C^*-algebras, Morita equivalence coincides with stable isomorphism, though it has the advantage that often one wants to make a specific choice of equivalence bimodule, not just know that one exists.

- Homotopy Equivalence. Two $*$-homomorphisms $f_0, f_1 \colon A \to B$ are said to be *homotopic* if there is a $*$-homomorphism $f \colon A \to C([0,1], B)$ which is equal to f_j after composing with evaluation at j, $j = 0, 1$. Two algebras are homotopy equivalent if there are $*$-homomorphisms $f \colon A \to B$ and $g \colon B \to A$ such that $f \circ g$ is homotopic to 1_B and $g \circ f$ is homotopic to 1_A.

- Stable Homotopy Equivalence. A and B are said to be stably homotopy equivalent if $A \otimes \mathcal{K}$ and $B \otimes \mathcal{K}$ are homotopy equivalent.

- KK-Equivalence. This is a rather complicated relation to explain, but it is implied by any of the above. There is a triangulated category, called KK, obtained from the category of separable C^*-algebras and $*$-homomorphisms between them, by requiring homotopy invariance, stability (under tensoring with $\mathcal{K}$), and split exactness. Two algebras A and B are called KK-equivalent if they become isomorphic in this category [1, §22]. That implies, for example, that they have the same K-theory groups.

The Commutative Case

If we specialize just to commutative C^*-algebras, which by Gelfand's Theorem (Theorem 3) are all of the form $C_0(X)$, X locally compact, then isomorphism, stable isomorphism, and Morita equivalence all coincide with homeomorphism for spaces. Homotopy equivalence has its usual meaning. Stable homotopy equivalence is more exotic; for finite CW complexes, $C(X)$ and $C(Y)$ are stably homotopy equivalent iff $k_*(X) \cong k_*(Y)$ (as $\mathbb{Z}[u]$-modules) and some other technical conditions are satisfied [6]. ($k_* =$ connective K-homology, $u =$ Bott map.) By [15], KK-equivalence amounts simply to having the same (complex) K-theory groups. Thus finite CW complexes X and Y with torsion-free homology are KK-equivalent if and only if the sum of the Betti numbers and the Euler characteristic are the same for both spaces.

Motivation for the Various Equivalence Relations

The use of Morita equivalence or stable isomorphism (or something even weaker, such as stable homotopy equivalence) as a basic equivalence relation in noncommutative topology requires some motivation.

Consider the case of locally compact groupoids G. It is frequently useful to study groupoids not up to isomorphism but up to similarity. (Two abstract groupoids are called similar or equivalent if they are equivalent as categories; in the locally compact case, one needs some topological compatibility also.) Similar groupoids have Morita equivalent C^*-algebras [11]. For example, if a group G acts freely on X, then the groupoid $G \times X$ is similar to the quotient space X/G (with trivial group action), and $C_0(X) \rtimes G$ is Morita equivalent to $C_0(X/G)$. If G acts transitively, then $G \times X$ is similar to a stabilizer group G_x, so $C_0(X) \rtimes G$ is Morita equivalent to $C^*(G_x)$, $x \in X$.

Geometry vs. Topology: Poincaré's View (C.R. Acad. Sci. 115 (1892), 633–636)

"On sait ce qu'on entend par l'ordre de connexion d'une surface et le rôle important que joue cette notion dans la théorie générale des fonctions, bien qu'elle soit empruntée à une branche toute différente des Mathématiques, c'est-à-dire à la géométrie de situation ou *Analysis situs*.

C'est parce que les recherches de ce genre peuvent avoir des applications en dehors de la Géométrie qu'il peut y avoir quelque intérêt à les poursuivre en les étendant aux espaces à plus de trois dimensions.

. . .

On peut se demander si les nombres de Betti suffisent pour déterminer une surface fermée au point de vue de l'*Analysis situs*, c'est-à-dire si, étant données deux surfaces fermées qui possèdent mêmes nombres de Betti, on peut toujours passer de l'une à l'autre par voie de déformation continue. . . ."

Geometry vs. Topology: The Noncommutative Case

Following Poincaré's lead, we can try to distinguish between noncommutative topology and noncommutative geometry as follows. In noncommutative (algebraic) topology, we try to find invariants for classifying noncommutative spaces up to equivalence relations weaker than homotopy equivalence. Since, for the reasons explained above, it is natural to want Morita invariance, we are forced to consider stable homotopy invariance, which is already somewhat close to K-theory [14]. As an answer to a question close to Poincaré's question about completeness of Betti numbers, we have:

Theorem 4 (15) *In a category $\mathcal{N}$ of separable C^*-algebras containing the inductive limits of type I C^*-algebras, two algebras are KK-equivalent if and only if they have the same invariants K_0 and K_1.*

In noncommutative geometry, on the other hand, we try to study analogues of metric structures or connections on noncommutative spaces. In most cases, these involve “smooth structures,” that is, dense subalgebras of a C^*-algebra consisting of “smooth elements,” and analogues of differential operators defined on these subalgebras. Examples of things one can study are:

- Connections and curvature, as studied in [3].

- Noncommutative de Rham theory, as studied in [4] and [8].

- Noncommutative spectral theory, that is study of spectral properties of analogues of the classical elliptic operators.

- Noncommutative complex geometry, as studied, say, in [10].

- Noncommutative Yang-Mills theory, as studied, say, in [5].

We will not go into most of these in these lectures for lack of time, but the reader is encouraged to look at the references.

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Lecture II. Noncommutative topology and equivariant topology

This lecture will deal primarily with applications of noncommutative topology to equivariant topology, that is, to the study of group actions on topological spaces. Even the special case of finite groups acting "locally linearly" on manifolds is of great interest.

We recall from Lecture 1 that when a locally compact group G acts on a locally compact space X, we have a locally compact groupoid $\mathcal{G} = G \times X$, and $C^*(\mathcal{G}) = G \ltimes C_0(X)$. It's convenient to remember that this C^*-algebra is generated (inside its multiplier algebra) by products ba, where $b \in C^*(G)$ and $a \in C_0(X)$. The order of the factors can be reversed using the basic commutation relation $gag^{-1} = g \cdot a$.

Equivariant K-theory

One of the most useful tools for studying equivariant topology is equivariant K-theory, introduced by G. Segal [12]. Let G be a compact group and X a compact G-space. A G-vector bundle over X is a (complex) vector bundle $p: E \to X$ (in the usual sense), equipped with an action of G on E such that $g \in G$ maps the fiber $E_x = p^{-1}(x)$ linearly to $E_{g \cdot x}$, for each $g \in G$ and $x \in X$. $K_G(X)$ is defined to be the Grothendieck group of isomorphism classes of such G-bundles. If $X = \text{pt}$, a G-vector bundle is just a finite-dimensional representation of G, so $K_G(\text{pt}) = R(G)$, the representation ring of G. Tensor product of G-vector bundles gives an external product $K_G(X) \otimes K_G(Y) \to K_G(X \times Y)$, and specializing to the case $X = \text{pt}$, we see $K_G(Y)$ is always an $R(G)$-module.

Just as with ordinary K-theory, equivariant K-theory can be made into a cohomology theory on the category of locally compact spaces and proper maps. For X a locally compact G-space, we define

$$K_G(X) = \ker\Big(K_G(X_+) \to K_G(\mathrm{pt})\Big),$$

where X_+ is the one-point compactification, and this agrees with the old definition if X is already compact, since then $X_+ = X \amalg \mathrm{pt}$. And again as with ordinary K-theory, we let $K_G^{-n}(X) = K_G(X \times \mathbb{R}^n)$, and get long exact cohomology sequences. The diagonal inclusion of X in $X \times X$, together with the external product, gives a cup-product

$$K_G^i(X) \otimes K_G^j(X) \to K_G^{i+j}(X)$$

that makes $K_G^*(X)$ into a graded commutative $R(G)$-algebra. Bott periodicity, $K_G^i(X) \cong K_G^{i+2}(X)$ is also true, but not elementary (as the easiest proof requires ideas of equivariant index theory).

Equivariant K-theory for Banach algebras

Equivariant K-theory also makes sense for (complex) Banach algebras A with a continuous G-action. If A is unital, $K^G(A)$ is defined to be the Grothendieck group of isomorphism classes of finitely generated projective (left) A-modules P, equipped with continuous G-actions compatible with the G-action on A. (In other words, P must be a G-equivariant direct summand in $A \otimes V$, for some finite-dimensional complex representation V of G.) We extend the theory to nonunital algebras in the usual way. Equivariant Bott periodicity holds in this context as well, i.e., $K^G(A \times \mathbb{R}^2) \cong K^G(A)$.

Theorem 5 (Equivariant Swan's Theorem) *Let X be a compact G-space, G a compact group. If $E \to X$ is a G-vector bundle, then $\Gamma(X, E)$ is a finitely generated projective (left) $C(X)$-module with compatible G-action, and in this way, one obtains a natural isomorphism $K_G^{-i}(X) \cong K_i^G(C(X))$.*

Sketch of proof [Segal]. The key fact is that there is a surjective G-vector bundle map $\varphi\colon X \times V \twoheadrightarrow E$, for some finite-dimensional complex representation V of G. Construct a G-invariant hermitian metric on $X \times V$ by taking any hermitian metric and averaging it. Then the orthogonal complement of $\ker\varphi$, for the invariant metric, is a G-invariant direct summand in $X \times V$ isomorphic to E. This makes it possible to write $\Gamma(X, E)$ as a G-equivariant direct summand in $C(X) \otimes V$ (as $C(X)$-modules). To construct φ, observe that given $x \in X$, by the Peter-Weyl Theorem, there is a finite set of sections s_j of E that generate a finite-dimensional G-subspace of $\Gamma(X, E)$ and such that $s_j(y)$ span E_y for y in a neighborhood of x. By compactness, there is therefore a finite subset of $\Gamma(X, E)$ generating a finite-dimensional G-subspace of $\Gamma(X, E)$ and generating $\Gamma(X, E)$ as a $C(X)$-module. These sections then define a map φ, as required. $\square$

Now we want to relate equivariant K-theory of a G-space X to the noncommutative topology of the crossed product $G \ltimes C_0(X)$, or equivalently, of the groupoid algebra $C^*(\mathcal{G})$, where $\mathcal{G} = G \times X$.

Theorem 6 (Green, Julg [4]) *Let A be a C^*-algebra equipped with a continuous action $G \to$ Aut A, where G is a compact group. Then there is a natural isomorphism*

$$K_i^G(A) \xrightarrow{\cong} K_i(G \ltimes A).$$

Sketch of proof. Clearly we may assume A is unital and $i = 0$. So we will construct a map Φ from finitely generated projective A-modules P with compatible G-action to finitely generated projective $G \ltimes A$-modules. Note: $C^*(G)$, and thus $G \ltimes A$, will not be unital if G is not finite, but it doesn't matter. Our map Φ will obviously be compatible with direct sums, so it suffices to deal with the case $P = A \otimes V$, V an irreducible G-module. (So as an A-module, P is free of rank $\dim V$.)

Since G is compact, $C^*(G)$ is a (possibly infinite) direct sum of matrix algebras, and there is a projection p in the summand corresponding to V such that $V \cong C^*(G)p$ as a left $C^*(G)$-module. So let

$$\Phi(A \otimes V) = (G \ltimes A)p,$$

p viewed also as a projection in $G \ltimes A$ (via the inclusion $C^*(G) \cong G \ltimes \mathbb{C} \hookrightarrow G \ltimes A = A \cdot C^*(G)$). The rest of the proof is rather routine. $\square$

The map in Theorem 6 is an isomorphism of $R(G)$-modules, if we let $R(G)$ act on $K_i(G \ltimes A)$ as follows. Without loss of generality, take A unital and $i = 0$, and consider the notation of the proof above. Then if P is a finitely generated projective A-module with compatible G-action, and V is a finite-dimensional G-module, $[V] \cdot [\Phi(P)] = [\Phi(P \otimes V)]$.

For applications to equivariant topology, we often need one more result, the Localization Theorem.

Theorem 7 (Segal [12]) *Let G be a compact Lie group and let $\mathfrak{p}$ be a prime ideal in $R(G)$. By a result of Segal* [11], *there is a subgroup H which is minimal among subgroups for which $\mathfrak{p}$ is induced from $R(H)$, and H is topologically cyclic and unique up to conjugacy. H is called the support of $\mathfrak{p}$. Then the inclusion $X^{(H)} \hookrightarrow X$ induces an isomorphism on equivariant K-theory localized at $\mathfrak{p}$. Here $X^{(H)}$ is the G-saturation of X^H, or equivalently, the union of the fixed sets for the conjugates of H.*

Equivalence relations in equivariant topology

In equivariant topology, there is interest in classifying G-spaces up to certain natural equivalence relations:

- Equivariant homeomorphism. The strongest possible equivalence relation, usually too strong to be useful.

- Equivariant homotopy equivalence. G-spaces X and Y are equivariantly homotopy equivalent if there are G-maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $g \circ f$ is G-homotopic to 1_X and $f \circ g$ is G-homotopic to 1_Y. Functors such as equivariant K-theory are preserved by this relation.

- Pseudoequivalence. Less well known is pseudoequivalence. G-spaces X and Y are pseudoequivalent if there is a G-map $f\colon X \to Y$ which, forgetting the G-action, is a homotopy equivalence. This relation is not symmetric, but it generates the equivalence relation called pseudoequivalence.

- Isovariant homotopy equivalence. A G-map is called *isovariant* if it preserves stabilizers. (Clearly, if f is equivariant, then the stabilizer of $f(x)$ contains the stabilizer of x, but the containment can be strict.) Two G-spaces are isovariantly homotopy equivalent if there are G-maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $g \circ f$ and $f \circ g$ are isovariantly G-homotopic to the identity.

A puzzle: pseudoequivalence invariance

A natural question is whether equivariant K-theory is a pseudoequivalence invariant. By passage to mapping cones, this can be reduced to a question in noncommutative topology. We will also discuss some related questions. These questions are highly nontrivial even for A commutative.

Questions 8 *1. If A is a G-C^*-algebra and A is contractible, is $G \ltimes A$ contractible? Is $K_*^G(A) = 0$? Is $K_*(G \ltimes A) = 0$?*

2. If α^t is a homotopy of G-actions on a C^*-algebra A, is $K_*^G(A, \alpha^t)$ constant in t? Is $K_*(G \ltimes_{\alpha^t} A)$ constant in t?

All these questions are in fact closely related. By Theorem 6, equivariant K-theory agrees with K-theory of the crossed product, at least for G compact. And A contractible certainly implies $K_*(A) = 0$, i.e., A is K-acyclic. In the situation of (2), one gets an action of G on the contractible algebra $C_0((0,1], A)$ by $(\alpha_g f)(t) = \alpha_g^t(f(t))$, so one is reduced to (1).

In fact, by [1], for $\mathbb{R}$-C^*-algebras, $K_*(A) = 0$ implies $K_*(\mathbb{R} \ltimes A) = 0$. And the answer to Question 2 is also affirmative if $G = \mathbb{R}$. But similar questions for other groups G are in fact related to the Baum-Connes Conjecture, and are taken up in [7] and [6].

Connection with Smith theory

The commutative case of Questions 8 is related to Smith theory. Suppose for simplicity that $G = \mathbb{Z}/p$ is cyclic of prime order. By [3], a finite CW complex Y is the fixed set of a G-action on a contractible finite CW complex X if and only if $\widetilde{H}_*(Y, \mathbb{F}_p) = 0$. Now by the Localization Theorem, Theorem 7, $K^*_G(X)_{\mathfrak{p}} \cong K^*_G(Y)_{\mathfrak{p}} \cong R(G)_{\mathfrak{p}} \otimes_{\mathbb{Z}} K^*(Y)$, if $\mathfrak{p}$ has all of G as its support, which means it does not contain the augmentation ideal. For example, suppose $p = 2$, ℓ is an odd prime, and $\mathfrak{p} = (\ell, t+1)$, so $R(G)/\mathfrak{p} \cong \mathbb{F}_\ell$ and $R(G)_{\mathfrak{p}} = \mathbb{Z}_{(\ell)}$. Note that we can construct Y so that $\widetilde{H}_*(Y, \mathbb{F}_p) = 0$ but $\widetilde{H}_*(Y, \mathbb{F}_\ell) \neq 0$. (For example, Y can be a $\mathbb{Z}/\ell$-Moore space.) Then it is quite possible to arrange for $R(G)_{\mathfrak{p}} \otimes_{\mathbb{Z}} K^*(Y)$ to have ℓ-torsion, giving a negative answer to Questions 8. Other counterexamples may be found in [7].

Invariants from equivariant index theory

We shall now move in a slightly different direction and consider invariants, that can be interpreted in terms of noncommutative topology, for group actions on manifolds. These invariants come from careful study of invariant elliptic operators. Examples of this approach may be found in [2], [10], [8], and [5]. We will be using the theory dual to equivariant K-theory, equivariant K-homology. Without going into details, the key thing for us will be that these are unified in equivariant KK-theory, a bifunctor $KK_i^G(\_\_,\_\_)$, with $KK^G(\mathbb{C},\mathbb{C}) = R(G)$, $KK_i^G(\mathbb{C},A) \cong K_i^G(A)$, $KK_i^G(A,\mathbb{C}) \cong K_G^i(A)$, and an $R(G)$-bilinear product

$$KK_i^G(A,B) \otimes_{R(G)} KK_j^G(B,C) \to KK_{i+j}^G(A,C).$$

G-invariant elliptic operators on a manifold X naturally give classes in $K_i^G(X) = K_G^i(C_0(X))$, $i \in \mathbb{Z}/2$.

A few facts about KK^G

Before we begin, it's worth recalling some facts about KK^G:

- The analogue of the Green-Julg Theorem (Theorem 6) holds in K-homology if G is discrete. In other words, for G discrete and A a G-C^*-algebra, $K^*_G(A) \cong K^*(G \ltimes A)$. This is useful if G is finite.

- For G compact, the Localization Theorem (Theorem 7) is valid for KK^G [8]. More precisely, if $\mathfrak{p}$ is a prime ideal in $R(G)$ with support H, and X and Y are finite G-CW complexes, then $KK^G_i(C(X), C(Y))_{\mathfrak{p}} \cong KK^G_i(C(X^{(H)}), C(Y^{(H)}))$.

- There is always an induction homomorphism (related to the Green-Julg homomorphism) $KK^G_i(A, B) \to KK_i(G \ltimes A, G \ltimes B)$. It is compatible with products.

For simplicity let's take G finite and let X be a compact topological manifold on which G acts by a locally linear action. Then for $H < G$, the fixed set X^H is a locally flat topological submanifold. We also assume X has a G-invariant Lipschitz structure — this is not much of a restriction; for example, it is obvious if everything is smooth or PL.

Fix a G-invariant (Lipschitz) Riemannian metric on X. Then we have a natural G-invariant elliptic operator, the Euler operator $D = d + d^*$ (with even-odd grading of forms), and if X is oriented and the action preserves orientation, then we also gave a G-invariant signature operator (really the same operator D, but with a different grading on the forms, coming from the Hodge $*$-operator, which uses the orientation). The G-index of the Euler operator is the G-Euler characteristic; the G-index of the signature operator is the G-signature.

Invariants from equivariant index theory (cont'd)

The G-Euler characteristic and the G-signature are actually pseudoequivalence invariants. To see this, consider a G-map of manifolds $M \to N$ which, nonequivariantly, is a homotopy equivalence. The map preserves Betti numbers, so gives an isomorphism of G-representations $H_i(M, \mathbb{R}) \to H_i(N, \mathbb{R})$ for each i, so it preserves the G-Euler characteristic (the alternating sum of these in $RO(G)$) as well as the G-signature (the difference of the "positive" and "negative" parts of H_{middle}).

But we get more interesting invariants by looking at more of the K^G-homology class of D, not just the G-index.

For the Euler operator, a formula for this class was given in [5] in terms of the "universal Euler characteristic," which assembles essentially all possible Euler characteristic data of fixed sets. Thus this class in $KO_0^G(X)$ is an invariant of isovariant homotopy equivalences.

The case of the G-signature operator D_{sign} is really more interesting, because one gets genuine "higher invariants."

Conjecture 9 (equivariant Novikov [9]) *Suppose M is an oriented G-manifold and $f\colon M \to Y$ is a G-map, where Y is a G-space which is equivariantly aspherical, i.e., Y^H is aspherical for all $H \leq G$. Then $f_*([D_{\mathsf{sign}}])$ in $K_*^G(Y)$ is an oriented pseudoequivalence invariant.*

If Y is a complete manifold of nonpositive curvature and G acts on Y by isometries, then Y is equivariantly aspherical (Cartan-Hadamard) and the conjecture is true(essentially Kasparov).

Furthermore, $[D_{\mathsf{sign}}]$ is computable, at least rationally, using the Localization Theorem [8]. When G is abelian and one localizes at $\mathfrak{p}$ with support H, one gets basically the signature operator class on M^H twisted by a certain characteristic class of the equivariant normal bundle.

Another (nonelementary) application

Further examination shows that in "good cases," the class of the equivariant signature operator, after inverting 2, gives the "normal invariant" term in the equivariant surgery sequence, and the "assembly map" is just given by the equivariant index theorem. Thus one ends up with:

Theorem 10 ([10]) *A topological orientation-preserving action of a finite group G on a compact simply connected topological manifold M, such that for all subgroups $H \subseteq K \subseteq G$, the fixed sets M^K and M^H are simply connected submanifolds of dimension $\neq 3$ and the inclusion of M^K in M^H is locally flat and of codimension ≥ 3, is determined up to finite indeterminacy by its isovariant homotopy type and the classes of the equivariant signature operators on all the fixed sets M^H.*

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Lecture III. Noncommutative topology and topology of singular spaces

This lecture will deal with applications of noncommutative topology to topology of singular spaces, that is, to the study of manifold-like spaces which are not everywhere locally Euclidean. Important examples of such spaces are "quotient spaces" of foliations, orbifolds, singular complex projective varieties, and $\mathbb{Z}/k$-manifolds. The case of the quotient space of a group action also brings us back to the subject of Lecture II. Incidentally, a related notion which will come up from time to time is that of a stratified space [9]. For our purposes, this will be a locally compact space with an increasing filtration

$$X^0 \subseteq X^1 \subseteq \cdots \subseteq X^n = X$$

by locally closed subsets, such that $X_0 = X^0$ and $X_j = X^j \setminus X^{j-1}$, $j \geq 1$, are manifolds, and X_{j-1} is open and dense in $X^j \setminus X^{j-2}$. In addition, one usually imposes conditions on how X_j is attached to X^{j-1}.

To explain what we will be doing, let's try to enunciate a general principle. To a singular space, we will try to attach a noncommutative C^*-algebra, in such a way that the noncommutative topology of the algebra reflects the extra structure of the space. For example, in the case of a stratified space as a above, we might want to attach a C^*-algebra A with a filtration

$$I^{-1} = 0 \subset I^0 \subseteq I^1 \subseteq \cdots \subseteq I^n = A$$

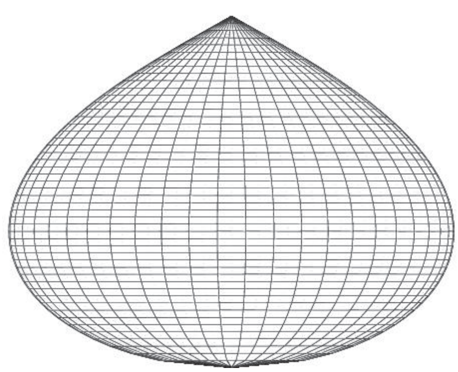
by ideals, such that I^j/I^{j-1} is related to $C_0(X_j)$, $j = 1, \cdots, n$, for example stably isomorphic to it, and such that the structure of A helps to understand X not just as an abstract space, but as a stratified space.

Filtration of the crossed product

Example: Suppose a compact group G acts smoothly on a manifold Y. Then there are only finitely many orbit types and the quotient space $X = Y/G$ is a stratified space, with stratification coming from the orbit types. A natural choice of A in this case is the crossed product $G \ltimes C_0(Y)$. This C^*-algebra comes with a filtration coming from the orbit type decomposition of Y, with each subquotient having spectrum which is a bundle over X_j. More precisely X_j comes from Y_j, which is a locally closed subset of Y with only one orbit type, say with all stabilizers conjugate to $H < G$. We have $Y_j/G \cong Y_j^H/(N_G(H)/H)$, and the groupoid $G \times Y_j$ is similar to the smaller groupoid $N_G(H) \times Y_j^H$, i.e., I^j/I^{j-1} is Morita equivalent to $N_G(H) \ltimes C_0(Y_j^H)$. The structure of the crossed product was computed by Wassermann [8]; it is a continuous trace algebra with spectrum an $\widehat{H}$-bundle over Y_j/G. We have already seen that the K-theory of A agrees with the equivariant K-theory of Y.

A simple example: orbifolds

One of the simplest kinds of singular spaces is an orbifold. This is a space X equipped with charts, like those for a manifold, except that the charts identify an open subset of X not with an open subset of $\mathbb{R}^n$ but with an open subset of $\mathbb{R}^n/G$, G a finite group acting orthogonally. We keep track of the G as well as of the topology of the quotient, and transition functions between charts are required to come from linear equivariant maps. Example: the teardrop:



In general, it is not true (the teardrop is a counterexample) that an orbifold, even if compact, must be a quotient of a manifold by a finite group. (Orbifolds that are such quotient spaces are usually called "good.") But nevertheless, every orbifold X is the quotient of a manifold $\widetilde{X}$ (namely the orthonormal frame bundle FX) by a locally free action of a compact Lie group G. ("Locally free" means all stabilizers are finite.) This suggests that to each orbifold X we should attach the orbifold C^*-algebra, which can be defined simply to be $G \ltimes \widetilde{X}$, though one could also give an intrinsic definition in terms of the orbifold charts of X. While the same orbifold can be written as a quotient of a manifold by a locally free compact group action in more than one way (clearly one can multiply both $\widetilde{X}$ and G by the same compact Lie group), the result $C^*(X)$ of this construction is well defined up to Morita equivalence [4]. And "index theory for orbifolds" can be formulated in terms of the noncommutative geometry of $C^*(X)$.

Foliations

The next important example is that of foliations. This example is quite important, but we will not go into too many details since Benameur will give a whole course on this subject.

Attached to a foliation $\mathcal{F}$ on a manifold M, we have a locally compact holonomy groupoid [10]. We have $\mathcal{G}^{(0)} = M$, and $\mathcal{G}_x^y \neq \emptyset$ if and only if they lie on the same leaf L of $\mathcal{F}$. More exactly, $\mathcal{G}_x^y$ is the set of holonomy classes of paths from x to y lying in L. Two paths are in the same holonomy class if they induce the same local diffeomorphism of a transversal to $\mathcal{F}$ at x to a transversal to $\mathcal{F}$ at y. Thus $\mathcal{G}_x^y$ is a quotient of $\pi(L)_x^y$, where $\pi(L)$ is the fundamental groupoid of the connected manifold L. For example, if $\mathcal{G}$ has leaves which are lines spiraling in to a closed leaf which is a circle, then for x on this circle, $\mathcal{G}^x$ is a line, not a circle. Following Connes, we define $C^*(M, \mathcal{F}) = C^*(\mathcal{G})$.

The simplest case of a foliation is a fibration $F \to M \to B$. In this case, all holonomy is trivial and $C^*(M, \mathcal{F})$ is Morita equivalent to $C_0(B)$, $B = M/\mathcal{F}$. For this reason, it makes sense in general to think of $C^*(M, \mathcal{F})$ (at least up to Morita equivalence) as defining the noncommutative space of leaves of the foliation.

The connection between foliation C^*-algebras and those defined by group actions is that if $\mathcal{F}$ comes from a locally free and generically free (i.e., free orbits are dense) action of a Lie group G on M, then $\mathcal{G} = G \times M$ and $C^*(M, \mathcal{F}) = G \ltimes C_0(M)$.

Incidentally, the theory of foliation C^*-algebras doesn't really require smoothness in the transverse direction, and thus one can replace foliations by laminations, spaces locally modeled by $\mathbb{R}^p \times T$, T any compact space. Here p is the leaf dimension.

Foliation index theory and applications

Now we are ready for some analysis. Suppose D is a differential operator on M (assumed compact) that is elliptic in the direction of the foliation $\mathcal{F}$. Then D can be viewed as giving a family of elliptic operators D_ℓ on the leaves ℓ, that vary continuously in the transverse direction in a suitable sense. In typical examples, one has a Riemannian metric on the tangent bundle $\mathcal{F}$ to the leaves, and we take D to be one of the associated "standard" elliptic operators (Euler, Dirac, signature, or Dolbeault). Then Connes-Skandalis define $\operatorname{Ind} D \in K_0(C^*(M, \mathcal{F}))$ and give a topological formula for it, which when $\mathcal{F}$ is a fibration with compact base B is the Atiyah-Singer Index Theorem for Families, for the index in $K^0(B)$ of a family of elliptic operators on F parameterized by B.

To get numerical invariants, we can apply a homomorphism $K_0(C^*(M,\mathcal{F})) \to \mathbb{R}$, for example the trace defined by an invariant transverse measure μ. (This was the case originally studied by Connes using von Neumann algebras.) Then one gets a formula of the form

$$\mathrm{Tr}_\mu(\mathrm{Ind}\, D) = \int \mathrm{Ind}_{\mathrm{top}}\, \sigma(D)\, d\mu,$$

where $\mathrm{Ind}_{\mathrm{top}}$ is the topological index and $\sigma(D)$ is the symbol of the operator.

As we mentioned before, see Benameur's course for more details on foliation index theory. But we'll give one example.

Theorem 11 (Connes) *Let $(M, \mathcal{F})$ be a compact laminated space with 2-dimensional oriented leaves and a smooth Riemannian metric g along the leaves. Let μ be an invariant transverse measure, C_μ its Rulle-Sullivan class (which is dual to tangential de Rham 2-forms). Let ω be the curvature 2-form of g. If $\langle [\omega], C_\mu \rangle > 0$, then $\mathcal{F}$ has a set of closed leaves of positive μ-measure. If $\langle [\omega], C_\mu \rangle < 0$, then $\mathcal{F}$ has a set of (conformally) hyperbolic leaves of positive μ-measure. If all the leaves are (conformally) parabolic, then $\langle [\omega], C_\mu \rangle = 0$ for every invariant transverse measure μ.*

The groupoid C^*-algebra of a $\mathbb{Z}/k$-manifold

We will now consider an application of groupoid C^*-algebras [6] to the $\mathbb{Z}/k$ index theorem [5]. This index theorem is a bit unusual in that the index only takes torsion values.

Definition 12 *A* $\mathbb{Z}/k$-manifold *is a smooth compact manifold with boundary, M^n, along with an identification of ∂M with a disjoint union of k copies of a fixed manifold N^{n-1}. It is oriented if M is oriented, the boundary components have the induced orientation, and the identifications are orientation-preserving.*

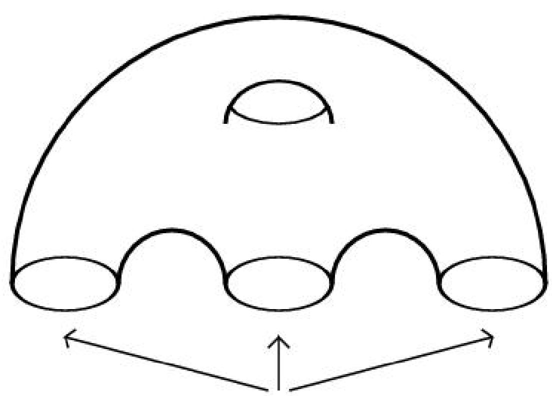


Figure: A $\mathbb{Z}/3$-manifold

This space doesn't look singular, but that's because it should be viewed as a desingularization of the space X obtained by collapsing the k copies of N down to a single copy of N. If $k = 2$ (and everything is oriented), the result is a nonorientable manifold; if $k \geq 3$, the result is not a manifold at all, as points in $N \subset X$ have neighborhoods of the form $\mathbb{R}^{n-1} \times c(k \cdot \text{pt})$ and not open subsets of $\mathbb{R}^n$. Here $c(k \cdot \text{pt})$ is the cone on k points (shown again for $k = 3$):

Thus X is a stratified space with two strata: the open nonsingular stratum $X_0 = M \smallsetminus N$ and the closed singular stratum $X_1 = N$. Such spaces were introduced by Sullivan as a way of giving geometric models for bordism classes (or homology cycles) with coefficients in $\mathbb{Z}/k$.

To do index theory on the stratified space X, we will introduce a groupoid C^*-algebra. First add infinite cylinders to M to make a noncompact manifold $\widetilde{M}$ without boundary, as shown:

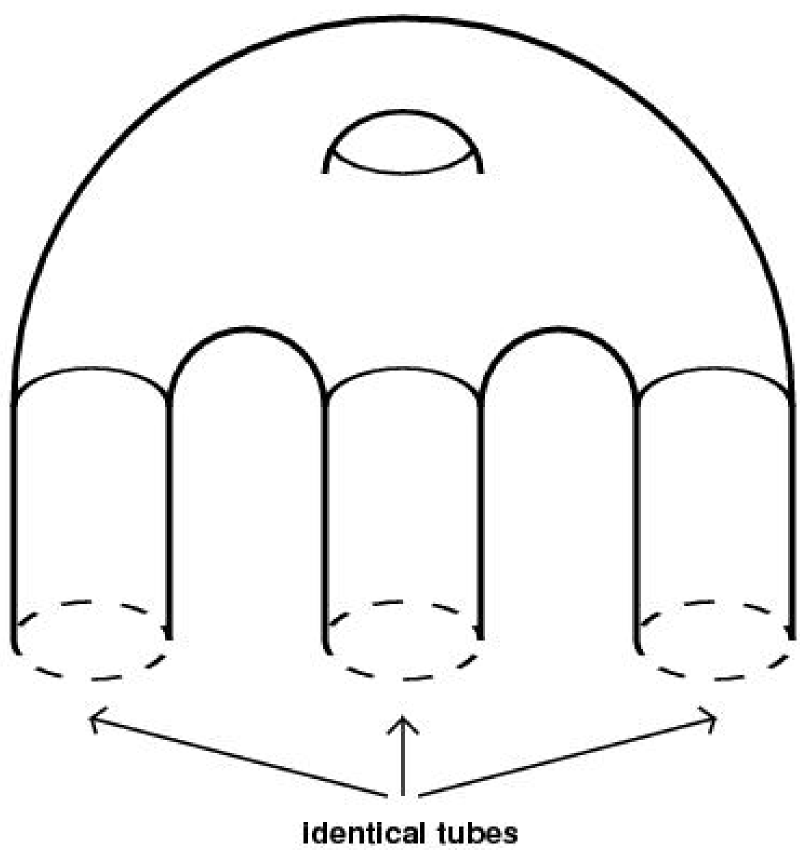


Figure: A $\mathbb{Z}/3$-manifold extended

Now we define a locally groupoid $\mathcal{G}$ with

$$\mathcal{G}^{(0)} = \widetilde{M} = M \cup_{k \cdot N} k \cdot N \times [0, \infty)$$

by means of the equivalence relation in which points in M (including those in ∂M) are equivalent only to themselves and points in any copy of $N \times (0, \infty)$ are equivalent to the corresponding points in all the other copies. Note that $C^*(\mathcal{G})$ can be described by an extension

$$0 \to C_0(N \times (0, \infty)) \otimes M_k(\mathbb{C}) \\ \to C^*(\mathcal{G}) \to C(M) \to 0.$$

There is a closely related C^*-algebra, which we'll call $C^*(\mathbb{C}; \mathbb{Z}/k)$, obtained by collapsing M and N to points:

$$0 \to C_0((0, \infty)) \otimes M_k(\mathbb{C}) \\ \to C^*(\mathbb{C}; \mathbb{Z}/k) \to \mathbb{C} \to 0.$$

The long exact sequence in K-theory shows $K_0(C^*(\mathbb{C}; \mathbb{Z}/k)) = 0$ and $K_1(C^*(\mathbb{C}; \mathbb{Z}/k)) = \mathbb{Z}/k$.

The $\mathbb{Z}/k$-index theorem

We will now state and prove the $\mathbb{Z}/k$-index theorem of Freed and Melrose [5]. Consider an elliptic differential operator D on M, which in a collar neighborhood of N is the restriction of a $(\mathbb{Z}/k \times \mathbb{R})$-invariant operator on $N \times \mathbb{Z}/k \times \mathbb{R}$. The standard elliptic operators will have this property if we make the right choice of metric on the collar neighborhood. Such an operator (acting, say, between sections of two vector bundles) gives a class in $K^0(C^*(\mathcal{G}))$. Via the obvious map $c\colon C^*(\mathbb{C}; \mathbb{Z}/k) \to C^*(\mathcal{G})$, we get a class in $K^0(C^*(\mathbb{C}; \mathbb{Z}/k)) \cong \mathbb{Z}/k$ called the analytical index of D.

The purpose of the index theorem is to show that the analytical index can be computed topologically.

Define the topological index of D by taking an embedding

$$\iota\colon (M, \partial M) \hookrightarrow (D^{2r}, S^{2r-1}),$$

r large, which is equivariant for the action of $\mathbb{Z}/k$ on the boundary on each side. This induces a map $X \to M_k$, where

$$M_k = D^{2r}/(\mathbb{Z}/k\text{-action on } S^{2r-1})$$

is a $\mathbb{Z}/k$-Moore space with its reduced homology all concentrated in degree $2r-1$ and equal to $\mathbb{Z}/k$. Note that ι induces a Gysin map on K-theory and thus a map of the class of the symbol of D to $\widetilde{K}(M_k) = \mathbb{Z}/k$. This is the topological index.

Theorem 13 ($\mathbb{Z}/k$ index theorem) *The analytical and topological indices agree in $\mathbb{Z}/k$.*

Sketch of proof. We mimic the KK proof of the usual Atiyah-Singer Theorem. For simplicity let's assume M is spin^c and D is Dirac with coefficients in a bundle E (which carries a compatible $\mathbb{Z}/k$-action on the boundary). So E defines a class $[E]$ in $K^0(X)$ (recall $X = M/\sim$) and

$$[D] = [E] \otimes_{C(X)} \alpha,$$

where $\alpha \in KK(C(X) \otimes C^*(\mathcal{G}), \mathbb{C})$ is the basic Dirac "fundamental" class. We have (by associativity of the Kasparov product and definition of the Gysin map):

$$\begin{aligned} \mathrm{Ind}_a(D) &= [c] \otimes_{C^*(\mathcal{G})} [D] \\ &= [c] \otimes_{C^*(\mathcal{G})} \Big([E] \otimes_{C(X)} \alpha \Big) \\ &= [E] \otimes_{C(X)} \Big([c] \otimes_{C^*(\mathcal{G})} \alpha \Big) \\ &= \mathrm{Ind}_{\mathrm{top}}(D). \quad \square \end{aligned}$$

Some cases still to be treated

Let me mention a few cases where the same sort of ideas might be helpful, though nobody's gotten them to work so far:

- Singular complex projective varieties. These have a natural stratification and satisfy Poincaré duality not for ordinary homology and cohomology but for intersection homology and cohomology. Can one do something similar in K-theory, and get it to match up with the K-theory of a noncommutative C^*-algebra attached to the stratified structure? A first step might be found in [3].

- Witt spaces. More generally, one can try to do the same thing with Witt spaces, a class of pseudomanifolds introduced in [7]. These have less structure than projective varieties, but Witt bordism (after inverting 2) is naturally isomorphic to KO-homology, which suggests that C^*-algebra ideas might work somehow.

- The singular foliation on $\mathfrak{g}^*$. Let G be a connected and simply connected nilpotent Lie group. The exponential map exp gives a diffeomorphism from the Lie algebra $\mathfrak{g}$ to G. Recall (say from Xu's lectures) that $\mathfrak{g}^*$ has a natural structure of Poisson manifold, for which the symplectic leaves are the coadjoint orbits. The problem is to relate a C^*-algebra coming from the singular foliation of $\mathfrak{g}^*$ by orbits to the noncommutative space represented by $C^*(G)$.

Motivation: the dual topology problem

To explain why this would be interesting and natural, recall that by the Kirillov orbit method, $\widehat{G}$ is in natural bijection with the space of orbits $\mathfrak{g}^*/G$. The map in one direction is given by the Kirillov character formula; given an irreducible representation π of G, there is a unique orbit $\mathcal{O} = \mathcal{O}(\pi)$ such that

$$\operatorname{Tr} \pi(\varphi) = \int_{\mathcal{O}} (\varphi \circ \exp)\widehat{\ }\, d\beta_{\mathcal{O}},$$

for $\varphi \in C_c^\infty(G)$. Here $\beta_{\mathcal{O}}$ is the canonical measure on the orbit $\mathcal{O}$ defined by the symplectic structure.

The map in the other direction is given by induced representations. Given $f \in \mathfrak{g}^*$, choose a polarization for f, i.e., a maximal isotropic subalgebra $\mathfrak{h}$ for $f([\_\_, \_\_])$, and induce e^{if} from H to G.

It turns out in this way that $\widehat{G} \cong \mathfrak{g}^*/G$ even as (non-Hausdorff) topological spaces, though bicontinuity of the Kirillov map is a difficult theorem [1].

If we could show directly that there were some way to define a "singular foliation algebra" $C^*(\mathfrak{g}^*, \mathcal{F})$ out of the singular foliation of $\mathfrak{g}^*$ coming from the Poisson structure, along with a Morita equivalence $C^*(\mathfrak{g}^*, \mathcal{F})$ from $C^*(G)$, then presumably it would be obvious that

$$C^*(\mathfrak{g}^*, \mathcal{F})\widehat{\ } \cong \mathfrak{g}^*/G$$

as topological spaces, and thus one would get an easier proof of the dual topology theorem.

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Lecture IV. Twisted K-theory

This lecture will give an introduction to twisted K-theory, and some of the reasons why it is interesting for geometry and physics. This topic will be touched on from other points of view in some of the other courses in the trimester.

Motivation: Poincaré duality in (ordinary) cohomology

To explain what we mean by twisted K-theory, it helps to recall what twisting means and does for ordinary cohomology, say with coefficients in $\mathbb{Z}$. Let's recall the idea of Poincaré duality. Suppose M^n is a compact (connected) oriented manifold. The choice of an orientation gives a fundamental class $[M] \in H_n(M, \mathbb{Z})$, and cap product with $[M]$ gives an isomorphism $H^j(M, \mathbb{Z}) \to H_{n-j}(M, \mathbb{Z})$.

The twisted case

But if M is not orientable, no such isomorphism can exist. For example, if $M = \mathbb{RP}^2$, $H^*(M, \mathbb{Z}) = \mathbb{Z}[a]/(2a, a^2)$, where a is a 2-torsion element in degree 2, and $H_1(M, \mathbb{Z}) \cong \mathbb{Z}/2$, $H_2(M, \mathbb{Z}) = 0$. So certainly there is no fundamental class in $H_2(M, \mathbb{Z})$ and no isomorphism $H^j(M, \mathbb{Z}) \to H_{n-j}(M, \mathbb{Z})$. However, we can remedy this by using the local coefficient system $\mathbb{Z}_w$ defined by the non-trivial oriented double cover $\widetilde{M} \to M$ with invariant

$$w = w_1(M) \in \mathrm{Hom}(\pi_1(M), \mathbb{Z}/2) = H^1(M, \mathbb{Z}/2).$$

Here w_1 is the first Stiefel-Whitney class. Now M *does* have a fundamental class $[M] \in H_n(M, \mathbb{Z}_w)$, and cap product with $[M]$ defines an isomorphism $H^j(M, \mathbb{Z}) \to H_{n-j}(M, \mathbb{Z}_w)$ or $H^j(M, \mathbb{Z}_w) \to H_{n-j}(M, \mathbb{Z})$. (One can put the twist in either homology or cohomology, as long as one group is twisted and the other is not.)

Poincaré duality in K-theory

The situation in K-theory and K-homology is similar, the only difference being that we can't just define twisted cohomology this time by means of a nonconstant locally constant sheaf. Instead, the simplest approach is to use continuous-trace C^*-algebras.

Poincaré duality is simplest in the case of a spin^c manifold M^n. For such manifolds, a Dirac operator D is defined and gives a fundamental class

$$[D] \in K_n(M) = KK_n(C(M), \mathbb{C}).$$

Kasparov product with this class defines Poincaré duality $K^j(M) \xrightarrow{\cong} K_{n-j}(M)$.

The spin^c condition fails if either w_1 or w_3 is non-zero. The simplest non-spin^c oriented manifold is $X^5 = SU(3)/SO(3)$. Except for H_0 and H_5 (which of course have to be $\mathbb{Z}$), the only non-zero homology is a $\mathbb{Z}/2$ in degree 2. Thus $\widetilde{K}_*(X)$ is $\mathbb{Z}/2$ in degree 0 and $\mathbb{Z}$ in degree 1. By UCT, $\widetilde{K}^*(X)$ vanishes in degree 0 and is $\mathbb{Z} \oplus \mathbb{Z}/2$ in degree 1. Any Poincaré duality map would have to give

$K^0(X) = \mathbb{Z} \cong K_1(X) = \mathbb{Z} \oplus \mathbb{Z}/2,$ contradiction.

However, the Dirac operator makes sense on the Clifford algebra bundle of the tangent bundle. So we get a fundamental class

$$[D] \in KK_n(\Gamma(M, \text{Cliff}(TM)), \mathbb{C}) := K_n^w(M)$$

and Kasparov product with $[D]$ gives $K^j(M) \xrightarrow{\cong} K_{n-j}^w(M)$. Here $C_w(M) = \Gamma(M, \text{Cliff}(TM))$ is a continuous-trace algebra over M with Dixmier-Douady class $w = w_3(M)$. In the spin^c case, it's Morita equivalent to $C(M)$, and the twist goes away.

Twisted theories and continuous-trace algebras

Abstracting from the examples we have seen, topologists define twisted cohomology theories as follows. In a reasonable category of spaces (say those homotopy equivalent to CW complexes), any cohomology theory $X \mapsto E^*(X)$ is representable by a representing object $\mathbf{E}$ called a spectrum. (There is no connection with the spectrum of an element of a Banach algebra, or the word "spectrum" meaning "dual space" for a C^*-algebra.) So, for example, $E^0(X) = [X, \mathbf{E}]$, meaning homotopy classes of (based) maps. A twisted E-group of X will be $E^0_{\mathcal{E}}(X) = \pi_0(\Gamma(X, \mathcal{E}))$, where $\mathcal{E}$ is a (possibly non-trivial) "principal $\mathbf{E}$-bundle" over X, i.e., there is a fibration

$$\mathbf{E} \to \mathcal{E} \to X.$$

So this agrees with the usual definition of $E^0(X)$ when $\mathcal{E}$ is trivial, i.e., is just a product bundle $X \times \mathbf{E}$. Since ordinary (topological) K-theory is closely associated to the algebra of compact operators, $\mathcal{K}$, and in fact one can construct $\mathbf{E}$ from the topological group

$$\{u \in U(\mathcal{L}(\mathcal{H})) : u \equiv 1 \bmod \mathcal{K}\},$$

this point of view suggests twisted K-theory should be associated with nontrivial bundles of algebras with fibers that are algebras of compact operators, i.e., with continuous-trace algebras. And of course the example we saw above, with Clifford algebra bundles, is also of this type, though with finite-dimensional fibers.

The Brauer group

Recall the notion of Brauer group in algebra. For a field F, $\operatorname{Br} F$ is constructed from central simple separable algebras over F, modulo Morita equivalence, with group operation coming from $\otimes_F$. There is a similar notion over a commutative ring R, and applying this definition to $R = C(X)$ gives $\operatorname{Br}_{\text{alg}} C(X) \cong \operatorname{Tors} H^3(X, \mathbb{Z})$ [4]. Again the equivalence relation is Morita equivalence over $C(X)$ and the group operation is $\otimes_{C(X)}$.

P. Green noticed that one can make slight changes in the definitions and construct a Brauer group out of continuous trace algebras over X, possibly infinite dimensional and nonunital. Then one gets a natural isomorphism $\operatorname{Br} C(X) \cong H^3(X, \mathbb{Z})$ (via the Dixmier-Douady class). Incidentally, inversion in the Brauer group is given by $A \mapsto A^{\text{op}}$. One can find an exposition of all of this in [12].

The graded case

Even though we won't go into this much here, there are good reasons to consider $\mathbb{Z}/2$-graded algebras and get a graded Brauer group. This allows a slightly more general kind of twisted K-theory, needed for Poincaré duality in K-theory on nonorientable manifolds (where $w_1 \neq 0$, in addition to possibly having $w_3 \neq 0$). The theory with finite-dimensional fibers was first worked out in [3], then generalized to the case of infinite-dimensional fibers and continuous-trace algebras in [11]. This theory works well in KK, since the Kasparov formalism is set up to work with $\mathbb{Z}/2$-graded algebras.

Computing twisted K-theory

To summarize, if X is locally compact, we have the Brauer group of continuous-trace algebras over X up to Morita equivalence over X. At least in the separable case, each class has a unique stable representative $CT(X, \delta)$. And the Dixmier-Douady class gives an isomorphism $\mathrm{Br}\, C_0(X) \cong H^3(X, \mathbb{Z})$. Given $\delta \in H^3$, the associated twisted K-theory is $K^*_\delta(X) = K_*(CT(X, \delta))$. How do we compute this, say if we understand the homotopy type of X?

Theorem 14 ([13]) *Suppose X is a finite CW complex and $\delta \in H^3(X, \mathbb{Z})$. Then there is a spectral sequence*

$$H^p(X, K^q(\mathit{pt})) \Rightarrow K_\delta^{p+q}(X),$$

for which the first nonzero differential is $d_3 = \_\_ \cup \delta + \mathrm{Sq}^3$.

Sketch of proof. Filter X by skeleta and consider the induced filtration of $CT(X, \delta)$. It is then easy to check that the E_2 term is as described. There cannot be any d_2 (since $E_2^{p,q} \neq 0$ only for q even). To compute d_3, observe that it has to be given by a universal formula involving cohomology operations and δ, and check on a few key examples (such as $CT(S^3, \delta)$) to determine the formula. $\square$

Example. If $X = S^3$ and $\delta = ky \neq 0$ in $H^3(X, \mathbb{Z}) = \mathbb{Z} \cdot y$, then cup product with δ is injective, so $K_\delta^0(S^3) = 0$ and $K_\delta^1(S^3) \cong \mathbb{Z}/k$ is torsion.

K-theory in string theory

We'll just give a quick introduction to a way K-theory shows up in physics. This was first noticed in [9] and [15]; another exposition, perhaps more readable, is in [10].

This application shows up in string theory, maybe a way of combining the quantum field theory of elementary particles with general relativity. In string theory, one initially considers maps from a 2-dimensional manifold (representing a string propagating in time) into a high-dimensional spacetime manifold X. There are various other background fields on X. One can have either closed strings (compact 2-manifolds without boundary) or open strings (really a misnomer, as these are compact 2-manifolds with boundary).

Geometry of branes

In the open case, strings begin and end on submanifolds Y of X called D-branes ("D" for "Dirichlet" [conditions], "brane" a word created out of "membrane," and meaning "manifold"). One of the most interesting parts of the theory is the analysis of "charges" on the D-branes and ways the branes can split apart or coalesce. The important part of this should be given by some sort of generalized homology theory with the D-branes as typical cycles. In fact each brane is to carry a Chan-Paton bundle, and (at least initially) both X and the branes should be spin^c manifolds. (This is because we need spinors to have a theory of fermions, and a certain anomaly must cancel.) Many will recognize here the Baum-Douglas approach [2] to topological K-homology. Thus we think of D-branes with their Chan-Paton bundles as giving K-homology classes in X, Poincaré dual to K-cohomology classes.

Twisted K-theory in physics

Then why twisted K-theory? As we mentioned above, spacetime carries a number of background fields. One of these is denoted H, and represents a class in $H^3(X, \mathbb{Z})$. This class plays an important role in the WZW model, in which X is a connected, simply connected, simple compact Lie group G, and thus $H^2(X, \mathbb{Z}) = 0$ and $H^3(X, \mathbb{Z}) \cong \mathbb{Z}$. In this model, H is usually the generator of $H^3(X)$. We neglected H above, but the anomaly cancellation argument really shows that w_3 of a brane should cancel the mod 2 reduction of H. This is reminiscent of what we said above about Poincaré duality in twisted K-theory for non-spin^c manifolds; there is no Dirac operator on the brane by itself, just on a certain bundle of algebras over it.

It is argued in [6] and [8] that the proper setting for D-brane charges is really H-twisted K-homology or K-theory of X (depending on point of view). In fact, the importance of twisted K-theory seems to be reinforced by work in [5] showing that one can recover the Verlinde algebra, which plays an important role in conformal field theory, from the structure of $K_G^H(G)$, the H-twisted equivariant K-theory of the simple compact Lie group G acting on itself by conjugation, with H the canonical generator of H^3.

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Lecture V. An application of noncommutative topology to string theory

As Julia already mentioned in his first lecture (and will probably discuss in greater detail later), one of the very interesting dualities in string theory is T-duality ("T" for torus). This kind of duality says that the field equations on two different spacetime manifolds give indistinguishable physics, even though a circle of radius R in one spacetime is replaced by one of radius $1/R$ (perhaps with a factor of 2π, depending on normalizations) in the other. Thus the duality interchanges small-scale behavior in one spacetime with behavior near ∞ in the other. This should be reminiscent of the Fourier transform, which gives an equivalence of representations of the Heisenberg commutation relations on configuration space and on its dual, momentum space. Indeed, T-duality has been related to the Fourier-Mukai transform [13].

Derivation of classical T-duality

T-duality was originally discovered [5] by calculations in local coordinates with Lagrange multipliers. Global considerations were not so relevant since spacetime was taken to be $X = Z \times S^1$, and only the metric structure changed under the duality.

Consider the simplest case. Take Σ a closed Riemannian 2-manifold and consider the Lagrangian for maps $x\colon \Sigma \to S^1_R$:

$$L(x) = \int_\Sigma g_0 \langle dx, dx \rangle \, d\mathrm{vol}_\Sigma,$$

g_0 coming from the metric on S^1_R. We can think of dx as a variable ω that ranges over 1-forms on Σ (with periods in $2\pi R\mathbb{Z}$) and consider instead

$$E(\omega, \theta) = \int_\Sigma \left(g_0 \langle \omega, \omega \rangle \, d\mathrm{vol}_\Sigma + \theta \, d\omega \right).$$

For an extremum of E for fixed non-zero θ, we need $d\omega = 0$, so we get back the original theory. But instead we can take the variation in ω:

$$\begin{aligned}\delta E &= \int_\Sigma (2g_0\langle\delta\omega,\omega\rangle\, d\mathrm{vol}_\Sigma + \theta\, d\delta\omega)\\ &= \int_\Sigma \delta\omega \wedge (2g_0 * \omega + d\theta)\,,\end{aligned}$$

so $*\omega = \frac{-1}{2g_0} d\theta$, $\omega = \frac{1}{2g_0} * d\theta$. Substituting back in the formula for E gives

$$\begin{aligned}E'(\theta) &= \int_\Sigma (g_0\langle\omega,\omega\rangle\, d\mathrm{vol}_\Sigma + \theta\, d\omega)\\ &= \int_\Sigma \left(\frac{1}{4g_0}\langle d\theta, d\theta\rangle\, d\mathrm{vol}_\Sigma + \frac{1}{2g_0}\theta\, d * d\theta\right)\\ &= -\frac{1}{4g_0}\int_\Sigma \langle d\theta, d\theta\rangle\, d\mathrm{vol}_\Sigma\end{aligned}$$

which is just like the original action (with θ replacing x) except for a switch from g_0 to $\frac{1}{4g_0}$.

T-duality and mirror symmetry

It was realized pretty soon (by numerous authors) that T-duality is closely related to mirror symmetry [14], which in its original form deals with the case where $X = \mathbb{R}^4 \times Y$, Y a Calabi-Yau 3-fold, and the symmetry interchanges Y with another Calabi-Yau, $Y^{\#}$, and roughly speaking, deformations of Kähler structure on one C-Y correspond to deformations of complex structure on the other. (In particular, the Hodge numbers $h^{1,1}$ and $h^{2,1}$ are switched.)

However, we will not consider here the parts of the field equations in string theory that force (at least when the background fields vanish) Y to be a Calabi-Yau, and will assume for the moment that X can be any manifold.

T-duality for bundles

For T-duality to make sense for a general space-time X, we want X to be "fibered by tori," so we assume for now that there is a principal T-bundle $p\colon X \to Z$, where $T = \mathbb{T}^n$ is an n-torus. T-duality should then in some sense replace the torus $T = \mathbb{R}^n/\Lambda$ by the dual torus $\widetilde{T} = \widetilde{\mathbb{R}}^n/\widetilde{\Lambda}$. It was discovered in [2] in this context that the topology of the bundle will change in general. Furthermore, the H-flux will usually change also.

To explain this, recall that D-brane charges live in the twisted K-theory (or twisted K-homology) of X. (Assuming X is spin^c and is compact, or compact $\times$ Euclidean, these are Poincaré dual except for a sign convention switch.) Since the physics on X and the T-dual $X^\#$ is supposed to be the same, this forces the relation

$$K_H^*(X) \cong K_{H^\#}^{*+n}(X^\#).$$

The isomorphism

$$K^*_H(X) \cong K^{*+n}_{H^\#}(X^\#).$$

on twisted K-theory should be determined by the duality and should be involutive, so that doing duality again brings us back to the starting point.

Note that there is a degree shift here by the dimension of the tori (mod 2). That is because there are really two kinds of D-brane charges, R-R and NS-NS, and the duality is expected to interchange them when n is odd. (This corresponds to switching from type IIA string theory to type IIB and vice versa.)

The topology change in passing from X to the T-dual $X^\#$ was first discovered when $X = S^2 \times S^1$, a trivial bundle over Z, but with H the usual generator of $H^3(X)$. In this case, calculations as above in local coordinates suggested that the T-dual should be S^3 with trivial H-flux. And indeed, $K^*_H(S^2 \times S^1)$ is $\mathbb{Z}$ in both even and odd degree, just like the K-theory of S^3!

An approach to T-duality for circle bundles through noncommutative topology

The following approach through noncommtuative topology, first explored in [2], explains T-duality for circle bundles in quite a nice way. One can also do everything purely topologically, as in [3].

Consider a principal $\mathbb{T}$-bundle $p \colon X \to Z$ and a fixed class $H \in H^3(X, \mathbb{Z})$. We already know that the pair (X, H) corresponds to a stable continuous-trace algebra $A = CT(X, H)$ with $\widehat{A} = X$. Now in general, the free action of $\mathbb{T}$ on X does not lift to an action of $\mathbb{T}$ on A. (It lifts iff $H \in p^*(H^3(Z))$.) However, if we think of $\mathbb{T}$ as $\mathbb{R}/\mathbb{Z}$, we can always lift to an $\mathbb{R}$-action α on A, which is unique up to the natural equivalence relation (exterior equivalence of actions).

Take the crossed product $A^{\#} = A \rtimes_{\alpha} \mathbb{R}$. By Connes' Thom isomorphism theorem [6], there is a natural isomorphism

$$K_{*+1}(A) \cong K_*(A^{\#}).$$

The left-hand group is $K_H^{*+1}(X)$, essentially by definition. But $A^{\#}$ has spectrum $X^{\#}$ a circle bundle over $Z = X/\mathbb{T}$, and in fact the fibers of $p^{\#}\colon X^{\#} \to Z$ can be naturally identified with $\widehat{\mathbb{Z}}$, the dual circle to the original $\mathbb{T}$. Furthermore, it's easy to show that $A^{\#}$ is a stable continuous-trace algebra. So if its Dixmier-Douady class is $H^{\#}$, we get the desired isomorphism

$$K_H^{*+1}(X) \cong K_{H^{\#}}^{*}(X^{\#}).$$

Furthermore, repeating the process brings us back to the starting point (by Takai duality).

It turns out this C^*-algebraic situation, exactly the one needed for T-duality, had been studied in [12] (without any physics applications in mind).

Theorem 15 ([12]) *Suppose* $\mathbb{R}$ *acts on a continuous-trace* A *with spectrum* X*, so that the induced action of* $\mathbb{R}$ *on* X *passes to a free action of* $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ *on* X *with quotient* Z*. Then the crossed product* $A^\#$ *is also a continuous-trace algebra, with spectrum* $X^\#$ *a principal bundle for the dual torus with the dual action. Furthermore, the characteristic classes of the circle bundles* $p\colon X \to Z$ *and* $p^\#\colon X^\# \to Z$ *and the Dixmier-Douady classes* $H \in H^3(X)$ *and* $H^\# \in H^3(X^\#)$ *satisfy:*

$$[p] = \left(p^\#\right)_! (H^\#), \quad [p^\#] = (p)_! (H).$$

An example

Suppose $X = S^{2n+1}$, $n \geq 1$, and $p\colon X \to Z = \mathbb{CP}^n$ is the Hopf fibration. If $n \geq 2$, $H^3(X) = 0$, and any continuous-trace algebra over X must be stably trivial. In general, if we T-dualize the pair $(X, 0)$ with respect to p, then of course $p_!(H) = 0$, so the dual bundle is trivial, i.e., $X^\# = \mathbb{CP}^n \times S^1$, with $p^\#$ just projection on the second factor. Note that $X^\#$ has much bigger cohomology, and thus K-theory, than X. But on the other hand, we need to have $(p^\#)_!(H^\#) = [p]$, which is the canonical generator of $H^2(\mathbb{CP}^2)$. So one can see that $H^\# = [p] \times y$, where y is the standard generator of $H^1(S^1)$. And sure enough, one can see that in the spectral sequence for computing $K^*_{[p]\times y}(\mathbb{CP}^n \times S^1)$, most of the cohomology cancels out and the twisted K-theory is just $\mathbb{Z}$ in both even and odd degree.

The case of higher-dimensional fibers

Let $p\colon X \to Z$ be a principal T-bundle as above, with T an n-torus, G its universal cover (a vector group). Also let $H \in H^3(X, \mathbb{Z})$. For the pair (X, H) to be dualizable, we want the T-action on X to be in some sense compatible with H. A natural hope is for the T-action on X to lift to an action on the principal PU-bundle defined by H, or equivalently, to an action on $CT(X, H)$. Equivariant Morita equivalence classes of such liftings (with varying H) define classes in the equivariant Brauer group [7]. Unfortunately

$$p^*\colon \mathrm{Br}(Z) \xrightarrow{\cong} \mathrm{Br}_T(X)$$

and so $\mathrm{Br}_T(X)$ is not that interesting. But $\mathrm{Br}_G(X)$, constructed from local liftings, is quite a rich object.

Theorem 16 *Let*

$$T \overset{\iota}{\hookrightarrow} X \overset{p}{\twoheadrightarrow} Z$$

be a principal T-bundle as above, with T an n-torus, G its universal cover (a vector group). The following sequence is exact:

$$\mathrm{Br}_G(X) \xrightarrow{F} \mathrm{Br}(X) \cong H^3(X, \mathbb{Z}) \xrightarrow{\iota^*} H^3(T, \mathbb{Z}).$$

Here F is the "forget G-action" map. Thus if $n \leq 2$, every stable continuous-trace algebra on X admits a G-action compatible with the T-action on X. In general, $\operatorname{image} F = \ker \iota^*$.

When such a G-action exists, we will construct a T-dual by looking at the C^*-algebra crossed product

$$CT(X, H) \rtimes G.$$

The desired K-theory isomorphism will come from Connes' Thom isomorphism theorem.

The case of fibers of dimension $n = 2$

From now on, we stick to the case $n = 2$ for simplicity. H^*_M denotes cohomology with Borel cochains in the sense of C. Moore.

Theorem 17 ([7], [10]) *If $n = 2$, there is a commutative diagram of exact sequences*:

$$
\begin{tikzcd}
 & H^0(Z,\mathbb{Z}) \arrow[d] & & 0 \arrow[d] & \\
H^2(X,\mathbb{Z}) \arrow[r] & H^2_M(G, C(X,\mathbb{T})) \arrow[rr, "\xi"] \arrow[d, dotted, "a"'] & & \ker F \arrow[r, "\eta"] \arrow[d] & 0 \\
 & C(Z, H^2_M(\mathbb{Z}^2,\mathbb{T})) & & \mathrm{Br}_G(X) \arrow[ll, "M"'] \arrow[d, "F"] & \\
 & H^1(Z,\mathbb{Z}) \arrow[d] & & H^3(X,\mathbb{Z}) \arrow[ll, dotted, "p_!"'] \arrow[d] & \\
 & 0 & & 0 &
\end{tikzcd}
$$

$M\colon \mathrm{Br}_G(X) \to C(Z, H^2_M(\mathbb{Z}^2,\mathbb{T})) \cong C(Z,\mathbb{T})$ is the Mackey obstruction map, $F\colon \mathrm{Br}_G(X) \to \mathrm{Br}(X)$ is the forgetful map, and $h\colon C(Z,\mathbb{T}) \to H^1(X,\mathbb{Z})$ sends a continuous function $Z \to S^1$ to its homotopy class.

Applications to T-duality

Theorem 18 *Let $p\colon X \to Z$ be a principal $\mathbb{T}^2$-bundle as above. Let $H \in H^3(X, \mathbb{Z})$ be an "H-flux" on X. Then:*

1. If $p_! H = 0 \in H^1(Z, \mathbb{Z})$, one can choose an element of $\mathrm{Br}_G(X)$ lifting H and with vanishing Mackey obstruction. Taking the crossed product by G gives a classical T-dual to (p, H), consisting of $p^\#\colon X^\# \to Z$, which is a another principal $\mathbb{T}^2$-bundle over Z, and $H^\# \in H^3(X^\#, \mathbb{Z})$, the "T-dual H-flux" on $X^\#$. One has an isomorphism

$$K^*_{H^\#}(X^\#) \cong K^*_H(X).$$

2. If $p_! H \neq 0 \in H^1(Z, \mathbb{Z})$, then a classical T-dual as above does not *exist. However, there is a "nonclassical" T-dual bundle of noncommutative tori over Z. It is* not unique, *but the non-uniqueness does not affect its K-theory, which is naturally $\cong K^*_H(X)$.*

An example

Let $X = T^3$, $p \colon X \to S^1$ the trivial $\mathbb{T}^2$-bundle. If $H \in H^3(X, \mathbb{Z}) \neq 0$, $p_!(H) \neq 0$ in $H^1(S^1)$. By Theorem 18, there is no classical T-dual to (p, H). (The problem is that non-triviality of H would have to give rise to non-triviality of $p^\#$, but there are no nontrivial $\mathbb{T}^2$-bundles over S^1.)

One can realize $CT(X, H)$ in this case as follows. Let $\mathcal{H} = L^2(\mathbb{T})$. Define the projective unitary representation $\rho_\theta : \mathbb{Z}^2 \to PU(\mathcal{H})$ by letting the first $\mathbb{Z}$ factor act by multiplication by z^k, the second $\mathbb{Z}$ factor act by translation by $\theta \in \mathbb{T}$. Then the Mackey obstruction of ρ_θ is $\theta \in \mathbb{T} \cong H^2(\mathbb{Z}^2, \mathbb{T})$. Let $\mathbb{Z}^2$ act on $C(\mathbb{T}, \mathcal{K}(\mathcal{H}))$ by α, which is given at the point θ by ρ_θ. Define the C^*-algebra

$$\begin{aligned} B &= \mathrm{Ind}_{\mathbb{Z}^2}^{\mathbb{R}^2} (C(\mathbb{T}, \mathcal{K}(\mathcal{H})), \alpha) \\ &= \Big\{ f : \mathbb{R}^2 \to C(\mathbb{T}, \mathcal{K}(\mathcal{H})) : \\ &\qquad f(t+g) = \alpha(g)(f(t)),\ \ t \in \mathbb{R}^2, g \in \mathbb{Z}^2 \Big\}. \end{aligned}$$

Then B is a continuous-trace C^*-algebra having spectrum T^3, having an action of $\mathbb{R}^2$ whose induced action on the spectrum of B is the trivial bundle $\mathbb{T}^3 \to \mathbb{T}$. The crossed product algebra $B \rtimes \mathbb{R}^2 \cong C(\mathbb{T}, \mathcal{K}(\mathcal{H})) \rtimes \mathbb{Z}^2$ has fiber over $\theta \in \mathbb{T}$ given by $\mathcal{K}(\mathcal{H}) \rtimes_{\rho_\theta} \mathbb{Z}^2 \cong A_\theta \otimes \mathcal{K}(\mathcal{H})$, where A_θ is the noncommutative 2-torus. In fact, the crossed product $B \rtimes \mathbb{R}^2$ is isomorphic to $C^*(H_k) \otimes \mathcal{K}$, where H_k is the integer Heisenberg-type group,

$$H_k = \left\{ \begin{pmatrix} 1 & x & \frac{1}{k}z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\},$$

a lattice in the usual Heisenberg group $H_{\mathbb{R}}$ (consisting of matrices of the same form, but with x, y, $z \in \mathbb{R}$).

The Dixmier-Douady invariant H of B is k times a generator of H^3. We see that the group C^*-algebra of H_k serves as a non-classical T-dual.

Further work

This just seems to be the beginning of the story, and several people are now working on noncommutative T-duality. Groupoids and stacks are appearing more and more. Block *et al*. [1] have been studying a Fourier-Mukai duality in a context like that of mirror symmetry in algebraic geometry, using derived categories. Bunke *et al*. [4] have a version of topological T-duality using stacks. Daenzer [8] has a version of noncommutative T-duality based on groupoids. Pande [11] has studied T-duality for S^1-bundles that are allowed to degenerate at isolated fixed points, and found an interesting duality between H-monopoles and Kaluza-Klein monopoles, which he explains in terms of gerbes.

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