Levi-Civita connections for noncommutative tori

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In honor of Henri, a long-time friend

Connections

One of the most basic notions in differential geometry is that of a connection. There are many equivalent points of view, but for our purposes we'll define connections this way. Let M be a C^∞ manifold and $p\colon E \to M$ a smooth vector bundle. Recall that a section of E is a (smooth) map $s\colon M \to E$ with $p \circ s = \mathrm{id}_M$. If E is a trivial bundle, then a section s is just a C^∞ (vector-valued) function on M and we can take directional derivatives of s. A connection is a way of doing this on a nontrivial bundle.

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$$\nabla\colon \mathcal{X}(M) \times \Gamma(E) \to \Gamma(E), \quad (X, s) \mapsto \nabla_X(s),$$

which is $C^\infty(M)$-linear in the variable X, i.e., $\nabla_{fX}(s) = f\nabla_X(s)$, and satisfies the Leibniz rule for derivatives.

Connections (cont'd)

This says that

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Now suppose that a metric is given on E, i.e., a smoothly varying family of inner products on the fibers $p^{-1}(x)$ of E so that we have a pairing

$$\langle \quad , \quad \rangle \colon \Gamma(E) \times \Gamma(E) \to C^\infty(M), \quad (s, s') \mapsto \langle s, s' \rangle.$$

We say ∇ is compatible with the metric if

$$X \cdot \langle s, s' \rangle = \langle \nabla_X s, s' \rangle + \langle s, \nabla_X s' \rangle. \tag{2}$$

This means that the inner product of parallel sections ($\nabla_X s = 0 \ \forall X$) is constant.

Levi-Civita's Theorem

Now suppose $E = TM$ is the tangent bundle of M. That means $\Gamma(E) = \mathcal{X}(M)$, so we can define the torsion of a connection ∇,

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]. \tag{3}$$

This is a bilinear map $\mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$.

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Theorem (Levi-Civita, 1917)

On a Riemannian manifold M, there is one and only one torsion-free connection on TM compatible with the metric.

The connection in this theorem is called the Levi-Civita connection.

Riemannian Curvature

Levi-Civita's Theorem gives an easy way to define curvature. On a Riemannian manifold, we let ∇ be the Levi-Civita connection, and then the Riemann curvature tensor is

$$R(X, Y) = \nabla_Y \nabla_X - \nabla_X \nabla_Y - \nabla_{[Y,X]}. \tag{4}$$

Thus $R \equiv 0$, i.e., the metric is flat $\Leftrightarrow$ ∇ is a Lie algebra homomorphism. It's a nontrivial fact that R is a tensor, i.e., $(X, Y, Z) \mapsto R(X, Y)Z$ is a section of $\mathrm{Hom}(TM^{\otimes 3}, TM)$.

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2. Assuming we know what a non-commutative manifold is, what is a vector field on such an object?

To answer (1), we'll define a (compact) noncommutative manifold to be given by a "nice" Fréchet subalgebra A^∞ of a unital C^*-algebra A. (The prototypes are noncommutative tori, to be discussed shortly.) The sections of a (smooth) vector bundle are replaced by a finitely generated projective A^∞-module. This is motivated by the fact that in the commutative case, $\Gamma(E)$ is such a module over $C^\infty(M)$.

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In the noncommutative case we could consider derivations of A^∞, the analogue of (2), but there is no reason why this should agree with (3). In any event, the space of derivations is a Lie algebra, but not necessarily a projective A^∞-module. So here we propose a novel solution: use both definitions at once!

Connes' Definition of Connections

In his famous 1980 *Comptes Rendus* paper, Connes proposed using (1) as a definition of connection in the noncommutative case. Here we replace s by an element of a projective A^∞-module (we're using left modules; Connes used right modules) and take for X an element of $\mathfrak{g}$, the Lie algebra of a group G acting on A^∞. Such X's are of course derivations, and can be viewed as *very special* vector fields. But we have to toss aside the analogue of $C^\infty(M)$-linearity in the variable X, since the space of X's isn't an A^∞-module. Connes also showed that (4) still works as a definition of curvature, and still has tensorial properties.

Noncommutative Tori

For any reasonable definition of noncommutative manifold, basic examples should be the noncommutative tori. Fix an $n \times n$ skew-symmetric matrix Θ and let A_Θ be the universal C^*-algebra on unitaries U_j, $j = 1, \cdots, n$, with $U_j U_k = \exp(2\pi\Theta_{jk}) U_k U_j$. This algebra carries a gauge action of $\mathbb{T}^n$ given by $t \cdot U_j = t_j U_j$, and the smooth vectors A_Θ^∞ for this action look like $\mathcal{S}(\mathbb{Z}^n)$ with convolution twisted by a 2-cocycle. The algebra A_Θ^∞ is our NC substitute for $C^\infty(\mathbb{T}^n)$. The infinitesimal generators of the gauge action are ∂_j with $\partial_j(U_k) = \delta_{jk}\, 2\pi\, i\, U_k$. From now on we'll fix Θ and drop the ∞ notation from A_Θ^∞, since we only care about "smooth functions."

Vector Fields on NC Tori

We are now working on A_Θ with the basic $*$-derivations ∂_j. Since tori are parallelizable, we would expect the "tangent bundle" on A_Θ to be trivial, so define

$$\mathcal{X} = \text{ the free rank-}n\ A_\Theta\text{-module on } \partial_1, \cdots, \partial_n.$$

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These correspond to definitions 2 and 3 of a vector field on an ordinary manifold.

The Basic Problem

We're now ready to define a Riemannian metric on A_Θ. We define this to be an A_Θ-valued inner product on $\mathcal{X}$ making it into a (pre)Hilbert C^*-module. (The idea of doing this is due to Rieffel.) But we also want the inner product to be "real" on "real" vector fields, so we add the requirement

$$\langle \partial_j, \partial_k \rangle = \langle \partial_j, \partial_k \rangle^* = \langle \partial_k, \partial_j \rangle. \tag{5}$$

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The big problem is that the axiom (1) for a connection only makes sense when X is a derivation, i.e., when $X \in \mathcal{D}$, though we need s to lie in an A_Θ-module, i.e., $s \in \mathcal{X}$.

Inner Derivations

What if $X = \operatorname{ad} a$ is an inner derivation? The axiom for a connection would require

$$\nabla_{\operatorname{ad} a}(bs) = [a, b]s + b\nabla_{\operatorname{ad} a}(s),$$

so

$$\nabla_{\operatorname{ad} a} \circ b - b\nabla_{\operatorname{ad} a} = [a, b].$$

This forces $\nabla_{\operatorname{ad} a}$ to be multiplication by a, up to something central.

The Theorem of Bratteli-Elliott-Jørgensen

Theorem (Bratteli-Elliott-Jørgensen)

Let Θ be "generic" (in a specific number-theoretic sense). Then $\mathcal{D}/\{$inner derivations$\}$ is just the linear span of $\partial_1, \cdots, \partial_n$. (Also, the center of A_Θ is just the scalars.)

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Note: In this case, there is a canonical splitting $\text{Inn}\, A_\Theta \to A_\Theta$ given by $\text{ad}\, a \mapsto a - \tau(a)$, where τ is the (unique) normalized trace.

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Note: In this case, there is a canonical splitting $\text{Inn}\, A_\Theta \to A_\Theta$ given by $\text{ad}\, a \mapsto a - \tau(a)$, where τ is the (unique) normalized trace. Therefore in the situation of the Theorem we will define $\nabla_{\text{ad}\, a}$ to be multiplication by $a - \tau(a)$, and then ∇ is determined once

$$\nabla_1 = \nabla_{\partial_1}, \cdots, \nabla_n = \nabla_{\partial_n}$$

are given.

Statement of the Theorem

Theorem (NC Levi-Civita)

Let Θ be generic in the sense of the B-E-J Theorem. Fix any Riemannian metric satisfying (5) on $\mathcal{X}_\Theta = \mathcal{X}(A_\Theta)$. Then there is a unique connection

$$\nabla\colon \mathcal{D}_\Theta \times \mathcal{X}_\Theta \to \mathcal{X}_\Theta$$

compatible with the metric, normalized as we've explained on inner derivations, and satisfying the symmetry condition

$$\nabla_j \partial_k = \nabla_k \partial_j.$$

(This is the "torsion-free" condition applied to $\partial_1, \cdots, \partial_n$. Torsion doesn't make sense for inner derivations.)

Sketch of Proof

Define

$$\langle \nabla_j \partial_k, \partial_\ell \rangle = \frac{1}{2} \left[\partial_j \langle \partial_k, \partial_\ell \rangle + \partial_k \langle \partial_\ell, \partial_j \rangle - \partial_\ell \langle \partial_j, \partial_k \rangle \right]. \qquad (6)$$

Then the axioms are all satisfied. In the other directions, the axioms force (6).

The Curvature Tensor

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Proposition

$R(X, Y) \equiv 0$ *if either* X *or* Y *is an inner derivation.*

Proof.

Direct calculation. □

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Proof.

Direct calculation. □

Thus the curvature is completely determined by the

$$R_{i,j,k,\ell} = \langle R(\partial_i, \partial_j)\partial_k, \partial_\ell \rangle.$$

Bianchi Identities, etc.

Theorem

The curvature satisfies the identities:

1. $R_{j,k,\ell,m} + R_{k,\ell,j,m} + R_{\ell,j,k,m} = 0$ (*Bianchi identity*)
2. $R_{j,k,\ell,m} = -R_{k,j,\ell,m}$.
3. $R_{j,k,\ell,m} = -R_{j,k,m,\ell}$.
4. $R_{j,k,\ell,m} = R_{\ell,m,j,k}$.

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3. $R_{j,k,\ell,m} = -R_{j,k,m,\ell}$.
4. $R_{j,k,\ell,m} = R_{\ell,m,j,k}$.

Proof.

Exactly as in the classical case. □

Metrics on A_θ Conformal to a Flat Metric

The simplest nontrivial example is the case of a noncommutative 2-torus, or an irrational rotation algebra A_θ. This is simple for θ irrational and satisfies the B-E-J condition for generic θ. Let's consider metrics "conformal" to the simplest flat metric $\langle \partial_j, \partial_k \rangle = \delta_{j,k}$. In other words, we assume

$$\langle \partial_j, \partial_k \rangle = e^h \delta_{j,k}, \quad h = h^* \in A_\theta.$$

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Then direct calculation gives

$$\begin{cases} \nabla_1 \partial_1 = -\nabla_2 \partial_2 = \frac{1}{2}(k_1 \partial_1 - k_2 \partial_2), \\ \nabla_2 \partial_1 = \nabla_1 \partial_2 = \frac{1}{2}(k_2 \partial_1 + k_1 \partial_2), \\ k_j = \partial_j(e^h) e^{-h}, \\ R_{1,2,1,2} = -\frac{1}{2}\left(\Delta(e^h) - \partial_1(e^h) e^{-h} \partial_1(e^h) - \partial_2(e^h) e^{-h} \partial_2(e^h)\right). \end{cases}$$

A Version of Gauss-Bonnet for A_θ

These formulas are the same as what one has in the classical case $\theta = 0$ for a metric on T^2 conformal to the flat metric on $\mathbb{R}^2/\mathbb{Z}^2$. But in the commutative case, k_j further simplifies to $\partial_j(h)$ and $R_{1,2,1,2}$ reduces to $-\frac{1}{2}e^h\Delta h$. This is not quite the Gaussian curvature since ∂_1 and ∂_2 are orthogonal but not normalized. Hence the Gaussian curvature in the commutative case is $e^{-2h}R_{1,2,1,2} = -\frac{1}{2}e^{-h}\Delta h$. Since the Riemannian volume form involves a factor of e^h, we see that the NC analogue of Gauss-Bonnet is this:

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Theorem (Gauss-Bonnet)

In the case of A_θ with metric $\langle \partial_j, \partial_k \rangle = e^h \delta_{j,k}$, $\tau(R_{1,2,1,2}e^{-h}) = 0$, regardless of the value of h.

More General Metrics on A_θ

A more complicated case is the one where the metric is given by $\langle \partial_j, \partial_k \rangle = e^{h_j}\delta_{j,k}$, i.e., the metric is given by

$$\begin{pmatrix} e^{h_1} & 0 \\ 0 & e^{h_2} \end{pmatrix}.$$

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Now the formulas are not as nice. One finds now that

$$\begin{aligned}
\nabla_1 \partial_1 &= \frac{1}{2}(k_1\partial_1 - k_2'\partial_2) \\
\nabla_2 \partial_2 &= \frac{1}{2}(-k_1'\partial_1 + k_2\partial_2) \\
\nabla_1 \partial_2 = \nabla_2 \partial_1 &= \frac{1}{2}(k_1''\partial_1 + k_2''\partial_2) \\
k_j &= \partial_j(e^{h_j})e^{-h_j},\ j = 1, 2, \\
k_1' &= \partial_1(e^{h_2})e^{-h_1}, \quad k_2' = \partial_2(e^{h_1})e^{-h_2}, \\
k_1'' &= \partial_2(e^{h_1})e^{-h_1}, \quad k_2'' = \partial_1(e^{h_2})e^{-h_2}.
\end{aligned}$$

The Curvature

One finds in this situation that

$$R_{1,2,1,2} = \frac{1}{2}[-\partial_2(k_2') - \partial_1(k_2'')]e^{h_2} + \frac{1}{4}[k_1\partial_1(e^{h_2}) - k_2'\partial_2(e^{h_2}) + k_1''\partial_2(e^{h_1}) - k_2''\partial_1(e^{h_2})].$$

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"Gauss-Bonnet" in this case would be the statement that $\tau(e^{-h_1/2}R_{1,2,1,2}e^{-h_2/2}) = 0$. I haven't been able to verify this in general but it's true in many special cases.

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Another thing one could do is compute the "Laplacian" for this metric and apply spectral analysis to it as in Connes-Moscovici and Fathizadeh-Khalkhali.