REVIEW OF *THE NOVIKOV CONJECTURE: GEOMETRY AND ALGEBRA* BY MATTHIAS KRECK AND WOLFGANG LÜCK, OBERWOLFACH SEMINARS, VOL. 33, BIRKHÄUSER, BASEL–BOSTON–BERLIN, 2005, ISBN 3-7643-7141-2

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The statement which nowadays goes under the name of the "Novikov Conjecture" was first formulated by Sergey Novikov in 1970 (though be careful—Novikov has made many conjectures during his career and sometimes this name is used to refer to one of the others!). We will come to the precise statement shortly, but suffice it to say for now that this is one of the crucial open questions in the topology of manifolds, perhaps *the* most important question not dealing with the special peculiarities of dimensions 2, 3, and 4. For a copy of Novikov's original formulation of the conjecture in Russian, as well as an English translation, the reader can consult [10, §11] or else §2 of "A history and survey of the Novikov Conjecture," in [4], by Steve Ferry, Andrew Ranicki, and this reviewer.

What has made the Novikov Conjecture so fascinating, and so central to contemporary mathematics, is not so much its precise statement, which on first sight appears rather technical, but the fact that it is so closely connected to so many other problems in topology, differential geometry, algebra, and even operator algebras and representation theory. The book under review, by Matthias Kreck and Wolfgang Lück, is the product of a one-week intensive course by the authors in Oberwolfach in January, 2004, on this circle of ideas.

The story of the Novikov Conjecture begins with the attempt to "classify" manifolds of an arbitrary dimension n. (For simplicity, let's restrict attention to connected compact manifolds without boundary, either in the smooth or in the topological category.) To do this in an optimal way would mean to give an explicit list of all such manifolds, together with invariants that could be used to distinguish them. For $n = 1$, there is only one compact connected manifold up to either homeomorphism or diffeomorphism, the circle, and for $n = 2$, there are two infinite families: the orientable surfaces, classified either by the genus $g = 0, 1, \cdots$ or by the Euler characteristic $\chi = 2 - 2g = 2, 0, -2, \cdots$, and the non-orientable surfaces M, classified either by the Euler characteristic $\chi(M)$ or else by the genus $\widetilde{g}$ of the oriented double cover $\widetilde{M}$. (These two numbers are related by the simple formula $\chi(M) = \frac{1}{2}\chi(\widetilde{M}) = 1 - \widetilde{g}$.) The classification of manifolds in dimension 2 is one of the most elegant classification theorems in all of mathematics, and it is frequently taught in beginning topology classes.

Unfortunately, no such simple classification can be given in higher dimensions. Part of the reason has to do with the fundamental group $\pi_1(M)$; there are simply too many possibilities for it. In fact, if $n \geq 4$, it is easy to see that *any* finitely presented group G (say with generators $a_1, \cdots, a_k$ and relations $r_1, \cdots, r_m$) is the fundamental group of a closed n-manifold. Simply start with a connected sum N^n of k copies of $S^1 \times S^{n-1}$; this has fundamental group the free group on k generators. Each relation r_j is an element of $\pi_1(N)$, and thus can be represented by a map $\gamma_j \colon S^1 \to N$. By transversality, we can homotope these maps so that they become embeddings with disjoint images, which then have disjoint tubular neighborhoods that are disjoint embedded copies of the cylinder $S^1 \times D^{n-1}$. Do "surgery" by cutting out each cylinder and replacing it by $D^2 \times S^{n-2}$, which has the same boundary, $S^1 \times S^{n-2}$. This has the effect of killing off each r_j, or in other words, "building in" each of the relations r_j in turn. The result is a connected compact smooth n-manifold M^n with fundamental group G. What makes things worse [than the fact that any G is the fundamental group of a manifold] is that the word problem for general finitely presented groups is unsolvable; so, in general, there is no algorithm for telling if the result M of this construction is simply connected or not.

Thus in high dimensions, we have to approach the classification problem differently, for instance, first fixing the fundamental group, then the homotopy type (which includes such information as the homology and homotopy groups), and only then trying to classify manifolds within a homotopy type. This program has had notable successes; for example, Smale's proof of the h-cobordism theorem [15] implied the [topological] Poincaré Conjecture in dimensions $n \geq 5$. In other words, any compact, smooth n-manifold homotopy-equivalent to S^n is homeomorphic to S^n, provided that $n \geq 5$. The same is now known to be true even if the manifold is only assumed to be a topological manifold, without necessarily having a smooth structure *a priori*, and is valid even in dimension $n = 4$ (this case requires deep work of Freedman). If one wants a classification up to diffeomorphism, then the results are somewhat more complicated—for example, Milnor and Kervaire ([8], [7]) showed that there are precisely 28 diffeomorphism classes of compact, smooth 7-manifolds homotopy-equivalent to S^7. Smooth manifolds homeomorphic to S^n but not diffeomorphic to it are usually called "exotic spheres."

In the case of manifolds with a fixed fundamental group G, the classification of high-dimensional manifolds within a homotopy type is given by the subject known as surgery theory, which involves some of the same techniques used for the results cited above, but also a much trickier study of transversality that takes the group G into account. This theory was first developed by Browder and Novikov in the simply connected case, then by Wall in general. (See [17] and the papers in [1] or [2] for general overviews, or the books of Ranicki, [11] and [13], and Wall's "bible" of the subject, [16], for all the details.) The main result of surgery theory is an exact sequence for computing the "structure set" $\mathcal{S}(M)$ of all n-manifolds homotopy-equivalent to a fixed closed (i.e., compact without boundary) manifold M^n, up to homeomorphism or diffeomorphism. (More precisely, one classifies homotopy equivalences $M' \xrightarrow{\varphi} M$, modulo the equivalence relation that two such are identified

if there is a homotopy-commutative diagram

$$
\begin{array}{ccc}
M' & \xrightarrow{\ \varphi\ } & M \\
 & \searrow{\scriptstyle\psi} & \uparrow{\scriptstyle\varphi'} \\
 & & M''
\end{array}
$$

with ψ a homeomorphism or diffeomorphism.) Roughly speaking[1], the surgery exact sequence has the form

$$
(1) \qquad \cdots \xrightarrow{\ \sigma\ } L_{n+1}(\mathbb{Z}G) \xrightarrow{\ \omega\ } \mathcal{S}(M) \xrightarrow{\ \nu\ } \mathcal{N}(M) \xrightarrow{\ \sigma\ } L_n(\mathbb{Z}G),
$$

where ν sends a manifold homotopy-equivalent to M to its "normal data" in $\mathcal{N}(M)$, measuring characteristic classes and the like, where $L_j(\mathbb{Z}G)$ are algebraically defined groups only depending on G and periodic in j with period 4, namely certain Grothendieck groups of quadratic forms on finitely generated free $\mathbb{Z}G$-modules, the so-called surgery obstruction groups, and where σ is the "surgery obstruction" map. In (1), $\mathcal{S}$ and $\mathcal{N}$ depend on whether one is classifying manifolds up to homeomorphism or up to diffeomorphism, but $L_j(\mathbb{Z}G)$ is the same in both cases. In fact, the whole theory in the topological case, including the exact sequence, can be constructed purely algebraically [12], but then one misses some of the geometric intuition for what it means.

The surgery exact sequence (1) in principle enables us to compute $\mathcal{S}(M)$, in other words, to classify manifolds in the homotopy type of M, but only if we can understand the surgery obstruction groups $L_j(\mathbb{Z}G)$ and the obstruction map σ. In some cases, say for finite G, the groups $L_j(\mathbb{Z}G)$ are completely computed (see the paper of Hambleton and Taylor in [1], for example), and thus surgery theory gives us a fairly explicit classification theory of manifolds in this case. But the problem, and this is where the Novikov Conjecture comes in, is that for general groups G, the group ring $\mathbb{Z}G$ can be quite complicated, and there is no obvious way to compute the surgery obstruction groups. Thus, in some sense, the surgery exact sequence (1) may simply replace one difficult problem by an equally difficult one, of computing $L_j(\mathbb{Z}G)$ and the obstruction map σ.

To put the Novikov Conjecture in context, it helps to recall a still older conjecture, the Borel Conjecture. This asserts that if a closed manifold M^n is aspherical, i.e., if its universal cover $\widetilde{M}$ is contractible, then M should be determined up to homeomorphism by its fundamental group. In other words, in this case, $\mathcal{S}^{\mathrm{top}}(M)$ should consist of only a single element, which means (because of the surgery exact sequence (1)) that we expect $\sigma\colon \mathcal{N}^{\mathrm{top}}(M) \to L_n(\mathbb{Z}\pi_1(M))$ to be an isomorphism. (Incidentally, the restriction to the topological category is essential here; M cannot be determined up to diffeomorphism by its fundamental group, because it is known that there are cases where taking the connected sum with an exotic sphere changes the smooth structure, though of course it does not change the homotopy type, or even the homeomorphism class.) When we think of things this way, the relationship between M and $\pi_1(M)$ is that M is a classifying space BG for the group $G = \pi_1(M)$, and we want $\mathcal{N}^{\mathrm{top}}(BG) \xrightarrow{\ \sigma\ } L_n(\mathbb{Z}G)$ to be an isomorphism. In

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[1]Experts will recognize that I am deliberately skipping over decorations on the L-groups, the difference between the periodic and connective L-groups, and the complications arising from the fact that $\mathcal{S}$ is only a pointed set, not a group.

this formulation, the manifold M itself plays no role in the Borel Conjecture, and everything depends only on the group G.

Now we are ready to explain the Novikov Conjecture and Borel Conjecture and their connection with the surgery obstruction groups $L_*(\mathbb{Z}G)$. It turns out that the surgery obstruction map $\sigma\colon \mathcal{N}(M) \to L_n(\mathbb{Z}G)$, when M is a closed n-manifold with fundamental group G, always factors through a so-called *assembly map* $A\colon H_n(BG; \mathbf{L}) \to L_n(\mathbb{Z}G)$. In fact, in the situation $M = BG$ of the Borel Conjecture, we have $\mathcal{N}^{\mathrm{top}}(M) \cong H_n(BG; \mathbf{L})$, and σ and A are the same map. Here $H_n(BG; \mathbf{L})$ denotes a certain generalized homology group of the classifying space BG; the homology with coefficients in the *surgery spectrum* $\mathbf{L}$. Rationally, we have $H_n(BG; \mathbf{L}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \bigoplus_k H_{n-4k}(G; \mathbb{Q})$, so the domain of the assembly map is roughly just the group homology of G. The Borel Conjecture boils down to the statement that the assembly map A should be an isomorphism when there is a closed manifold model for BG; the Novikov Conjecture (whose original formulation had to do with homotopy invariance of certain combinations of rational characteristic classes) boils down to the statement that the assembly map A should be *rationally injective* for any group G. (It can't be injective on the torsion for all groups, since if G is a cyclic group of odd order, $H_*(BG; \mathbf{L})$ has odd torsion but $L_*(\mathbb{Z}G)$ does not.)

As I mentioned before, part of what makes the Novikov Conjecture so interesting is the fact that it connects with many other areas of mathematics. This can be explained by the fact that assembly maps, similar to the one we have discussed for surgery theory, $A\colon H_n(BG; \mathbf{L}) \to L_n(\mathbb{Z}G)$, show up in other areas of mathematics as well. Thus there are also assembly maps in algebraic K-theory,

$$A^K\colon \; H_n(BG; \mathbf{K}(\mathbb{Z})) \to K_n(\mathbb{Z}G),$$

and in topological K-theory of C^*-algebras,

$$A^{C^*}\colon \; K_n(BG) = H_n(BG; \mathbf{K}^{\mathrm{top}}) \to K_n(C_r^*(G)).$$

There are conjectures that these are also rationally injective, and these conjectures are usually known as the Novikov Conjecture for algebraic K-theory and the Strong Novikov Conjecture, respectively. (The latter name comes from the fact that the C^*-algebraic Novikov Conjecture implies the usual Novikov Conjecture, by an argument due to Mishchenko and Kasparov, as well as results about the classification of manifolds of positive scalar curvature, by an argument of the reviewer.) In fact, there are also analogues of the Borel Conjecture, too, usually known as the Farrell-Jones Conjecture and the Baum-Connes Conjecture, respectively. In the case where G is torsion-free (as it would have to be if G is the fundamental group of an aspherical manifold), these assert that the assembly maps $A\colon H_n(BG; \mathbf{K}(\mathbb{Z})) \to K_n(\mathbb{Z}G)$ and $A\colon H_n(BG; \mathbf{K}^{\mathrm{top}}) \to K_n(C_r^*(G))$ are also isomorphisms.

Now that we have explained the basics of the subject, it is time to discuss the Kreck-Lück book itself. It is not an authoritative treatise on the Novikov Conjecture, but rather an outline and discussion of the Novikov Conjecture and related areas, intended for those who already have a reasonable background in topology. The book covers a remarkably large amount of mathematics, but it does move quickly and usually only gives sketches, but not full details, of proofs. The book can be subdivided into four main sections, which involve increasing levels of complexity:

(1) a rough discussion of the Novikov Conjecture, emphasizing geometric implications and the connection with the signature (chapters 0–4);

(2) a quick discussion of Whitehead torsion and the s-cobordism theorem (chapters 5–8);
(3) a mini-course on surgery theory, leading to the surgery exact sequence, the assembly map, and a sketch of a proof of the Novikov Conjecture and the Borel Conjecture for free abelian groups (chapters 9–16);
(4) a more technical mini-course on a fancier approach to assembly maps, using equivariant stable homotopy theory, followed by a review of the status of the original Novikov Conjecture as well as its variants, such as the Farrell-Jones Conjecture and the Baum-Connes Conjecture (chapters 17–24).

In addition, the book comes with a good set of exercises (chapter 25) and hints for their solution (chapter 26), plus a very comprehensive bibliography and an index.

For someone who wants to learn about the Novikov Conjecture and related topics, this book provides a very natural starting point. In fact, the first three sections of the book could be used for a second-year graduate course on the classification of manifolds. The first section of the book is quite approachable even to graduate students who have only had a single year of topology. The second section (on Whitehead torsion and the s-cobordism) could be supplemented by the classic texts [3], [14], and [9]. The third and fourth sections of this book require more preparation, both in algebra and in homotopy theory. But they are still a lot more readable for beginners than much of the research literature. The reader could also consult the two volumes [4] and [5] on the Novikov Conjecture, as well as the articles in the Handbook of K-theory [6], for more on the algebraic K-theory side of things. And for general perspectives on surgery and the Novikov Conjecture, the texts [13] and [17] are highly recommended. (Ranicki's style is more formal than Kreck-Lück; Weinberger's a lot more informal.)

In general, I very much enjoyed reading this book, and appreciated the care with which the authors supplemented the text with exercises and examples (such as the problem, posed in chapter 0, of classifying spin manifolds of dimension ≤ 6 with fundamental group $\mathbb{Z} \oplus \mathbb{Z}$), which will greatly enhance the usefulness of the book for someone who wants to use it for self-study. My only major criticism is that it is a shame that the book wasn't proofread more carefully. There are misprints everywhere, most of them just minor annoyances, but some of them rather serious. I will just mention three examples, but a more extensive list of errata may be found at

<center>http://www.math.umd.edu/~jmr/KreckLueckErr.pdf</center>

The first example is that in the statement of Exercise 6.1 on page 216, one should have $o(P_*) = \sum_{n \geq 0} (-1)^n [P_n]$, but the factor $(-1)^n$ is missing. The properties of the finiteness obstruction all depend on the fact that one has an *alternating* sum. Secondly, in the discussion of G-CW-complexes on pages 153–154, the hypotheses on G are never made explicit. Definition 19.1 and Remark 19.2 seem to be given for G discrete, but then Example 19.4 seems to refer to the case of G a Lie group, for which the definition of properness has to be modified. Or on page 189, it is asserted that restricting to dimensions ≥ 5 in the Stable Gromov-Lawson-Rosenberg Conjecture is essential, whereas this is not true at all. What *is* true is that there are some exceptional Seiberg-Witten obstructions to positive scalar curvature in dimension 4, but they are unstable, and do not affect the stable conjecture. Thus the *unstable* version of the conjecture could only be valid in dimensions 5 and up, but even here,

there are counterexamples due to Schick for certain special fundamental groups, which again do not affect validity of the stable conjecture.

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