Number 348

Jonathan Rosenberg
and Claude Schochet

The Künneth theorem and the universal coefficient theorem for equivariant K-theory and KK-theory

Memoirs
of the American Mathematical Society

Providence · Rhode Island · USA

July 1986 · Volume 62 · Number 348 (second of 6 numbers) · ISSN 0065-9266

Memoirs of the American Mathematical Society
Number 348

Jonathan Rosenberg
and Claude Schochet

The Künneth theorem and the universal coefficient theorem for equivariant K-theory and KK-theory

Published by the
AMERICAN MATHEMATICAL SOCIETY
Providence, Rhode Island, USA

July 1986 · Volume 62 · Number 348 (second of 6 numbers)

MEMOIRS of the American Mathematical Society

SUBMISSION. This journal is designed particularly for long research papers (and groups of cognate papers) in pure and applied mathematics. The papers, in general, are longer than those in the TRANSACTIONS of the American Mathematical Society, with which it shares an editorial committee. Mathematical papers intended for publication in the Memoirs should be addressed to one of the editors:

Ordinary differential equations, partial differential equations, and applied mathematics to JOEL A. SMOLLER, Department of Mathematics, University of Michigan, Ann Arbor, MI 48109

Complex and harmonic analysis to LINDA PREISS ROTHSCHILD, Department of Mathematics, University of California at San Diego, La Jolla, CA 92093

Abstract analysis to VAUGHAN F. R. JONES, Department of Mathematics, University of California, Berkeley, CA 94720

Classical analysis to PETER W. JONES, Department of Mathematics, Box 2155 Yale Station, Yale University, New Haven, CT 06520

Algebra, algebraic geometry, and number theory to LANCE W. SMALL, Department of Mathematics, University of California at San Diego, La Jolla, CA 92093

Geometric topology and general topology to ROBERT D. EDWARDS, Department of Mathematics, University of California, Los Angeles, CA 90024

Algebraic topology and differential topology to RALPH COHEN, Department of Mathematics, Stanford University, Stanford, CA 94305

Global analysis and differential geometry to TILLA KLOTZ MILNOR, Department of Mathematics, Hill Center, Rutgers University, New Brunswick, NJ 08903

Probability and statistics to RONALD K. GETOOR, Department of Mathematics, University of California at San Diego, La Jolla, CA 92093

Combinatorics and number theory to RONALD L. GRAHAM, Mathematical Sciences Research Center, AT&T Bell Laboratories, 600 Mountain Avenue, Murray Hill, NJ 07974

Logic, set theory, and general topology to KENNETH KUNEN, Department of Mathematics, University of Wisconsin, Madison, WI 53706

All other communications to the editors should be addressed to the Managing Editor, WILLIAM B. JOHNSON, Department of Mathematics, Texas A&M University, College Station, TX 77843-3368

PREPARATION OF COPY. Memoirs are printed by photo-offset from camera-ready copy prepared by the authors. Prospective authors are encouraged to request a booklet giving detailed instructions regarding reproduction copy. Write to Editorial Office, American Mathematical Society, Box 6248, Providence, RI 02940. For general instructions, see last page of Memoir.

SUBSCRIPTION INFORMATION. The 1986 subscription begins with Number 339 and consists of six mailings, each containing one or more numbers. Subscription prices for 1986 are $214 list, $171 institutional member. A late charge of 10% of the subscription price will be imposed on orders received from nonmembers after January 1 of the subscription year. Subscribers outside the United States and India must pay a postage surcharge of $18; subscribers in India must pay a postage surcharge of $15. Each number may be ordered separately; *please specify number* when ordering an individual number. For prices and titles of recently released numbers, see the New Publications sections of the NOTICES of the American Mathematical Society.

BACK NUMBER INFORMATION. For back issues see the AMS Catalogue of Publications.

Subscriptions and orders for publications of the American Mathematical Society should be addressed to American Mathematical Society, Box 1571, Annex Station, Providence, RI 02901-1571. *All orders must be accompanied by payment.* Other correspondence should be addressed to Box 6248, Providence, RI 02940.

MEMOIRS of the American Mathematical Society (ISSN 0065-9266) is published bimonthly (each volume consisting usually of more than one number) by the American Mathematical Society at 201 Charles Street, Providence, Rhode Island 02904. Second Class postage paid at Providence, Rhode Island 02940. Postmaster: Send address changes to Memoirs of the American Mathematical Society, American Mathematical Society, Box 6248, Providence, RI 02940.

Printed in the United States of America.

The paper used in this journal is acid-free and falls within the guidelines established to ensure permanence and durability.

TABLE OF CONTENTS

Library of Congress Cataloging-in-Publication Data

Rosenberg, J. (Jonathan), 1951–

The Künneth theorem and the universal coefficient theorem for equivariant K-theory and KK-theory.

(Memoirs of the American Mathematical Society, ISSN 0065-9266; no. 348)

"July 1986."

Bibliography: p.

1. K-theory. 2. Spectral sequences (Mathematics) 3. Algebra, Homological. I. Schochet, Claude, 1944– . II. Title. III. Title: Universal coefficient theorem for equivariant K-theory and KK-theory. IV. Series.

QA3.A57 no. 348 510 s [514'.23] 86-10959
[QA612.33]
ISBN 0-8218-2349-3

ABSTRACT

If G is a compact connected Lie group with $\pi_1(G)$ torsion-free, and if A and B are suitable C^*-algebras equipped with continuous G-actions, then we construct a Künneth spectral sequence of the form

$$E^2_{p,*} = \mathrm{Tor}_p^{R(G)}(K^G_*(A), K^G_*(B)) \implies K^G_*(A \otimes B),$$

where $A \otimes B$ is given the diagonal G-action. This generalizes the Künneth spectral sequence for equivariant K-theory of spaces, as constructed by Hodgkin, Snaith, and McLeod. Then we construct a Universal Coefficient spectral sequence

$$E_2^{p,*} = \mathrm{Ext}^p_{R(G)}(K^G_*(A), K^G_*(B)) \implies KK^G_*(A,B)$$

for the equivariant Kasparov bivariant K-functor. We discuss several applications, for instance to the question of determining when G-algebras with $K^G_*(A) \cong K^G_*(B)$ (as R(G)-modules) are KK^G-equivalent.

__1980 Mathematics Subject Classification__ (1985 Revision)

46L80, 46M20, 55U20, 55U25, 55N15, 55S25, 46L55.

__Key words and phrases__

Equivariant K-theory, Künneth Theorem, Universal Coefficient Theorem, Hodgkin spectral sequence, Pimsner-Voiculescu exact sequence, Kasparov KK-functor, KK-equivalence, homology operations, representation ring of a compact Lie group, actions of compact groups on C^*-algebras.

This page intentionally left blank

SECTION 1: INTRODUCTION

Suppose that G is a compact Lie group and that A and B are C^*-algebras upon which G acts (referred to here as G-algebras). With modest hypotheses on A and B, G.G. Kasparov [Ka2] has defined groups $KK_j^G(A,B)$, $j \in \mathbb{Z}_2$, which are equivariant generalizations of the renowned Kasparov groups $KK_*(A,B)$ which play a fundamental role in the modern theory of C^*-algebras. The groups $KK_j^G(A,B)$ seem destined for a similar role. For example, if M^n is a compact smooth manifold on which G acts by diffeomorphisms, then the G-algebra of pseudo-differential operators of order 0 on M determines an index element of $KK_1^G(C(S^*M),\mathcal{K})$, where $\mathcal{K} = \mathcal{K}(H)$ is the algebra of compact operators on a Hilbert space with some G-action and S^*M is the cosphere bundle. Determination of this element gives a strong form of the Atiyah-Singer equivariant index theorem for families. Note that the usual equivariant K-theory groups arise via the identification

$$K_j^G(B) \cong KK_j^G(\mathbb{C},B),$$

and, in particular,

$$K_G^{-j}(X) \cong KK_j^G(\mathbb{C},C(X))$$

for X a compact G-space.

This paper is concerned, first of all, with effective methods of computation for equivariant K-theory and KK-theory.

Received by the editors July 3, 1985.

Research partially supported by NSF grants 81-20790 (JR,CS), DMS 84-00900 (JR) and DMS 84-01367 (CS).

To this end, we henceforth assume that the compact Lie group G is connected and has torsion-free fundamental group; we call this the Hodgkin condition, to recognize the pioneering work of L. Hodgkin [Ho]. Suppose that one knows $K_*^G(A)$ and $K_*^G(B)$. Does this determine $K_*^G(A \otimes B)$? If G is the trivial group and A is suitably restricted then we have shown [Sc2] that there is a Künneth short exact sequence

$$0 \longrightarrow K_*(A) \otimes K_*(B) \longrightarrow K_*(A \otimes B) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(K_*(A), K_*(B)) \longrightarrow 0$$

which splits unnaturally. Here is the equivariant analogue. The category $\tilde{\mathcal{B}}_G$ is defined at the beginning of Section 2. It contains, in particular, all commutative G-algebras. We also define a somewhat larger category $\mathcal{C}_G$, containing all Type I G-algebras. For $G = \mathbb{T}$ or $SU(2)$, $\tilde{\mathcal{B}}_G = \mathcal{C}_G$.

THEOREM 5.1. (Künneth Spectral Sequence). Let G be a compact Lie group satisfying the Hodgkin condition. For $A \in \tilde{\mathcal{B}}_G$ and B a G-algebra, there is a spectral sequence of $R(G)$-modules strongly converging to $K_*^G(A \otimes B)$ with

$$E^r_{p,*} = \mathrm{Tor}_p^{R(G)}(K_*^G(A), K_*^G(B)).$$

The spectral sequence has the canonical grading, so that $\mathrm{Tor}_p^{R(G)}(K_s^G(A), K_t^G(B))$ has total degree $p+s+t \pmod 2$. The spectral sequence is natural with respect to pairs (A,B) in the category. If G has rank r then $E^2_{p,q} = 0$ for $p>r+1$ and $E^{r+2} = E^\infty$.

COROLLARY. Let G be a compact Lie group satisfying the Hodgkin condition. Suppose that $A \in \tilde{\mathcal{B}}_G$ and that $K_*^G(A)$ or $K_*^G(B)$ is $R(G)$-free (or more generally $R(G)$-flat). Then there is a natural isomorphism

$$\alpha(A,B)\colon K_*^G(A) \otimes_{R(G)} K_*^G(B) \longrightarrow K_*^G(A \otimes B).$$

The spectral sequence (5.1) was known to Hodgkin-Snaith-McLeod [Ho,Mc,Sn] for A and B commutative (with minor restrictions on the spaces involved which we have removed). Localized versions of the spectral sequence also hold. For instance, if G is a compact Lie group satisfying the Hodgkin condition, p is a prime ideal with $R(G)_p$ a principal ideal domain, then there is a natural short exact sequence which determines $K_*^G(A\otimes B)_p$ in terms of $K_*^G(A)_p$ and $K_*^G(B)_p$.

The following special case deserves special attention.

THEOREM 6.1. (Hodgkin spectral sequence). Let G be a compact Lie group satisfying the Hodgkin condition, let H be a closed subgroup, and let B be a G-algebra. Then there is a spectral sequence of R(G)-modules which strongly converges to $K_*^H(B)$ with

$$E_*^2 \cong \mathrm{Tor}_*^{R(G)}(R(H), K_*^G(B)).$$

In particular, there is a strongly convergent spectral sequence

$$E_{p,*}^2 = \mathrm{Tor}_p^{R(G)}(\mathbb{Z}, K_*^G(B)) \implies K_*(B).$$

This reduces to Kasparov's generalization [Ka3, §7] of the Pimsner-Voiculescu sequence [PV] by taking

$$G = \mathbb{T}^r,$$

$$B = \mathbb{Z}^r \ltimes A.$$

The Pimsner-Voiculescu sequence is recovered by setting r = 1.

Our other basic theorem is a generalization of the Universal Coefficient Theorem of [RS1, RS2] which determined $KK_*(A,B)$ in terms of $K_*(A)$ and $K_*(B)$. More precisely, we proved that there was a natural short exact sequence of the form

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B)) \longrightarrow KK_*(A,B) \xrightarrow{\ \gamma\ } \mathrm{Hom}_{\mathbb{Z}}(K_*(A), K_*(B)) \longrightarrow 0$$

which splits unnaturally. The map γ is the natural Kasparov pairing. It generalizes to the equivariant setting to yield a natural map

$$\gamma\colon KK_*^G(A,B) \longrightarrow \mathrm{Hom}_{R(G)}(K_*^G(A), K_*^G(B)).$$

<u>THEOREM</u> <u>9.2.</u> (Universal Coefficient Spectral Sequence) Let G be a compact Lie group which satisfies the Hodgkin condition. For $A \in \tilde{\mathcal{B}}_G$ and B a G-algebra, there is a spectral sequence of R(G)-modules which strongly converges to $KK_*^G(A,B)$ with

$$E_2^{p,*} \cong \mathrm{Ext}^p_{R(G)}(K_*^G(A), K_*^G(B)).$$

The spectral sequence has the canonical grading, so that $\mathrm{Ext}^p_{R(G)}(K_s^G(A), K_t^G(B))$ has homological degree p and total degree p+s+t (mod 2). The edge homomorphism

$$KK_*^G(A,B) \longrightarrow E_2^{0,*} \cong \mathrm{Hom}_{R(G)}(K_*^G(A), K_*^G(B))$$

is the map γ. The spectral sequence is natural with respect to pairs (A,B) in the category. If G has rank r then $E_2^{p,q} = 0$ for $p>r+1$ and $E_{r+2} = E_\infty$.

<u>COROLLARY.</u> Let G be a compact Lie group which satisfies the Hodgkin condition. Suppose that $A \in \tilde{\mathcal{B}}_G$ and that either $K_*^G(A)$ is R(G)-projective or that $K_*^G(B)$ is R(G)-injective. Then there is a natural isomorphism

$$\gamma(A,B)\colon KK_*^G(A,B) \longrightarrow \mathrm{Hom}_{R(G)}(K_*^G(A), K_*^G(B)).$$

We apply the Universal Coefficient spectral sequence (9.2) in several ways. First we consider KK^G-equivalence. In our previous work we showed that if A and B are C^*-algebras in $\mathcal{C}$ with $K_*(A) \cong K_*(B)$, then A is KK-equivalent to B. The equivariant version of this is false in general: in Example 10.6 we construct commutative $\mathbb{T}$-algebras A and B such that $K_*^{\mathbb{T}}(A) \cong K_*^{\mathbb{T}}(B)$ as $R(\mathbb{T})$-modules but $K_*(A) \neq K_*(B)$, so that A and B can't be K-equivalent, much less $KK^{\mathbb{T}}$-equivalent.

On the positive side, we are able to offer some results. Here is a theorem whose hypotheses are frequently satisfied in practice.

<u>THEOREM</u> <u>10.3.</u> Suppose that G is a Hodgkin group, A and B are G-algebras in $\tilde{\mathcal{B}}_G$ with $K_*^G(A) \cong K_*^G(B) \cong M$, and suppose that M has homological or injective dimension $\leqslant 1$. Then A and B are KK^G-equivalent (and the equivalence covers the given

isomorphism).

In the same vein, we prove the following theorem.

<u>THEOREM</u> <u>10.8.</u> Let G be a compact Lie group satisfying the Hodgkin condition, and let $A \in \tilde{B}_G$ with $K_*^G(A)$ having homological or injective dimension ≤ 1 (as an R(G)-module). Then

a) A is KK^G-equivalent to a G-algebra in $\tilde{B}_G$ of the form $C^0 \oplus C^1$, where $K_i^G(C^j) = 0$ unless $i=j$.

b) If B is any G-algebra with $K_1^G(B) = 0$, then there are <u>split</u> exact sequences of the form

$$0 \longrightarrow K_i^G(A) \otimes_{R(G)} K_0^G(B) \longrightarrow K_i^G(A \otimes B) \longrightarrow \mathrm{Tor}_1^{R(G)}(K_{i-1}^G(A), K_0^G(B)) \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{Ext}^1_{R(G)}(K_{i-1}^G(A), K_0^G(B)) \longrightarrow KK_i^G(A,B) \xrightarrow{\ \gamma\ } \longrightarrow \mathrm{Hom}_{R(G)}(K_i^G(A), K_0^G(B)) \longrightarrow 0.$$

c) In particular, there are <u>split</u> exact sequences

$$0 \longrightarrow K_0^G(A) \otimes_{R(G)} \mathbb{Z} \longrightarrow K_0(A) \longrightarrow \mathrm{Tor}_1^{R(G)}(K_1^G(A), \mathbb{Z}) \longrightarrow 0$$

and

$$0 \to \mathrm{Ext}^1_{R(G)}(K_1^G(A), R(G)) \to K_G^0(A) \to \mathrm{Hom}_{R(G)}(K_0^G(A), R(G)) \to 0.$$

The splittings of these sequences are not natural.

When G is the trivial group, $R(G) = \mathbb{Z}$ and the Universal Coefficient spectral sequence collapses to the short exact sequence of [RS1, Theorem 4.2]. Specializing still further gives the Universal Coefficient Theorem of Brown [B1] for the Brown-Douglas-Fillmore functor $K^*(A)$. If one takes $A = C_0(X)$, X a locally compact G-space, and $B = \mathbb{C}$, then one obtains a spectral sequence of the form

$$\mathrm{Ext}^*_{R(G)}(K^*_G(X), R(G)) \Longrightarrow K^G_*(X),$$

which seems to be new even when X is a smooth manifold with a differentiable G-action. This could be important in understanding the index theory of G-equivariant elliptic operators on X, since (at least "roughly") $K^*_G(X)$ classifies G-vector bundles over X and $K^G_*(X)$ classifies elliptic G-operators over X. For instance, the fact that

$$K^G_*(X) \cong \mathrm{Hom}_{R(G)}(K^*_G(X), R(G))$$

when $K^*_G(X)$ is R(G)-free generalizes [Pe, Part II, Proposition 3.9 and Theorem 5.2]. In fact, several geometric applications of this spectral sequence were given in [IP], although they dealt only with the simplest possible case: $G = \mathbb{T}$ and R(G) localized to make it a principal ideal domain. Presumably the general Universal Coefficient spectral sequence could be used for similar applications with other compact groups, or for analyzing phenomena even in the case of $\mathbb{T}$ that can be traced to $\mathrm{Ext}^2_{R(G)}$.

In order to make the discussion somewhat more concrete, we pause to discuss some interesting cases where we can control $K^*_G(X)$, X a (locally) compact G-space, and hence get some information about $K^G_*(X)$ and the KK^G-equivalence class of C(X).

Suppose that V is a finite-dimensional complex vector space with a linear G-action, and let $X = \mathbb{P}(V)$, the projective space of V. By the equivariant Bott periodicity theorem, $K^*_G(V) \cong R(G)$ (concentrated in degree 0) and $K^*_G(X)$ is a free R(G)-module (concentrated in degree 0), so all spectral sequences collapse. Similarly, if S is the unit sphere in V and G acts by isometries, so that S is a G-space, then the equivariant short exact sequence

$$0 \longrightarrow C_0((0,\infty)\times S) \longrightarrow C_0(V) \longrightarrow C(\{0\}) \longrightarrow 0$$

implies that there is an exact sequence

$$0 \longrightarrow K^1_G(S) \longrightarrow R(G) \xrightarrow{\sigma} R(G) \longrightarrow K^0_G(S) \longrightarrow 0.$$

Most of the time σ is injective and then $K_G^1(S) = 0$ and $K_G^0(S)$ has homological dimension 1. Otherwise $\sigma = 0$ and $K_G^*(S)$ is R(G)-free.

Here is another type of example. Let T be a torus, and let us suppose that T is embedded as a maximal torus in a simply connected compact Lie group G. Let H be any closed subgroup of G. Then G/H is a T-space (one can get circle actions on 3-dimensional lens spaces, for instance), and

$$K_T^*(G/H) \cong R(T)\otimes_{R(G)}K_G^*(G/H) \cong R(T)\otimes_{R(G)}R(H),$$

concentrated in degree 0 (though possibly with big homological dimension).

We briefly mention one other consequence of our results: for groups G satisfying Hodgkin's condition one obtains a fairly definitive answer to a question raised in [Pa]. By a theorem of Green [Gr2] and Julg [Ju1], one knows that for any compact group G and any G-algebra A, there is a natural isomorphism

$$K_*^G(A) \cong K_*(G\ltimes A)$$

where $G\ltimes A$ denotes the C^*-crossed product. (The case A = C(X) and G finite had actually been treated much earlier by Atiyah.) Paulsen points out (and in fact this is done much more generally in [Ka3]) that the analogous result for the dual theory holds if G is finite, but not in general, and he raises the question of determining the precise relationship between $K_G^*(A)$ and $K^*(G\ltimes A)$. We see in fact that for good connected groups, the relationship is given by a spectral sequence

$$\mathrm{Ext}^*_{R(G)}(K_*(G\ltimes A),R(G)) \Longrightarrow K_G^*(A),$$

while we have a short exact sequence

$$0 \longrightarrow \mathrm{Ext}_{\mathbf{Z}}(K_*(G\ltimes A),\mathbf{Z}) \longrightarrow K^*(G\ltimes A) \longrightarrow \mathrm{Hom}_{\mathbf{Z}}(K_*(G\ltimes A),\mathbf{Z}) \longrightarrow 0.$$

which splits, unnaturally. It's no wonder one sees no obvious connection between $K^*(G\ltimes A)$ and $K_G^*(A)$.

We proceed next to the study of $K_*^G(-;\mathbb{Z}_n)$, equivariant K-theory mod n. We show that the ring of R(G)-linear homology operations is the exterior algebra over R(G) on the Bockstein element. Finally, we classify admissible multiplications; these correspond to elements of $R(G)\otimes\mathbb{Z}_n$ and (if n is odd) exactly one of them is (graded) commutative.

We note that if G is a <u>finite</u> group of odd order then M. Bökstedt (unpublished thesis [Bo]) has obtained a short exact UCT sequence for $K_G^n(X)$, where X is a finite G-CW-complex. These results would seem to be orthogonal to our own, as R(G) is always of infinite homological dimension if G is of finite order. According to I. Madsen and M. Rothenberg [Mad], Bökstedt also has results in the compact Lie case which are along the same lines as our own. We have not seen these results.

We wish to thank the Mathematical Sciences Research Institute for supporting us in 1984-5, during which time this paper was completed. We are grateful to A. Wassermann for telling us about Theorem 3.7 (ii).

SECTION 2: FUNDAMENTAL FAMILIES

Let G be a compact Lie group. We will be interested in certain full subcategories of the category of separable nuclear C^*-algebras equipped with a continuous action of G by *-automorphisms. For simplicity, we'll call such an object a G-algebra. The morphisms in this category are G-equivariant *-homomorphisms. A certain G-algebra which plays a vital role is $\mathcal{K}$, the algebra of compact operators on the representation space of an infinite-dimensional unitary representation of G, which we will generally take to be an infinite direct sum of copies of the regular representation of G. The spectrum $\hat{A}$ of a G-algebra has the structure of a G-space in a natural manner. When we regard the spectrum as a G-space with its canonical structure, we use the term G-spectrum of A. Open G-invariant subsets correspond to G-invariant ideals, and closed G-invariant subsets correspond to G-invariant quotients. We remind the reader that the G-structure on $\hat{A}$ does *not* determine the G-action on A in general. This is a difficult point which enormously complicates the study of G-algebras. We shall return to this matter later in this section.

Occasionally we shall wish to vary the group G. Therefore it's useful to record for future reference that if A is a G-algebra and H is a closed subgroup of G, then A may also be viewed by restriction as an H-algebra. In the other direction, if B is an H-algebra, we may define the induced G-algebra (cf. [RR,§3]) to be

$$\mathrm{Ind}_{H\uparrow G}A = C(G\times_H A) = \{f\in C(G,A): f(gh) = h^{-1}\cdot f(g), \text{ for } g\in G,\ h\in H\}.$$

Here G operates by left translation.

Unless stated otherwise, spaces are to be locally compact, second countable, and equipped with continuous G-actions. (If X is such a space, then $C_o(X)$ is a typical abelian G-algebra.) $\mathcal{A}_G$ denotes the category of abelian G-algebras. $\mathcal{C}_G$ denotes the smallest subcategory of G-algebras containing the separable Type I G-algebras, and closed under G-kernels, G-quotients, G-extensions, inductive limits, crossed products by $\mathbb{R}$- or $\mathbb{Z}$-actions commuting with the G-action, exterior equivalence, and G-stable isomorphism. $\mathcal{B}_G$ denotes the "bootstrap category", the smallest subcategory of $\mathcal{C}_G$ which contains $\mathcal{A}_G$ and is closed under the same operations as $\mathcal{C}_G$. Since we are dealing with nuclear C^*-algebras, all tensor products may be taken to be spatial. Then if A and B are G-algebras, G acts on $A \otimes B$ by the diagonal action. A and B are said to be G-stably isomorphic if $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$, where $\cong$ means "G-equivariant *-isomorphism". By Kasparov's stabilization theorem [Ka1] (see also (3.1) below), the question of which unitary representation of G is used in constructing the G-action on $\mathcal{K}$ is irrelevant, and we may take the action to be an infinite number of copies of the regular representation on $L^2(G)$.

We say that the group G satisfies the Hodgkin condition if G is connected and if $\pi_1(G)$ is torsion-free. As pointed out by Hodgkin [Ho, p. 68], this condition implies but is strictly stronger than assuming that R(G) has finite global homological dimension. The global dimension of R(G) is then rank(G)+1. Groups satisfying the Hodgkin condition have another very important property. If T is a maximal torus of G, then R(T) is a free R(G)-module of finite rank (by [Pi] and [St]), and hence the ring extension $R(G) \hookrightarrow R(T)$ is faithfully flat. We shall use this fact (cf. Theorem 3.7) to reduce many questions about K-theory of G-algebras to questions about T-algebras. Accordingly, it is useful to introduce the category $\tilde{\mathcal{B}}_G$ of G-algebras in $\mathcal{C}_G$ which, when viewed as T-algebras, lie in $\mathcal{B}_T$. Note that

$$\mathcal{A}_G \subseteq \mathcal{B}_G \subseteq \tilde{\mathcal{B}}_G \subseteq \mathcal{C}_G.$$

The Hodgkin condition on G might seem a little mysterious. However, from our point of view, it has a very natural explanation given in the following proposition.

PROPOSITION 2.1. Let G be a compact connected Lie group. Then G has no non-trivial projective representations if and only if $\pi_1(G)$ is torsion-free.

PROOF: By [Mo] and [Wi], G has no non-trivial projective representations iff $H^2_M(G,\mathbb{T}) = 0$, (where M indicates Moore's

"Borel cochain" cohomology theory) iff $H^3_M(G,\mathbb{Z}) = 0$ iff $H^3_{top}(BG,\mathbb{Z}) = 0$ (where top denotes usual topological (singular) cohomology). However, $H^*_{top}(BG;\mathbb{Z})$ vanishes modulo torsion in odd degrees, since if T is a maximal torus in G and W is the corresponding Weyl group, then

$$H^*(BG,\mathbb{Q}) \cong H^*(BT,\mathbb{Q})^W$$

and $H^*(BT,\mathbb{Q})$ is a polynomial algebra on generators in degree 2. Therefore, by the classical universal coefficient theorem,

$$H^3(BG,\mathbb{Z}) \cong \text{Tors } H_2(BG,\mathbb{Z})$$

where Tors denotes the torsion subgroup. Since G is connected, BG is simply-connected and, by the Hurewicz theorem,

$$\text{Tors } H_2(BG,\mathbb{Z}) \cong \text{Tors } \pi_2(BG) \cong \text{Tors } \pi_1(G). \quad \square$$

DEFINITION 2.2. A collection of G-algebras F is a C_G-fundamental family if every Type I G-algebra may be constructed from elements of F by taking extensions, kernels, quotients, inductive limits, tensor product with the trivial G-algebra $C_o(\mathbb{R})$, by changing the G-action up to exterior equivalence, and G-stable isomorphism. (The operations may be applied any countable number of times in any order.) Similarly, F is called an A_G-fundamental family if every abelian G-algebra may be constructed out of F.

For instance, if the group G is trivial, then the family $\{C_o(\mathbb{R}^n)\otimes K\}$ is a fundamental family, by the well-known structure theory of Type I algebras (cf. [Sc2, §2]). In fact, we may be even more frugal and use the one-element family $\{\mathbb{C}\}$, since we are allowed to tensor with $C_o(\mathbb{R})$, hence with $C_o(\mathbb{R}^n)$, and since we may tensor with K.

We recall some basic information on the C^*-algebra associated to a group and a cocycle. Suppose that G is a locally compact abelian group, and suppose that $\omega \in Z^2(G,\mathbb{T})$ is a normalized cocycle. Then we form $L^1(G,\omega)$ which is just $L^1(G)$ as a Banach space, but with a convolution product twisted by ω. It is the universal object for maps $\pi:G \longrightarrow U(H)$ with the property that

$$\pi(s)\pi(t) = \omega(s,t)\pi(s+t).$$

Then $C^*(G,\omega)$ is the usual completion. Its isomorphism class only depends upon the cohomology class of ω in $H^2(G,\mathbb{T})$. Note that $C^*(G,\omega)$ is unital if and only if G is a discrete group.

The cocycle ω is called <u>totally skew</u> if $\omega(x,y) = \omega(y,x)$ for all $y \in G$ implies $x = 0$. Changing ω within its cohomology class if necessary, we may always assume ω is lifted from a totally skew multiplier on a quotient group G/K, where K is uniquely determined by the cohomology class of ω [BK, Theorem 3.1]. Then ω determines a continuous injection $h_\omega : G/K \longrightarrow (G/K)\hat{}$ with dense range, and ω is Type I if and only if h_ω is bicontinuous [BK, Theorem 3.2]. We record here the following general result due to Baggett and Kleppner [BK], Kleppner (unpublished), and Pukanszky [Pu, Ch. I, Proposition 2.1]; see also Green [Gr1, Prop. 33].

<u>THEOREM</u> <u>2.3.</u> Let G be a locally compact abelian group and let ω be a totally skew cocycle on G. The C^*-algebra $C^*(G,\omega)$ is simple, and it is Type I if and only if it is isomorphic to $\mathcal{K}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. If so, and if G is a discrete abelian group then $\mathcal{K}(\mathcal{H})$ is unital, hence $\mathcal{H}$ has finite dimension and G is finite.

Our interest lies in the Type I setting. So suppose that F is a free abelian group of finite rank with cocycle ω. Let K denote the radical of the associated skew form. Then $C^*(F/K,\omega)$ is a unital algebra. If it is Type I then Theorem 2.3 implies that F/K is a finite group, so that K is of finite index in F. We record this as a corollary.

<u>COROLLARY</u> <u>2.4.</u> Suppose that F is a free abelian group of finite rank with a Type I cocycle ω. Then the radical K of the associated skew form has finite index in F, and $F/K \cong C \times \hat{C}$ for some finite group C with ω the dual pairing $C \times C \longrightarrow \mathbb{T}$. □

If $G = \mathbb{T}^n$ is a torus with n>1 and if H is a closed subgroup, then it is not necessarily true that the group $H^2_M(G,C(G/H,\mathbb{T}))$, in which obstructions to exterior equivalence of two G-actions on an algebra with G-spectrum G/H live [RR, Corollary 0.13], must vanish. (See [Ro3].) Nevertheless, we can prove the following:

THEOREM 2.5. Let G be a torus (of any dimension) and let

$$\alpha : G \longrightarrow \mathrm{Aut}(A)$$

be an action of G on a continuous-trace algebra A such that $\hat{A} \cong G/H$ as a G-space for some closed subgroup H of G. Then the Dixmier-Douady class of A is trivial, i.e., A is stably isomorphic (as an algebra) to $C(G/H)\otimes K$. Further, the action $\alpha\otimes\mathrm{id}$ is exterior equivalent to $\beta\otimes\mathrm{id}$, where β is the natural action of G on $C^*(K^\perp,\sigma)$ and $K\subseteq H$ is an appropriate subgroup. Thus, up to stable isomorphism and exterior equivalence, A is G-isomorphic to $C(G/H,\omega)\otimes K$. Here $C(G/H,\omega)$ denotes $\mathrm{Ind}_{H\uparrow G}\mathrm{End}(V)$, where V is an irreducible ω-representation of H.

PROOF: Since G acts transitively on $\hat{A}$ with isotropy group H, the Mackey machine implies that

$$(A\rtimes_\alpha G)^\wedge \cong (H,\omega)^\wedge,$$

where $\omega \in H^2(H,\mathbb{T})$ is the Mackey obstruction. Regardless of H and ω, $(H,\omega)^\wedge$ is a countable set. Thus, replacing (A,α) by $(A\otimes K,\alpha\otimes\mathrm{id})$ if necessary, we have $A\rtimes G$ isomorphic to a countable direct sum of copies of K. Since we have stabilized A, Takai duality implies that

$$A \cong (A\rtimes_\alpha G)\rtimes_{\hat{\alpha}}\hat{G}.$$

Here $\hat{G}$ is a free abelian group. By the Mackey machine again, $\hat{A}$ may be computed in terms of the $\hat{G}$-orbits on the countable set $(A\rtimes_\alpha G)^\wedge$.

Since $\hat{A} \cong (G/H)$ is connected, there can be only one $\hat{G}$-orbit, for otherwise $A\rtimes_\alpha G$ would split as a direct sum of two $\hat{G}$-invariant ideals, which would give a non-trivial decomposition of $\hat{A}$ as a disjoint union of two connected components. Let $K^\perp$ be the common stability group in $\hat{G}$ of the points in $(A\rtimes_\alpha G)^\wedge$. (By Pontrjagin duality, each subgroup of $\hat{G}$ is the annihilator of a unique closed subgroup of G.) Then if $\sigma \in H^2(K^\perp,\mathbb{T})$ is the Mackey obstruction for the action of $K^\perp$ on $A\rtimes_\alpha G$, we have

$$\hat{A} \cong (K^{\perp},\sigma)^{\wedge}.$$

Since A is type I, σ is also Type I. Let L be the radical of the associated form. Then $C^*(K^{\perp}/L,\sigma)$ is Type I and unital. By Corollary 2.4, L has finite index in $K^{\perp}$. Thus (since $\hat{A} \cong G/H$)

$$\dim (G/H) = \dim (K^{\perp})^{\wedge} = \dim (G/K)$$

and dim H = dim K. In fact, as the G-action on A must be dual to the action of $\hat{G}$ on $(A\rtimes_{\alpha}G)$, we see that H = K if σ is trivial and, more generally, $H^{\perp} = L$. Since $(A\rtimes_{\alpha}G)^{\wedge}$ is discrete, each spectrum-fixing automorphism of $A\rtimes_{\alpha}G$ is inner [RR, Theorem 0.5(b)], and by [RR, Theorem 0.11] the only obstruction to exterior equivalence of the action of $\hat{G}$ on $A\rtimes_{\alpha}G$ with the standard action of $\hat{G}$ on $C(\hat{G}/K^{\perp},\mathcal{K})$ is a class in $H^2(\hat{G},C((A\rtimes_{\alpha}G)^{\wedge},\mathbb{T}))$. Now

$$H^2(\hat{G},C((A\rtimes_{\alpha}G)^{\wedge},\mathbb{T})) = H^2(\hat{G},C(\hat{G}/K^{\perp},\mathbb{T})) = H^2(\hat{G},U(\hat{G}/K^{\perp},\mathbb{T}))$$

(in the notation of Moore [Mo], this being true since $\hat{G}/K^{\perp}$ is discrete)

$$\cong H^2(K^{\perp},\mathbb{T}))$$

by [Mo, Theorem 6] (Moore's version of Shapiro's Lemma), and our obstruction class corresponds to σ. Thus A is stably isomorphic to $C^*(K^{\perp},\sigma)$. Now this is a continuous-trace algebra with spectrum G/H and fibres isomorphic to End(V), where V is the unique (by the Stone-von Neumann theorem) irreducible σ-representation of the finite group $K^{\perp}/H^{\perp} \cong (H/K)^{\wedge}$. Since End(V) is finite-dimensional, the Dixmier-Douady class of $C^*(K^{\perp},\sigma)$ must be a torsion class in $H^3(G/H,\mathbb{Z})$ (by an observation of Serre [Gr3]). However, G/H is a torus, so its cohomology is torsion-free. Thus the Dixmier-Douady class of A vanishes.

In fact, we have proved somewhat more. By Takai duality, not only is

$$A\otimes\mathcal{K} \cong C^*(K^{\perp},\sigma)\otimes\mathcal{K}$$

but also the action $\alpha\otimes\mathrm{id}$ must be exterior equivalent to $\beta\otimes\mathrm{id}$, where β is the natural action of G on $C^*(K^\perp,\sigma)$. □

REMARK 2.6. We note that $C^*(K^\perp,\sigma)$ may be identified with the induced algebra

$$\mathrm{Ind}_{H\uparrow G}C^*(K^\perp/H^\perp,\sigma) \cong \mathrm{Ind}_{H\uparrow G}\mathrm{End}(V).$$

Since $(H/K)\hat{}$ carries a non-degenerate cocycle, it is canonically self-dual, and V may also be viewed as the space of an irreducible projective representation of (H/K).

Next we attempt to mimic the classical [Di, Ch. 4] argument to produce a small fundamental family for G-algebras.

The following result is a necessary prerequisite to understanding the structure of Type I G-algebras. For simplicity, we restrict to the case of compact Lie groups, since this is the only case we are interested in. The same result when G is any compact, metrizable group could be deduced from [Ph, §8.1], which gives a very different argument using only general topology. However, our proof has the advantage of giving a more concrete description of a G-invariant continuous-trace ideal in A, at least when $G = \mathbb{T}$.

THEOREM 2.7. Let G be a compact Lie group (not necessarily connected) and let A be any Type I (separable) G-algebra. Then A contains a non-zero G-invariant ideal of continuous-trace.

PROOF: We assume initially that G is a compact connected Lie group. Without loss of generality, we may assume that A is liminary, since the largest liminary ideal of A [Di, 4.2.6] is invariant under all automorphisms of A, and in particular under G. We now try as much as possible to imitate the proof of [Di, Lemmes 4.4.2 and 4.4.4] in a G-equivariant way. Using [Di, Lemme 4.4.3], choose a non-zero element y" of A_+, say with $\|y''\| \leq 1$, with $\mathrm{Tr}(\pi(y''))<\infty$ for all $\pi \in \hat{A}$. Let

$$y' = \int_G g\cdot y''dg.$$

Then y' has all the same properties as y" and it is also

G-invariant. Define $f:\hat{A}\longrightarrow\mathbb{R}$ by $f(\pi) = \mathrm{Tr}(\pi(y'))$. Then f is everywhere finite, lower semicontinuous, and not identically zero. Let $U = \{\pi\in\hat{A} : \mathrm{Tr}(\pi(y'))\geq\alpha>0\}$. Then U is a Baire space, so the function f has a point of continuity in U for suitable α, say at $\pi_o \in \hat{A}$. So f is continuous and non-zero at π_o.

Let λ be the largest eigenvalue of $\pi_o(y')$, and let $\phi:\mathbb{R}\longrightarrow[0,1]$ be a weakly increasing continuous function with $\phi(t) = 1$ for $\lambda\leq t$ and with $\phi(t) = 0$ for $t\leq\mu$, where $\mu>0$ is greater than or equal to the second-largest eigenvalue of $\pi_o(y')$. Using the functional calculus, set $y = \phi(y')$. Then $\|y\| = 1$, y is still G-invariant, and $\pi_o(y)$ is a finite-rank projection. Furthermore, $\pi(y)$ is of finite rank for all $\pi\in\hat{A}$ (since $\pi(y')$ can have only finitely many eigenvalues greater than μ, and each occurs with finite multiplicity), and the map $\pi \mapsto \mathrm{Tr}(\pi(y))$ is also continuous at π_o, by [Di, 4.4.2(i)].

To proceed further, note that since G is compact and $\hat{A}$ is T_1, the orbit $G\pi_o$ is closed in $\hat{A}$ by [MR, Lemma 4.1], hence corresponds to a G-quotient B of A with G-spectrum homeomorphic to G/H, where H is the stabilizer group of π_o in G, via the implication (1)⇒(6) in [Gl, Theorem 1]. Since $\pi_o(y)$ is a finite-rank projection and y is G-invariant, y maps to a G-invariant projection $\bar{y}$ in B.

To continue in this fashion for arbitrary G is difficult, so assume now that $G = \mathbb{T}$ is a one-dimensional torus. Then by Theorem 2.5, $B\otimes\mathcal{K}$ is isomorphic to $C(G/H)\otimes\mathcal{K}$, and the action α of G on $B\otimes\mathcal{K}$ is exterior equivalent to the translation action β, where $\beta_t f(\tilde{g}) = f(\tilde{t}^{-1}\tilde{g})$, for $f\in C(G/H,\mathcal{K})$. We claim that $\bar{y}$ dominates a G-invariant rank-one projection p in B. To see this is trivial if G = H, so we may assume H is finite cyclic. We may as well assume that A (hence B) is stable, so that the action α of G on $B \cong C(G/H,\mathcal{K})$ is given by

$$\alpha_t f(\tilde{g}) = v_t(\tilde{g})f(\tilde{t}^{-1}\tilde{g})v_t(\tilde{g})^* \qquad (t\in G,\ \tilde{g}\in G/H)$$

for some cocycle

$$v:G \longrightarrow C(G/H,\mathcal{U}),$$

where $\mathcal{U}$ is the infinite-dimensional unitary group with the strong topology. The fact that y is G-invariant implies that $y(\tilde{g})$ commutes with $v_h(\tilde{g})$ for $h \in H$. Since H is a finite cyclic group, it is easy to choose $p(\tilde{1})$ to be a rank-one subprojection of $\bar{y}(\tilde{1})$ commuting with $\{v_h(\tilde{1}) : h \in H\}$. Then

$$p(\tilde{t}) = v_{t^{-1}}(\tilde{1})^{-1}p(\tilde{1})v_{t^{-1}}(\tilde{1}) \qquad t \in G$$

gives a well-defined rank-one projection $p \in C(G/H, \mathcal{K})$, since given $\tilde{t} \in G/H$, $v_{t^{-1}}(\tilde{1})$ is well-defined modulo unitaries commuting with $p(\tilde{1})$. From the cocycle identity for v we compute that

$$v_t(\tilde{g})p(\tilde{t}^{-1}\tilde{g}) = v_t(\tilde{g})v_{g^{-1}t}(\tilde{1})^{-1}p(\tilde{1})v_{g^{-1}t}(\tilde{1})$$

$$= (v_{g^{-1}}(\tilde{1})^{-1}v_{g^{-1}t}(\tilde{1}))v_{g^{-1}t}(\tilde{1})^{-1}p(\tilde{1})v_{g^{-1}t}(\tilde{1})$$

$$= v_{g^{-1}}(\tilde{1})^{-1}p(\tilde{1})v_{g^{-1}t}(\tilde{1})$$

$$= v_{g^{-1}}(\tilde{1})^{-1}p(\tilde{1})v_{g^{-1}}(\tilde{1})(v_{g^{-1}}(\tilde{1})^{-1}v_{g^{-1}t}(\tilde{1}))$$

$$= p(\tilde{t})v_t(\tilde{g}),$$

which shows that p is G-invariant.

Having produced a G-invariant rank-one projection p in B, we let $x \in A_+$ be any element projecting to p. Averaging x under G, we may assume that x is G-invariant, and also we may assume that $\|x\| = 1$. Then $y^{1/2}xy^{1/2}$ is G-invariant and dominated by y, so the function

$$\pi \mapsto \mathrm{Tr}(\pi(y^{1/2}xy^{1/2}))$$

is continuous at π_o, by [Di, 4.4.2(i)], and of course G-invariant. Since

$$\pi_o(y^{1/2}xy^{1/2}) = \pi_o(\bar{y}p\bar{y}) = \pi_o(p)$$

is of rank one, we conclude as in [Di, 4.4.2(ii)] that $y^{1/2}xy^{1/2}$

may be modified by spectral calculus to obtain an element $z \in A_+$ with $\pi(z)$ a rank-one projection for π near π_o. As in [Di,4.4.4], z defines a non-trivial continuous-trace ideal of A, and since z is G-invariant, so is this ideal. This proves the theorem in the case $G = \mathbb{T}$.

Next we consider the case G a compact Lie group. Let G_o denote the connected component of the identity, and consider its action on A. We may assume that A is liminary, as before. By the previous case, and reasoning as in [Di, 4.4.5], we see that for every closed subgroup H of G_o with $H \cong \mathbb{T}$, $\hat{A}$ contains a dense open H-invariant Hausdorff subset. But G_o contains a finite number of one-parameter subgroups $H_1, \ldots, H_r$, each isomorphic to $\mathbb{T}$, which generate G_o <u>algebraically</u> (no closures needed). So choosing such a dense open H_i-invariant Hausdorff subset U_i of $\hat{A}$ for each i and letting $U = U_1 \cap \ldots \cap U_r$, we obtain a dense G_o-invariant open Hausdorff subset of $\hat{A}$. (Density of U follows from the fact that $\hat{A}$ is a Baire space.)

Choose representatives g_i for the (finitely many) cosets of G_o in G, and let $Y = \cap g_i U$. Again by the Baire property, Y is a dense open G-invariant Hausdorff subset of $\hat{A}$ and corresponds to some non-trivial G-invariant ideal J. Let C be some continuous-trace ideal in J and let D be its G-saturation. Then D is a non-trivial G-invariant ideal with Hausdorff spectrum and D has local rank-one projections, thus D is continuous-trace. This completes the proof of the theorem when G is a compact Lie group. □

<u>THEOREM</u> <u>2.8.</u> Let G be a compact Lie group, not necessarily connected. If F is a collection of G-algebras and if each continuous-trace G-algebra A with $\hat{A} = G/H$ (H running over closed subgroups of G, G acting by translation) may be constructed from F as described above, then F is a C_G-fundamental family. Similarly, if F is the collection of commutative G-algebras of the form C(G/H), then F is an A_G-fundamental family.

<u>PROOF:</u> We shall prove only the statement about C_G-fundamental families. The corresponding statement about A_G-fundamental families is much easier, and uses (a proper subset of) exactly the same arguments.

Let A be a Type I G-algebra. By repeated use of Theorem 2.8, we see that A has a composition series $\{A_\alpha\}$ where $A_0 = 0$, $A_{\alpha+1}/A_\alpha$ is a non-zero G-invariant ideal of A/A_α with continuous-trace, and

$$A_\alpha = \varinjlim \{A_\beta : \beta<\alpha\}$$

for α a limit ordinal. Since we are assuming that A is separable, we have $A_\alpha = A$ for some countable ordinal α. So it suffices to show that any continuous-trace G-algebra is generated by F.

The next step is the reduction to the case of one orbit type. Suppose that A is a continuous-trace G-algebra with spectrum X. Then X contains an open subset with one orbit type, by [MZ, p. 222]. We divide by the corresponding G-invariant ideal and repeat the argument on the quotient algebra. By a limit argument we can reduce to the case of a finite number of orbit types. By iterated extensions, we can reduce to the case of a single orbit type. So suppose that X has a single orbit type- say all stability groups are conjugate to H. Then X^H is a free $N_G(H)/H$ -space and

$$X \cong G\times_{N_G(H)} X^H.$$

By Gleason's cross-section theorem [MZ, pp. 219-221], X^H has a covering by open sets U_α, $N_G(H)$-isomorphic to $(N_G(H)/H)\times S_\alpha$ for certain locally compact S_α. Taking the induced cover of X, then extracting a countable subcovering, we see from passage to limits that we may assume that $X = (G/H)\times S$, S locally compact. Taking one-point compactifications, we may assume without loss of generality that S is compact. Then S is a projective limit of finite simplicial complexes, and so X is a limit of finite G-complexes (e.g., in the sense of [May]) built out of G-cells of the form $(G/H)\times\mathbb{R}^n$. So we have reduced to the case where A is a continuous-trace algebra with $\hat{A} = (G/H)\times\mathbb{R}^n$, where G acts transitively on G/H and trivially on $\mathbb{R}^n$.

Now the problem is to classify continuous-trace G-algebras with G-spectrum $(G/H)\times\mathbb{R}^n$. Taking the algebra to be G-stable and using the fact that $\mathbb{R}^n$ is contractible (and hence that all continuous-trace algebras with spectrum $\mathbb{R}^n$ are stably commutative), we may assume

$$A \cong W \otimes C_o(\mathbb{R}^n) \qquad (*)$$

where W is a stable continuous-trace algebra with spectrum G/H. Our difficulty is that we don't know yet that the isomorphism (*) can be made equivariant. For that we need the following proposition.

<u>PROPOSITION</u> <u>2.9.</u> Let G be a compact Lie group, not necessarily connected, and let (A,α) be a stable continuous-trace G-algebra with G-spectrum $(G/H)\times\mathbb{R}^n$. Then there is a stable continuous-trace G-algebra (W,ω) with G-spectrum G/H and an exterior equivalence

$$(A,\alpha) \approx (W\otimes C_o(\mathbb{R}^n), \omega\otimes id).$$

<u>PROOF:</u> Let W be the quotient of A corresponding to the closed G-invariant subset $(G/H)\times\{0\} \subset (G/H)\times\mathbb{R}^n$. Then W is a G-algebra and $A \cong W\otimes C_o(\mathbb{R}^n)$, though not necessarily equivariantly. Let ω be the G-action on W. We want to compare α with $\omega\otimes id$. The cocycle

$$g \mapsto \alpha_g^{-1}(\omega_g \otimes id)$$

takes values in the inner automorphisms of A, by [RR, Theorem 0.8] if G is connected, but actually in general via [RR, Theorem 0.5(b)] since the map

$$H^2((G/H)\times\{0\},\mathbb{Z}) \longrightarrow H^2((G/H)\times\mathbb{R}^n,\mathbb{Z})$$

is always an isomorphism.

By [RR, Theorem 0.4], there is only one obstruction to exterior equivalence of α with $\omega\otimes id$, and it lies in the Moore cohomology group $H^2_M(G,C((G/H)\times\mathbb{R}^n,\mathbb{T}))$. Now we argue as in [Ro3, Theorem 3.9]. Let $C(X,\mathbb{T})_o$ denote the connected component of the identity in $C(X,\mathbb{T})$. The commuting diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & C((G/H)\times\mathbb{R}^n,\mathbb{T})_o & \longrightarrow & C((G/H)\times\mathbb{R}^n,\mathbb{T}) & \longrightarrow & H^1((G/H)\times\mathbb{R}^n,\mathbb{Z}) & \longrightarrow 0 \\ & \Big\downarrow \mathrm{res}_0 & & \Big\downarrow \mathrm{res} & & \Big\downarrow \cong & \\ 0 \longrightarrow & C((G/H)\times\{0\},\mathbb{T})_o & \longrightarrow & C((G/H)\times\{0\},\mathbb{T}) & \longrightarrow & H^1((G/H)\times\{0\},\mathbb{Z}) & \longrightarrow 0 \end{array}$$

and the five-lemma applied to the corresponding long exact cohomology sequence imply that res induces an isomorphism on $H^2_M(G,-)$ if res_0 does. Consider the commuting diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & H^0((G/H)\times\mathbb{R}^n,\mathbb{Z}) & \longrightarrow & C((G/H)\times\mathbb{R}^n,\mathbb{R}) & \longrightarrow & C((G/H)\times\mathbb{R}^n,\mathbb{T})_o & \longrightarrow 0 \\ & \Big\downarrow \cong & & \Big\downarrow \mathrm{res}_2 & & \Big\downarrow \mathrm{res}_1 & \\ 0 \longrightarrow & H^0((G/H)\times\{0\},\mathbb{Z}) & \longrightarrow & C((G/H)\times\{0\},\mathbb{R}) & \longrightarrow & C((G/H)\times\{0\},\mathbb{T})_o & \longrightarrow 0. \end{array}$$

By "averaging" of cocycles, $H^j_M(G,V) = 0$ for $j>0$ and V any Fréchet space with linear G-action (such as $C((G/H)\times M,\mathbb{R})$ for M a trivial G-space) so that res_2 is an isomorphism on $H^j_M(G,-)$ for $j>0$. By the five-lemma, res_1 is an isomorphism on $H^2_M(G,-)$. Thus we deduce that the restriction map res induces an isomorphism

$$\mathrm{res}_* : H^2_M(G,C((G/H)\times\mathbb{R}^n,\mathbb{T})) \longrightarrow H^2_M(G,C((G/H)\times\{0\},\mathbb{T})).$$

Since W with action ω is constructed as the quotient of A with action α, the image of the obstruction is zero. This completes the proof of Proposition 2.9 and thus of Theorem 2.8. □

For arbitrary compact groups G (even satisfying the Hodgkin condition), identifying explicitly all the G-algebras in a G-fundamental family requires being able to list all the closed subgroups H of G and being able to compute $H^2_M(H,\mathbb{T})$ for each. For G simply connected (e.g., SU(N), Spin(N), Sp(N)), the algebras $C(G\times_H \mathrm{End}(V))$ (V a finite-dimensional projective representation of H, H a closed subgroup of G) form a G-fundamental family. This is conceptually satisfactory and perhaps it is the best possible result at this level of generality. However, further progress is possible in the case of $G = T^n$ and, in particular, if $G = \mathbb{T}$.

COROLLARY 2.10. Suppose that G is a torus (of any dimension). Let $\mathcal{F}$ be the collection of G-algebras of the form $C(G/H,\omega)$, where H is a closed subgroup of G and ω is a G/H cocycle. Then $\mathcal{F}$ is a $\mathcal{C}_G$-fundamental family.

PROOF: This follows from Theorems 2.5 and 2.8. □

COROLLARY 2.11. If $G = \mathbb{T}$, the G-algebras of the form C(G/H), where $H = \{1\}$, $\mathbb{Z}_n$, or G, are a $\mathcal{C}_G$-fundamental family.

PROOF: For any closed subgroup H of $\mathbb{T}$, $H^2_M(H,\mathbb{T}) = 0$. □

COROLLARY 2.12. If $G = \mathbb{T}$, then $\mathcal{B}_G = \mathcal{C}_G$, and if $G = SU(2)$, then $\tilde{\mathcal{B}}_G = \mathcal{C}_G$.

PROOF: This is a restatement of (2.11). □

SECTION 3: THE KÜNNETH SPECTRAL SEQUENCE- SPECIAL CASES

This section is devoted to the examination of the Künneth map

$$\alpha(A,B):\ K_*^G(A)\otimes_{R(G)}K_*^G(B) \longrightarrow K_*^G(A\otimes B).$$

We concentrate upon the special case when $K_*^G(B)$ is $R(G)$-flat. In that case the Künneth spectral sequence (if it were to exist) would predict that $\alpha(A,B)$ is an isomorphism. We show that this is indeed the case for appropriate G and for $A\in \tilde{B}_G$.

On the category of G-algebras the bifunctor $KK_*^G(\ ,\ )$ is defined and satisfies homotopy and exactness axioms in both variables [Ka2]. Since it's easy to see how it behaves with respect to c_o-direct sums, the argument of Milnor [Sc3, §5] shows that KK_*^G commutes with countable inductive limits in the second variable and satisfies a $\varprojlim^1$ sequence with respect to inductive limits in the first variable. The proof is exactly the same as in the non-equivariant case; see [RS2], Theorems 1.12 and 1.14. We write $KK_*^G(\mathbb{C},B) = K_*^G(B)$; K_*^G is equivariant K-theory in the usual sense of the term. We also write $K_G^*(X)$ for $K_*^G(C_o(X))$. This is equivariant K-theory with compact supports as defined in [Se2]. <u>Caution:</u> Kasparov writes $K_G^*(B)$ for $KK_*^G(\mathbb{C},B)$, even though the functor is *covariant* in B. We choose to adhere to the usual convention of writing covariant functors with indices down.

The groups $KK_*^G(A,B)$ depend upon A and B as G-algebras, so that if one changes the G-structure, one would expect the Kasparov groups to change. However, it is only the exterior equivalence classes of the G-actions that matter (see 3.1 below).

As usual, we denote the (complex) representation ring of G by R(G). This coincides with $KK_0^G(\mathbb{C},\mathbb{C})$. For any G-algebras A and B, $KK_*^G(A,B)$ is an R(G)-module via the intersection pairing. This module action on $K_G^*(X)$ is the same as Segal's. As explained in the introduction, we will be interested in a Künneth theorem for

K_*^G and in a universal coefficient theorem for $KK_*^G(\ ,\ )$. In both cases, these are to be interpreted in the sense of [Ad1] and [Ad2, Section 13]- that is, we want spectral sequences involving Ext and Tor of the coefficient ring $R(G)$.

PROPOSITION 3.1. Let $\alpha, \beta : G \longrightarrow \mathrm{Aut}(A)$ be two actions of a compact (metrizable) group G on a separable C^*-algebra A which are exterior equivalent. Then (A,α) and (A,β) are KK^G-equivalent.

PROOF: By assumption, there is a cocycle $u:G \longrightarrow U(M(A))$ (the unitary group of the multiplier algebra of A) such that $\beta_t = (\mathrm{Ad}\ u_t)\alpha_t$. We construct $\lambda \in KK^G((A,\alpha),(A,\beta))$ to be the class of the Kasparov triple $(E\oplus 0,0)$, where E is the right Hilbert A-module A with A-valued inner product

$$\langle x,y\rangle = x^*y$$

and G-action

$$t\cdot x = u_t\alpha_t(x)$$

and where A acts on the left and right in the obvious way. Note that as required,

$$\begin{aligned} ts\cdot x &= u_{ts}\alpha_{ts}(x) \\ &= u_t\alpha_t(u_s)\alpha_t(\alpha_s(x)) \\ &= t\cdot(u_s\alpha_s(x)) \\ &= t\cdot(s\cdot x). \end{aligned}$$

The G-actions on A and on E are compatible since

$$\begin{aligned} (t\cdot x)\alpha_t(y) &= u_t\alpha_t(x)\alpha_t(y) \\ &= t\cdot(xy) \end{aligned}$$

and

$$\beta_t(x)(t\cdot y) = u_t\alpha_t(x)u_t{}^*u_t\alpha_t(y)$$

$$=u_t\alpha_t(xy)$$

$$=t\cdot(xy).$$

Similarly we construct $\mu \in KK^G((A,\alpha),(A,\beta))$ to be the class of $(F\oplus 0,0)$ where F is the A-bimodule A with the G-action

$$t\cdot x = u_t{}^*\beta_t(x) = \alpha_t(x)u_t{}^*.$$

We claim that λ and μ are inverses of each other. Indeed, $\mu\otimes_{(A,\beta)}\lambda$ is the class in $KK^G((A,\alpha),(A,\alpha))$ of $(F\otimes_A E\oplus 0,0)$, which is clearly just $1_{(A,\alpha)}$, and similarly $\lambda\otimes_{(A,\alpha)}\mu = 1_{(A,\beta)}$. □

<u>LEMMA</u> <u>3.2.</u> Suppose that G satisfies the Hodgkin condition. Then for any G-algebras A,B and for any action of G on $M_n(\mathbb{C})$, there are natural isomorphisms

$$KK_*^G(A,B) \cong KK_*^G(A\otimes M_n,B)$$

$$KK_*^G(A,B) \cong KK_*^G(A,B\otimes M_n).$$

The same holds for actions on $\mathcal{K}$ other than the standard one.

<u>PROOF:</u> An action of G on M_n is given by a continuous homomorphism $G \longrightarrow Aut(M_n) = PU(n)$, and an action on $\mathcal{K}$ is given by a homomorphism $G \longrightarrow PU(\mathcal{H})$. The Hodgkin condition guarantees that such actions come from ordinary unitary representations. Thus one can apply [Ka2, § 5, Theorem 1]. □

It's worth pointing out one other fact we'll use many times.

<u>REMARK</u> <u>3.3.</u> If H is a closed subgroup of G and A is any G-algebra, then there's a G-equivariant isomorphism of $C(G/H)\otimes A$ (with the diagonal action) with $C(G\times_H A)$ (with action by left translation), where as in §2,

$$C(G\times_H A) = \{f \in C(G,A) \;:\; f(gh) = h^{-1}f(g),\ g\in G,\ h\in H\},$$

namely:

$$\phi(f\otimes a)(g) = f(g)(g^{-1}a).$$

It follows that

$$K_*^G(C(G/H)\otimes A) \cong K_*^G(C(G\otimes_H A)).$$

By [Gr2] and [Ju1], this group is isomorphic to $K_*(G\ltimes C(G\times_H A))$, which by [Gr1] is just $K_*(H\ltimes A) \cong K_*^H(A)$, since $G\ltimes C(G\ltimes_H A)$ and $H\ltimes A$ are stably isomorphic and the R(G)-module structures are easily seen to be compatible. Thus,

$$K_*^H(A) \cong K_*^G(C(G/H)\otimes A)$$

with G acting diagonally on $C(G/H)\otimes A$, for any G-algebra A and any closed subgroup H of G. Compare also [RR], §3.

<u>PROPOSITION</u> <u>3.4.</u> Assume that $K_*^G(B)$ is a flat R(G)-module. Then $K_*^G(-)\otimes_{R(G)}K_*^G(B)$ and $K_*^G(-\otimes B)$ are additive homology theories (on the category of G-algebras), and $\alpha(-,B)$ is a natural transformation of theories

$$\alpha(-,B)\colon\ K_*^G(-)\otimes_{R(G)}K_*^G(B) \longrightarrow K_*^G(-\otimes B).$$

<u>PROOF:</u> Each theory is clearly homotopy invariant. Since G-algebras are nuclear, the functor $(-)\otimes B$ preserves exact sequences and countable direct sums. Thus $K_*^G(-\otimes B)$ is a homology theory (with no use of flatness). If $K_*^G(B)$ is a flat R(G)-module then the functor $(-)\otimes_{R(G)}K_*^G(B)$ is exact, and hence the functor $K_*^G(-)\otimes_{R(G)}K_*^G(B)$ satisfies the exactness and additivity axioms. Finally, the naturality of α follows from the naturality of the Kasparov product. □

<u>PROPOSITION</u> <u>3.5.</u> Suppose that $K_*^G(B)$ is R(G)-flat.

a) If J is an invariant ideal of the G-algebra A and if two of the maps $\alpha(J,B)$, $\alpha(A,B)$, $\alpha(A/J,B)$ are isomorphisms, then so is the third map.

b) If $A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow \cdots$ is a countable direct system of G-algebras with limit A and $\alpha(A_i,B)$ is an isomorphism for each i, then $\alpha(A,B)$ is an isomorphism.

c) If $\alpha(A,B)$ is an isomorphism, then so is $\alpha(A\otimes C_o(\mathbb{R}^n),B)$.

d) If $\alpha(A,B)$ is an isomorphism and if A is G-stably equivalent to A', then $\alpha(A',B)$ is an isomorphism.

e) If $\alpha(A,B)$ is an isomorphism and if A is exterior equivalent to $\tilde{A}$, then $\alpha(\tilde{A},B)$ is an isomorphism.

f) If $\alpha(A,B)$ is an isomorphism and if $A\rtimes\Gamma$ denotes the crossed product by the group $\Gamma = \mathbb{R}$ or $\mathbb{Z}$ whose action commutes with the G-action, then $\alpha(A\rtimes\Gamma,B)$ is an isomorphism.

<u>PROOF:</u> Parts a) and b) depend upon Proposition 3.4. For part d), we note that both sides are invariant under G-stable isomorphism. Part c) holds by Bott periodicity. For part f) in the case of the group $\mathbb{R}$, one uses Connes' Thom isomorphism and the fact that

$$K_*^G(A\rtimes\mathbb{R}) \cong K_*((A\rtimes\mathbb{R})\rtimes G) \cong K_*((A\rtimes G)\rtimes\mathbb{R})$$

$$\cong K_{*+1}(A\rtimes G) \cong K_{*+1}^G(A)$$

(using [Gr2] or [Ju1] plus the fact that the G and $\mathbb{R}$ actions commute.) Crossed products by $\mathbb{Z}$ follow from the case of crossed products by $\mathbb{R}$ (as pointed out by Connes), or else using a similar argument with the Pimsner-Voiculescu sequence and the 5-lemma (cf. [RS2, 2.7]). Part e) follows from Proposition 3.1. □

<u>PROPOSITION 3.6.</u> Suppose that $K_*^G(D)$ is R(G)-flat and that $\alpha(A,B)$ is an isomorphism for all A in some $\mathcal{C}_G$-fundamental family $\mathcal{F}$. Then $\alpha(A,B)$ is an isomorphism for all $A \in \mathcal{C}_G$. If $\alpha(A,B)$ is an isomorphism for all A in some $\mathcal{A}_G$-fundamental family, then $\alpha(A,B)$ is an isomorphism for all $A \in \mathcal{B}_G$.

<u>PROOF:</u> This is an immediate consequence of Proposition 3.5. □

Proposition 3.6 is the key to our strategy: we focus upon

showing that $\alpha(A,B)$ is an isomorphism for $K_*^G(B)$ $R(G)$-flat and for A in one of the fundamental families constructed in §2. Furthermore, things are greatly simplified if we restrict attention to the case where G is a torus. This turns out to be no restriction because of the following theorem, due in the full generality of (ii) to A. Wassermann [Wa], and discovered in the form (i) by one of us (J.R.) in 1982.

<u>THEOREM</u> <u>3.7.</u> Let G be a compact Lie group in the Hodgkin class and let T be a maximal torus in G. Then for all G-algebras A,

(i) $K_*^T(A) \cong R(T)\otimes_{R(G)} K_*^G(A)$,

and more generally, for all G-algebras A and B,

(ii) $KK_*^T(A,B) \cong R(T)\otimes_{R(G)} KK_*^G(A,B)$.

<u>PROOF:</u> Statement (ii) is due to A. Wassermann, and we understand he intends to publish the proof in the near future [Wa]. Therefore we content ourselves here with sketching an argument for (i), which is what we shall need later in this section. There are three major ingredients:

a) the theorem of Pittie and Steinberg (see [St]) that for G a Hodgkin group, R(T) is a free R(G)-module of finite rank, and

b) the theorem of McLeod [Mc], which relies on (a), that for G a Hodgkin group, the Künneth pairing

$$R(T)\otimes_{R(G)} R(T) \longrightarrow K_G^*(G/T \times G/T) \cong K_T^*(G/T)$$

is an isomorphism (a key case of our Theorem 3.10 below), and

c) the use of elliptic operators on G/T, where G is any connected compact Lie group, to show that the restriction map

$$r\colon K_*^G(A) \longrightarrow K_*^T(A)$$

is split injective.

Note that (c) does not require the Hodgkin condition, and that by [St], $R(T)$ is also free over $R(G)$ when $G = SO(2n+1)$. However, (b) works only for Hodgkin groups.

Part (c) was proved by Atiyah [At, Proposition 4.9] in the case where A is commutative, and generalized to the case of arbitrary G-algebras by Julg [Ju2]. However, since Julg does not use the machinery of *equivariant* Kasparov groups, it seems more in keeping with the spirit of this article to proceed as follows.

Since G/T may be given the structure of a smooth complex projective variety, with G-invariant complex structure corresponding to a choice of positive roots for (G,T), we may form the corresponding $\bar{\partial}$-operator (the Dirac operator would do just as well), and on the Dolbeault complex,

$$D = \bar{\partial}+\bar{\partial}^*: \Omega^{0,\mathrm{even}}(G/T) \rightleftarrows \Omega^{0,\mathrm{odd}}(G/T)$$

is elliptic and G-invariant if we use a G-invariant metric. Then $(1+D^2)^{-1/2}D$ is bounded on L^2 forms and Fredholm, and defines a class

$$[D] \in KK_0^G(C(G/T),\mathbb{C}).$$

For any G-algebra A, the Kasparov product map

$$K_*^T(A) \cong K_*^G(C(G/T)\otimes A) \cong$$

$$\cong KK_*^G(\mathbb{C},C(G/T)\otimes A) \xrightarrow{-\otimes_{C(G/T)}[D]} KK_*^G(\mathbb{C},A) \cong K_*^G(A)$$

is the "holomorphic induction map" of [Ju2] which Julg shows is a left inverse to r (without assuming G is a Hodgkin group).

The fact that G/T has a G-invariant complex structure and a G-invariant (Kähler) metric implies by [CS] and by [Ka3, §8] that there is a Poincaré duality isomorphism

$$\delta: KK^G(C(G/T),\mathbb{C}) \longrightarrow KK^G(\mathbb{C},C(G/T))$$

taking $[D]$ to the class of the trivial line bundle, which corresponds to the identity element $1 \in R(T)$ under the usual isomorphism

$$KK^G(\mathbb{C},C(G/T)) = K_G^0(G/T) \cong R(T).$$

In fact, δ is given by Kasparov product with a certain canonical class in the group

$$KK^G(\mathbb{C},C(G/T)\otimes C(G/T)) \cong K_G^0(G/T \times G/T),$$

and so we have a similar Poincaré duality isomorphism

$$\delta_B\colon KK^G(C(G/T),B) \longrightarrow KK^G(\mathbb{C},C(G/T)\otimes B)$$

for any G-algebra B. Note by associativity properties of the Kasparov product that for any

$$x \in K_0^G(C(G/T)) \cong KK^G(\mathbb{C},C(G/T))$$

and for any

$$y \in K_0^G(B) \cong KK^G(\mathbb{C},B),$$

we have

$$\delta^{-1}(x)\otimes_{\mathbb{C}} y = \delta_B{}^{-1}(x\otimes_{\mathbb{C}} y) \text{ in } KK^G(C(G/T),B).$$

We apply all this in the case where $B = C(G/T)$. Thus we have an $R(G)$-module isomorphism

$$\delta_{G/T}\colon KK^G(C(G/T),C(G/T)) \longrightarrow KK^G(\mathbb{C},C(G/T)\otimes C(G/T)),$$

and by McLeod's Theorem, the right-hand side is identified with

$$K_T^0(G/T) \cong R(T)\otimes_{R(G)} R(T).$$

Let $W = N_G(T)/T$ be the Weyl group of (G,T). The Pittie-Steinberg Theorem provides a free basis $\{e_w\}_{w\in W}$ for $R(T)$ over $R(G)$, with $e_{1_G} = 1$. Thus there must be unique elements $\{b_w\}_{w\in W}$ of $R(T) \cong KK^G(\mathbb{C},C(G/T))$ such that

$$\delta_{G/T}(1_{C(G/T)}) = \Sigma_{w\in W}\, b_w \otimes_{\mathbb{C}} e_w.$$

(The notation here is ambiguous, but $\otimes_{\mathbb{C}}$ denotes Kasparov product over the G-algebra $\mathbb{C}$, which corresponds to $\otimes_{R(G)}$ for the corresponding modules.)

Letting (for $w \in W$)

$$a_w = \delta^{-1}(b_w) \in K_0^G(G/T),$$

we deduce that we have a unique decomposition of $1_{C(G/T)}$ as

$$1_{C(G/T)} = \Sigma_{w\in W}\, a_w \otimes_{\mathbb{C}} e_w.$$

Note also that $\{a_w\}$ and $\{e_w\}$ must be dual to each other with respect to the Kasparov product, since for $u\in W$,

$$\begin{aligned} e_u &= e_u \otimes_{C(G/T)} 1_{C(G/T)} \\ &= e_u \otimes_{C(G/T)} (\Sigma_{w\in W} a_w \otimes_{\mathbb{C}} e_w) \\ &= \Sigma_{w\in W} (e_u \otimes_{C(G/T)} a_w) \otimes_{\mathbb{C}} e_w \end{aligned}$$

by associativity of the product, and thus (since the e_w's are a free basis for R(T) over R(G)) that

$$e_u \otimes_{C(G/T)} a_w = \begin{cases} 1_{R(G)} & \text{if } u = w \\ 0 & \text{otherwise.} \end{cases}$$

Now consider any G-algebra A and some

$$x \in K_*^T(A) \cong K_*^G(A \otimes C(G/T)) \quad \text{(by Remark 3.3).}$$

We have

$$x = x \otimes_{C(G/T)} 1_{C(G/T)} = \Sigma_{w\in W} (x \otimes_{C(G/T)} a_w) \otimes_{\mathbb{C}} e_w,$$

and furthermore, for any $x_w \in K_*^G(A)$ such that

$$x = \Sigma_{w\in W} x_w \otimes_{\mathbb{C}} e_w,$$

we must have

$$x\otimes_{C(G/T)} a_w = (\Sigma_{u\in W} x_u \otimes_{\mathbb{C}} e_u) \otimes_{C(G/T)} a_w$$

$$= \Sigma_{u\in W} x_u \otimes_{\mathbb{C}} (e_u \otimes_{C(G/T)} a_w)$$

$$= x_w,$$

which shows that the x_w's are uniquely determined. Thus

$$K_*^T(A) \cong K_*^G(A) \otimes_{R(G)} R(T),$$

at least as R(G)-modules.

To check that the R(T)-module structures on $K_*^T(A)$ and on $K_*^G(A)\otimes_{R(G)} R(T)$ coincide, it is enough to observe that the restriction map

$$r\colon K_*^G(A) \longrightarrow K_*^T(A)$$

sends Kasparov products for G-algebras to Kasparov products for T-algebras, and is also given on the level of R(G)-modules by Kasparov product over the G-algebra $\mathbb{C}$ with

$$e_1 = 1_{R(T)}.$$

Thus if we use a raised dot to denote the module action of R(T), we have for $x \in K_*^G(A)$ and $u,w \in W$,

$$e_w \cdot r(x) = x\otimes_{\mathbb{C}} e_w,$$

$$e_u \cdot (x\otimes_{\mathbb{C}} e_w) = (e_u e_w)\cdot r(x) = x\otimes_{\mathbb{C}} (e_u e_w).$$

Since $K_*^T(A)$ is spanned as an R(G)-module by elements of the form $x\otimes_{\mathbb{C}} e_w$, and since the e_u's span R(T) as an R(G)-module, we see that the module action of R(T) on $K_*^T(A)$ coincides with the R(T)-module structure on

$$R(T)\otimes_{R(G)}K_*^G(A).$$

(Since R(G) and R(T) are commutative, we use right and left modules interchangeably.) This completes the proof. □

Note, incidentally, that as indicated in [Se1], p. 127,

$$R(G) \cong R(T)^W$$

even for G connected but not Hodgkin. This suggests that one ought to have

$$K_*^G(A) \cong K_*^T(A)^W$$

for any G-algebra A. As pointed out in [Mc, Remark 4.5], this fails even for G = SU(2) if $K_*^G(A)$ has torsion, but this follows immediately from Theorem 3.7 if G is Hodgkin and if $K_*^G(A)$ is R(G)-free (equivalently, if $K_*^T(A)$ is R(T)-free).

We shall see in Theorem 3.10 below how to pass from Theorem 3.7 and results for tori to results for general Hodgkin groups.

<u>THEOREM</u> <u>3.8.</u> Suppose that G is a torus, H is a closed subgroup of G, and B is a C^*-algebra with $K_*^G(B)$ R(G)-free. Then the natural map

$$\alpha\colon R(H)\otimes_{R(G)}K_*^G(B) \longrightarrow K_*^H(B)$$

is an isomorphism.

Note that we are using the identification

$$K_*^G(C(G/H)\otimes B) \cong K_*^H(B)$$

mentioned in Remark 3.3 and that α corresponds to α(C(G/H),B) under this identification, so that this theorem is a special case of the Künneth Theorem for the group G.

We begin the proof of Theorem 3.8.

PROPOSITION 3.9. Suppose that T is a torus and that $K_*^T(B) = 0$. Then $K_*^H(B) = 0$ for any closed subgroup H of T.

We note that Proposition 3.9 implies Theorem 3.8. Indeed, if B is as in Theorem 3.8, we can by Proposition 4.1 and Remark 4.2 below choose $F = C_o(Y)$, with Y a disjoint union of copies of $\mathbb{R}$ and of $\mathbb{R}^2$, with trivial T-action, and a morphism of T-algebras

$$\mu: F \longrightarrow SB \otimes \mathcal{K}$$

inducing an isomorphism on K_*^T. Consider the mapping cone exact sequence

$$0 \longrightarrow S^2 B \otimes \mathcal{K} \longrightarrow C\mu \longrightarrow F \longrightarrow 0,$$

and note that by construction, $K_*^T(C\mu) = 0$. Thus, if Proposition 3.9 holds, $K_*^H(C\mu) = 0$. Since α is obviously an isomorphism for F, application of the five-lemma to the diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & R(H)\otimes_{R(T)}K_*^T(F) & \longrightarrow & R(H)\otimes_{R(T)}K_{*-1}^T(B) & \longrightarrow & 0 \\
\downarrow{\alpha} & & \downarrow{\alpha} & & \downarrow{\alpha} & & \downarrow{\alpha} \\
K_*^H(C\mu) & \longrightarrow & K_*^H(F) & \longrightarrow & K_{*-1}^H(B) & \longrightarrow & K_*^H(C\mu)
\end{array}$$

shows that

$$\alpha: R(H)\otimes_{R(T)}K_*^T(B) \longrightarrow K_*^H(B)$$

is an isomorphism. Conversely, 3.9 is clearly a special case of 3.8, so that the two are equivalent.

PROOF (of Proposition 3.9): Without loss of generality we may assume that B has been T-stabilized. More precisely, we tensor B with $\mathcal{K}(\mathcal{H})$, where $\mathcal{H}$ is the direct sum of infinitely many copies of $L^2(T)$, so that each irreducible representation of T and of T/H appears with infinite multiplicity. Then B is also H-stable, since each irreducible representation of H extends to a representation of T, and the fixed-point algebra B^H is (T/H)-stable. As explained by Bratteli [Bra], if B is J-stable, then

$$B^J \text{ is stably isomorphic to } B \rtimes J \qquad (*)$$

for J = H, T, or T/H. Thus

$$\begin{aligned} K_*^T(B) &\cong K_*(B\rtimes T) && \text{by [Ju1] or [Gr2]} \\ &\cong K_*(B^T) && \text{by } (*) \\ &\cong K_*((B^H)^{T/H}) && \text{by definition} \\ &\cong K_*((B^H \rtimes (T/H)) && \text{by } (*) \\ &\cong K_*^{T/H}(B^H) && \text{as above} \end{aligned}$$

and similarly,

$$K_*^H(B) \cong K_*(B\rtimes H) \cong K_*(B^H)$$

so that it suffices to prove that if $K_*^{T/H}(B^H) = 0$ then $K_*(B^H) = 0$. Since T/H is another torus, we have reduced the proof of Proposition 3.9 down to the special case H = {1}, in which case we must demonstrate that if $K_*^T(B) = 0$, then $K_*(B) = 0$. This follows from iterations of the Pimsner-Voiculescu exact sequence [PV] or, more elegantly, the Kasparov spectral sequence [Ka3, §7] which has the form

$$E^2 = H_*(\pi, K_*(A)) \Rightarrow K_*(A\rtimes\pi)$$

for π a free abelian group. Take $A = B\rtimes T$, $\pi = \hat{T}$, and use Takai duality and the Julg-Green identification to obtain the spectral sequence

$$E^2 = H_*(\pi, K_*^T(B)) \Rightarrow K_*(B).$$

If $K_*^T(B) = 0$ then $E^2 = 0$ and so $K_*(B) = 0$, and thus Proposition 3.9 and Theorem 3.8 are established. □

Finally, we summarize the results of this section.

THEOREM 3.10. Suppose that G is a compact Lie group satisfying the Hodgkin condition, and suppose that B is a G-algebra with $K_*^G(B)$ R(G)-free. Then $\alpha(A,B)$ is an isomorphism for all commutative G-algebras A, and in fact for all $A \in \tilde{\mathcal{B}}_G$. If G is of rank 1 (i.e., if $G = \mathbb{T}$ or SU(2)) then $\alpha(A,B)$ is an isomorphism for all $A \in \mathcal{C}_G$.

We note that the missing link in the higher rank situation is the map $\alpha(C^*(G/H,\omega),B)$ where ω is a non-trivial cocycle. If these maps are isomorphisms then $\alpha(A,B)$ is an isomorphism for all $A \in \mathcal{C}_G$.

PROOF: Proposition 3.6 shows us that we may restrict to consideration of G-fundamental families. First suppose that G is a torus. Then Theorem 2.8 and Corollary 2.10 show that it suffices to consider $\alpha(A,B)$ where $A = C^*(G/H,\omega)$, if we are interested in all $A \in \mathcal{C}_G$, or to consider $\alpha(A,B)$ with $A = C(G/H)$, if we are interested in $A \in \mathcal{B}_G$. But $\alpha(C(G/H),B)$ is an isomorphism for any H by Theorem 3.8. This proves the result about $\mathcal{B}_G$, and by Corollary 2.12, also about $\mathcal{C}_G$ when $G = \mathbb{T}$.

In the general case, we must consider the map

$$\alpha^G(A,B) : K_*^G(A)\otimes K_*^G(B) \longrightarrow K_*^G(A\otimes B).$$

Apply the functor $R(T)\otimes_{R(G)}(-)$ to the map and one obtains (by Theorem 3.7) the map $\alpha^T(A,B)$. Now if $K_*^G(B)$ is R(G)-free, then

$$K_*^T(B) \cong R(T)\otimes_{R(G)}K_*^G(B)$$

is R(T)-free, so $\alpha^T(A,B)$ is an isomorphism by the first part of the proof. The fact that $R(T)\otimes_{R(G)}(-)$ is fully faithful (since R(T) is free over R(G)) implies that $\alpha^G(A,B)$ is an isomorphism. □

REMARK 3.11. In Theorem 3.10, we may replace the requirement that $K_*^G(B)$ be R(G)-free by the condition that it be R(G)-projective. The reason is that if we let $B_1 = c_0\oplus(c_0\otimes C_0(\mathbb{R}))$ with trivial

G-action, so that each $K_j^G(B_1)$ is free of countable rank, then $K_*^G(B \oplus B_1) \cong K_*^G(B) \oplus K_*^G(B_1)$ is also R(G)-free of countably infinite rank, by "Eilenberg's Lemma" [Ba]. Hence $\alpha(A,B_1)$ and $\alpha(A,B \oplus B_1)$ are isomorphisms by Theorem 3.10, so $\alpha(A,B)$ must be an isomorphism as well.

SECTION 4: GEOMETRIC PROJECTIVE RESOLUTIONS

In this section we construct a geometric resolution for an arbitrary G-algebra B. This tool is then used to construct the Künneth spectral sequence in section 5.

PROPOSITION 4.1. Let B be any unital G-algebra. Then there exists a C^*-algebra F with trivial G-action and with $K_*(F)$ $\mathbb{Z}$-free (hence with $K_*^G(F)$ R(G)-free) and a map μ: $F \longrightarrow B\otimes\mathcal{K}$ inducing a surjection

$$K_*^G(F) \longrightarrow K_*^G(B) \longrightarrow 0.$$

Here $\mathcal{K} = \mathcal{K}(H)$ with H the Hilbert space of of some unitary representation of G (an infinite direct sum of copies of $L^2(G)$ will do). If $K_*^G(B)$ is R(G)-free then the map μ may be chosen to induce an isomorphism on K_*^G. Moreover, F may be taken to be $C_0(Y)$ with Y a disjoint union of points and lines.

PROOF: For B a unital G-algebra, $K_0^G(B)$ consists of formal differences $[(P_1,\beta_1)] - [(P_2,\beta_2)]$, where P_j is a finitely generated projective B-module and β_j is a compatible G-action. By the Kasparov stabilization theorem [Ka1] each P_j can be embedded equivariantly as a summand of H_B, the Hilbert C^*-module of an infinite direct sum of copies of B tensored with all finite-dimensional representations of G, each appearing infinitely often. After tensoring with a finite-dimensional representation of G if necessary, we may assume that $[P_j,\beta_j]$ is represented by a G-fixed projection in $B\otimes\mathcal{K}$. Alternatively, we have

$$K_0^G(B) \cong K_0(G\ltimes B) \cong K_0(G\ltimes(B\otimes\mathcal{K})) \cong K_0((B\otimes\mathcal{K})^G),$$

using [Ju1], [Gr2], and [Ro1] or [Bra]. Thus $K_0^G(B)$ is generated by (since B is separable) countably many G-fixed projections in $B\otimes\mathcal{K}$, which we may make disjoint and commuting by the argument of [Sc2]. Taking one copy of $\mathbb{C}$ for each such projection, we get a map from an abelian algebra F_0 with discrete spectrum and trivial

G-action to $B\otimes K$ inducing a surjection on K_0^G.

Similarly, generators of $K_1^G(B)$ may be represented by homomorphisms $C_o(\mathbb{R}) \longrightarrow (B\otimes K)^G$, and by making the elements of $B\otimes K$ involved disjoint from those used to constuct F_0 we obtain an algebra F which is a sum of copies of $\mathbb{C}$ and of $C_o(\mathbb{R})$ with trivial G-action and a map $F \longrightarrow B\otimes K$ inducing a surjection on K_*^G. If $K_*^G(B)$ is R(G)-free then we choose a set of generators and hit those by generators of $K_*^G(F)$. □

We note that if A is a G-algebra then CA and SA are G-algebras in a natural fashion- with trivial action in the new coordinate. There are, of course, other possible G-structures, but we understand CA and SA to mean this particular G-structure. Similarly, if $f:A \longrightarrow B$ is a G-map, then the mapping cone Cf is a G-algebra and the mapping cone sequence is a sequence of G-maps.

<u>REMARK</u> <u>4.2.</u> Because of [Ro2, Theorem 4.1], if we replace B by its suspension $SB = B\otimes C_o(\mathbb{R})$ we may obtain a map $SF \longrightarrow SB\otimes K$ with the properties of Proposition 4.1 without assuming that B is unital. This will be convenient since we'll have to iterate uses of this Proposition and units will be lost at each stage of the construction. Then we obtain the following proposition.

<u>PROPOSITION</u> <u>4.3.</u> Let B be a G-algebra. Then there exist G-algebras F, W with $K_*^G(F)$ R(G)-free (of finite rank if $K_*^G(B)$ is finitely R(G)-generated) and a short exact sequence of G-algebras

$$0 \longrightarrow S^2B\otimes K \longrightarrow W \longrightarrow SF \longrightarrow 0$$

such that the associated long exact K_*^G sequence splits into short exact sequences

$$0 \longrightarrow K_{j+1}^G(W) \longrightarrow K_j^G(F) \longrightarrow K_j^G(B) \longrightarrow 0.$$

Furthermore, F is a sum of copies of $\mathbb{C}$ and of $C_o(\mathbb{R})$ with trivial G-action.

<u>PROOF:</u> Choose ϕ: $SF \longrightarrow B\otimes K$ as explained above and let W be the mapping cone $C\phi$.□

THEOREM 4.4. (Geometric Realization of Projective Resolutions) Suppose that G is a compact group whose representation ring R(G) has finite global homological dimension (e.g., suppose that it satisfies the Hodgkin condition). Let B be a G-algebra. Then there exists a C^*-algebra D with a natural isomorphism

$$K_*^G(A \otimes B) \cong K_*^G(A \otimes D)$$

for all A, and a finite filtration

$$0 = D_0 \subset D_1 \subset \cdots \subset D_{k+1} = D$$

by G-invariant ideals so that each $K_*^G(D_j/D_{j-1})$ is R(G)-projective. The filtration gives rise to a projective R(G)-resolution

$$0 \to K_*^G(D/D_k) \to K_*^G(D_k/D_{k-1}) \to \cdots \to K_*^G(D_1/D_0) \to K_*^G(B) \to 0.$$

PROOF: We may assume that B is G-stable and that all algebras appearing in the proof have been G-stabilized. Using (4.3), there is a map $\phi_1 : F_1 \longrightarrow SB$ where F_1 is as in (4.1) and the associated map $K_*^G(F_1) \longrightarrow K_*^G(SB)$ is surjective. The mapping cone sequence

$$0 \longrightarrow S^2B \longrightarrow W_1 \longrightarrow F_1 \longrightarrow 0$$

has associated to it the short exact sequence of R(G)-modules

$$0 \longrightarrow K_*^G(W_1) \longrightarrow K_*^G(F_1) \longrightarrow K_*^G(SB) \longrightarrow 0$$

which is the beginning of a projective resolution for $K_*^G(B)$. Repeat the process commencing with a K_*^G surjection $\phi_2 : F_2 \longrightarrow SW_1$ to obtain

$$0 \longrightarrow S^2W_1 \longrightarrow W_2 \longrightarrow F_2 \longrightarrow 0$$

with F_2 as in (4.1) and associated short exact sequence

$$0 \longrightarrow K_*^G(W_2) \longrightarrow K_*^G(F_2) \longrightarrow K_*^G(SW_1) \longrightarrow 0.$$

Continuing, we obtain a sequence of K_*^G- surjections $\phi_j : F_j \longrightarrow SW_{j-1}$ and each F_j as in (4.1) with associated mapping cone sequence

$$0 \longrightarrow S^2W_{j-1} \longrightarrow W_j \longrightarrow F_j \longrightarrow 0$$

The K-theory exact sequences splice together to yield the long exact sequence

$$0 \longrightarrow K_{j+k}^G(W_k) \longrightarrow K_{j+k}^G(F_k) \longrightarrow \ldots \longrightarrow K_{j+1}^G(F_1) \longrightarrow K_j^G(B) \longrightarrow 0.$$

Let us fix some k which is greater than the global homological dimension of R(G) and carry out the above procedure k times. Then there is a sequence of inclusions of ideals

$$S^{2k}B \subset S^{2k-2}W_1 \subset S^{2k-4}W_2 \subset \ldots \subset S^2W_{k-1} \subset W_k.$$

Let $\pi : W_k \longrightarrow W_k/S^{2k}B$ and let $D = D_{k+1} = C\pi$ be the mapping cone of π. Then there is a short exact sequence

$$0 \longrightarrow S(W_k/S^{2k}B) \longrightarrow D \longrightarrow W_k \longrightarrow 0,$$

from which we see that D is KK^G-equivalent to B. Let $D_j = S(S^{2k-2j}W_j/S^{2k}B)$ for $0<j<k+1$, $D_0 = 0$. Then

$$D_j/D_{j-1} \cong S^{2k-2j+1}(W_j/S^2W_{j-1}) \cong S^{2k-2j+1}F_j$$

and so $K_*^G(D_j/D_{j-1})$ is R(G)-free. The natural maps

$$K_i^G(D_j/D_{j-1}) \cong K_i^G(SF_j) \longrightarrow K_i^G(S^2W_{j-1}) \longrightarrow$$

$$\longrightarrow K_i^G(S^2F_{j-1}) \cong K_{i-1}^G(SF_{j-1}) \cong K_{i-1}^G(D_{j-1}/D_{j-2}))$$

induce a long exact sequence of R(G)-modules

$$0 \longrightarrow K_{j+k}^G(D/D_k) \longrightarrow K_{j+k-1}^G(D_k/D_{k-1}) \longrightarrow \ldots$$

$$\ldots \longrightarrow K_j^G(D_1/D_0) \longrightarrow K_j^G(B) \longrightarrow 0.$$

Each module $K_*^G(D_j/D_{j-1})$ is R(G)-projective by construction. The module $K_*^G(D/D_k)$ is R(G)-projective since k exceeds the global homological dimension of R(G). So this is the required geometric projective resolution of $K_*^G(B)$. □

We note for future reference that in particular cases some module $K_*^G(W_j)$ might be R(G)-projective for some smaller value of j; at that point the process may be terminated (and certainly should be, as the resulting spectral sequence will be simpler).

If R(G) has infinite global homological dimension (e.g., if G is a finite group) then the process described above leads to an R(G)-projective resolution of infinite length of the form

$$\cdots \longrightarrow K_*^G(F_k) \longrightarrow \cdots \longrightarrow K_*^G(F_1) \longrightarrow K_*^G(B) \longrightarrow 0.$$

This could be used to set up the Künneth spectral sequence. However, the spectral sequence would not converge and would be useless. Computing equivariant K- and KK-groups for such groups seems to be quite difficult, even in the commutative setting.

SECTION 5: CONSTRUCTION OF THE KÜNNETH SPECTRAL SEQUENCE

In this section we use the geometric projective resolutions of Section 4 to construct a spectral sequence which strongly converges to $K_*^G(A\otimes B)$ under the hypothesis that G satisfies the Hodgkin condition. In fact, the spectral sequence will eventually collapse: $E^r_{p,q} = E^\infty_{p,q}$ for r = 1 +(gl. dim. R(G)). For example, in the case of G = {1}, we have r = 2 and the Künneth spectral sequence reduces down to the usual Künneth Theorem for K-theory of [Sc2]. If $G = \mathbb{T}$, then r = 3. We shall examine this case further in Section 10.

The spectral sequence may be constructed with minimal hypotheses. For it to be interesting and effective in computation, however, one must be able to identify its E^2 term in terms of computable algebraic invariants. To make this identification, we must assume that $A \in \tilde{\mathcal{B}}_G$, the category of "bootstrap" G-algebras. (This assumption is analogous to the situation in the non-equivariant setting.) The result is the following theorem.

THEOREM 5.1. (Künneth Theorem, general case). Let G be a compact Lie group which satisfies the Hodgkin condition. For $A \in \tilde{\mathcal{B}}_G$ and B a G-algebra there is a spectral sequence of R(G)-modules strongly converging to $K_*^G(A\otimes B)$ with

$$E^r_{p,q} = \mathrm{Tor}_p^{R(G)}(K_*^G(A), K_*^G(B)).$$

The spectral sequence has the canonical grading (so that $\mathrm{Tor}_p^{R(G)}(K_s^G(A), K_t^G(B))$ has total degree p+s+t (mod 2)). The edge homomorphism

$$K_*^G(A)\otimes_{R(G)}K_*^G(B) \cong E^2_{0,*} \longrightarrow E^\infty_{0,*} \longrightarrow K_*^G(A\otimes B)$$

is the Künneth pairing $\alpha(A,B)$. The spectral sequence is natural with respect to pairs (A,B) in the category. If G has rank r then $E^2_{p,q} = 0$ for p>r+1 and $E^{r+2} = E^\infty$.

If $K_*^G(B)$ is R(G)-projective (e.g., if it is R(G)-free), then

$$\mathrm{Tor}_p^{R(G)}(K_*^G(A),K_*^G(B)) = 0 \quad \text{for } p>0$$

and so the spectral sequence degenerates to the statement that the natural map

$$\alpha(A,B):\ K_*^G(A)\otimes_{R(G)}K_*^G(B) \longrightarrow K_*^G(A\otimes B)$$

is an isomorphism. This, of course, was precisely the subject of Theorem 3.10 and Remark 3.11. We shall deduce our general results here from fairly standard methods in homological algebra and the results that we have established.

<u>PROOF:</u> Let us fix a G-algebra A in the category $\tilde{B}_G$. Theorem 4.4 (Geometric resolutions) applied to the G-algebra B gives us a C^*-algebra D with a natural isomorphism

$$K_*^G(A\otimes B) \cong K_*^G(A\otimes D)$$

for all A, and a finite filtration

$$0 = D_0 \subset D_1 \subset \dots \subset D_{k+1} = D$$

by G-invariant ideals so that each $K_*^G(D_j/D_{j-1})$ is R(G)-projective. The filtration gives rise to a projective R(G)-resolution

$$0 \longrightarrow K_{j+k}^G(D/D_k) \longrightarrow K_{j+k-1}^G(D_k/D_{k-1}) \longrightarrow \dots$$

$$\dots \longrightarrow K_j^G(D_1/D_0) \longrightarrow K_j^G(B) \longrightarrow 0.$$

Tensor the filtration

$$0 = D_0 \subset D_1 \subset \dots \subset D_{k+1} = D$$

with A to obtain the filtration

$$0 = A\otimes D_0 \subset A\otimes D_1 \subset \dots \subset A\otimes D_{k+1} = A\otimes D.$$

Apply Section 6 of [Sc1]. We obtain a spectral sequence strongly convergent to $K_*^G(A\otimes D) \cong K_*^G(A\otimes B)$ and with

$$E^1_{p,q} = K^G_{p+q}(A\otimes(D_p/D_{p-1})).$$

By Theorem 3.10, Remark 3.11, and the fact that $K_*^G(D_p/D_{p-1})$ is $R(G)$-projective, we may identify E^1 with $K_*^G(A)\otimes_{R(G)}K_*^G(D_p/D_{p-1})$. To be more precise about the gradings, we identify

$$E^1_{p,q} \cong K^G_{p+q}(A)\otimes_{R(G)}K^G_0(D_{p+1}/D_p) \oplus K^G_{p+q+1}(A)\otimes_{R(G)}K^G_1(D_{p+1}/D_p).$$

The d_1 differential in the spectral sequence is easily identified as the map

$$\mathrm{id}\otimes\delta : K^G_t(A)\otimes K^G_*(D_{p+1}/D_p) \longrightarrow K^G_t(A)\otimes K^G_*(D_p/D_{p-1})$$

where

$$\delta: K^G_*(D_{p+1}/D_p) \longrightarrow K^G_{*-1}(D_p/D_{p-1})$$

is the map in the $R(G)$-projective resolution of $K^G_*(B)$. By the definition of $\mathrm{Tor}^{R(G)}_*$, we obtain

$$E^2_{p,q} = \text{homology of } (\mathrm{id}\otimes\delta) \cong$$

$$\mathrm{Tor}^{R(G)}_p(K^G_0(A),K^G_q(B)) \oplus \mathrm{Tor}^{R(G)}_p(K^G_1(A),K^G_{q+1}(B)).$$

Often we write the spectral sequence in reverse with

$$E_2^{p,*} = \mathrm{Tor}^{R(G)}_p(K^G_*(A),K^G_*(B));$$

this corresponds merely to thinking of D as filtered not by the ideals D_j but rather by the quotients D/D_{k+1-j} and reindexing accordingly. □

SECTION 6: SOME CONSEQUENCES OF THE KÜNNETH SPECTRAL SEQUENCE

Our proof of the Universal Coefficient spectral sequence uses our Künneth spectral sequence in a non-trivial fashion. Accordingly, we pause in our development to record some consequences of the Künneth spectral sequence which we require. We also insert a simple consequence of the Künneth spectral sequence which builds along the lines suggested by Iberkleid-Petrie [IP] in their study of smooth actions of the circle group on manifolds. As an added diversion, we indicate how the Künneth spectral sequence implies the theorem of Pimsner and Voiculescu [PV]. This is quite similar to the proof of Kasparov [Ka3], though his spectral sequence arises in a quite different manner. Other applications will be deferred to Sections 10 and 11.

<u>THEOREM</u> <u>6.1.</u> (Hodgkin Spectral Sequence). Let G be a compact Lie group satisfying the Hodgkin condition, let H be a closed subgroup, and let B be a G-algebra. Then there is a spectral sequence which strongly converges to $K_*^H(B)$ with

$$E_*^2 \cong \mathrm{Tor}_*^{R(G)}(R(H), K_*^G(B)).$$

In particular, there is a strongly convergent spectral sequence

$$E_{p,*}^2 \cong \mathrm{Tor}_p^{R(G)}(\mathbb{Z}, K_*^G(B)) \implies K_*(B),$$

corresponding to the case H = {1}.

<u>PROOF:</u> This is just the Künneth spectral sequence for the pair (C(G/H),B). □

An important consequence of (6.1) is that if $K_*^G(B) = 0$, then $K_*^H(B) = 0$ for all closed subgroups H of G.

<u>COROLLARY</u> <u>6.2.</u> Let H be a closed subgroup of a torus T of any dimension, and let B be some T-algebra. Then there is a strongly convergent spectral sequence

$$E^2 = \mathrm{Tor}_*^{R(T)}(R(H),K_*^T(B)) \Rightarrow KK_T^*(C(T/H),B).$$

PROOF: The group T/H is also a torus, and so it is parallelizable, in fact in a T-equivariant way. So by Poincaré duality ([CS] and [Ka3, §8, Theorem 2]),

$$KK_T^*(C(T/H),B) \cong KK_T^*(\mathbb{C},C(T/H)\otimes B) \cong K_*^T(C(T/H)\otimes B).$$

By the equivariant Künneth Theorem and the fact $K_*^T(C(T/H)) \cong R(H)$ (as an R(T)-module), we obtain the corollary. □

Suppose that G is a compact Lie group which satisfies the Hodgkin condition. Then (unless G is trivial) R(G) is not a principal ideal domain. However, if we localize by some suitable family P of prime ideals, then $R(G)_P$ is a pid. Then we have the following theorem.

THEOREM 6.3. Suppose that G is a compact Lie group satisfying the Hodgkin condition, P is a collection of prime ideals in R(G), and that A and B are G-algebras with $A \in \tilde{B}_G$. Let $R = R(G)_P$. Then there is a spectral sequence which strongly converges to $K_*^G(A\otimes B)_P$, with

$$E_{p,*}^2 = \mathrm{Tor}_p^R(K_*^G(A)_P,K_*^G(B)_P).$$

If R is a principal ideal domain, then there is a natural short exact sequence

$$0 \longrightarrow K_*^G(A)_P \otimes_R K_*^G(B)_P \longrightarrow K_*^G(A\otimes B)_P \longrightarrow \mathrm{Tor}_1^R(K_*^G(A)_P,K_*^G(B)_P) \longrightarrow 0.$$

PROOF: The Künneth spectral sequence localizes to yield a spectral sequence which converges strongly to $K_*^G(A\otimes B)_P$ and with

$$E_{p,*}^2 = \mathrm{Tor}_p^R(K_*^G(A)_P,K_*^G(B)_P),$$

since Tor respects localization. If R is a pid, the Tor_p terms vanish for p>1 and the spectral sequence degenerates to the short exact sequence shown. □

Theorem 6.3 is parallel to results of C. Phillips [Ph], who deals with equivariant K-theory for finite groups.

Next we indicate how our equivariant Künneth theorem implies the Pimsner-Voiculescu exact sequence [PV]. Suppose that A is a C^*-algebra with a *-automorphism ϕ inducing an action α of $\mathbb{Z}$. The problem is to compute $K_*(\mathbb{Z}\ltimes_\alpha A)$ in terms of the action of ϕ_* on $K_*(A)$. Let $\hat{\alpha}$ denote the dual action of $\mathbb{T}$ on $\mathbb{Z}\ltimes_\alpha A$. Then Takai duality takes the form

$$A\otimes\mathcal{K} \cong \mathbb{T}\ltimes_{\hat{\alpha}}(\mathbb{Z}\ltimes_\alpha A)$$

and so

$$K_*(A) \cong K_*(\mathbb{T}\ltimes_{\hat{\alpha}}(\mathbb{Z}\ltimes_\alpha A)) \cong K_*^{\mathbb{T}}(\mathbb{Z}\ltimes_\alpha A).$$

The $R(\mathbb{T})$-action on $K_*(A)$ comes from tensoring by characters of $\mathbb{T}$ or, via Fourier transform, from the action of $\mathbb{Z}$ on A, so if we write $R(\mathbb{T}) = \mathbb{Z}[t,t^{-1}]$, then t acts as ϕ_*. Then $K_*(\mathbb{Z}\ltimes_\alpha A)$ may be computed by the Hodgkin spectral sequence (with $H = \{1\}$), which has E^2 term

$$\mathrm{Tor}_p^{R(\mathbb{T})}(\mathbb{Z},K_*(A)) \cong \mathrm{Tor}_p^{R(\mathbb{T})}(\mathbb{Z},K_*^{\mathbb{T}}(\mathbb{Z}\ltimes_\alpha A)).$$

Now $\mathbb{Z}$ has homological dimension 1 over $R(\mathbb{T})$, and hence $\mathrm{Tor}_p^{R(\mathbb{T})}(\mathbb{Z},-) = 0$ for $p>1$. Thus the spectral sequence collapses. In fact, the free resolution

$$0 \longrightarrow R(\mathbb{T}) \xrightarrow{t-1} R(\mathbb{T}) \longrightarrow \mathbb{Z} \longrightarrow 0$$

yields the exact sequence

$$0 \longrightarrow \mathrm{Tor}_p^{R(\mathbb{T})}(\mathbb{Z},K_*(A)) \longrightarrow K_*(A) \xrightarrow{t-1} K_*(A) \longrightarrow \mathbb{Z}\otimes_{R(\mathbb{T})}K_*(A) \longrightarrow 0.$$

Thus there is a short exact sequence

$$0 \longrightarrow \mathrm{Cok}(\phi_*-1) \longrightarrow K_*(\mathbb{Z}\ltimes_\alpha A) \longrightarrow \mathrm{Ker}(\phi_*-1) \longrightarrow 0$$

and this implies the Pimsner-Voiculescu long exact sequence

$$\cdots \longrightarrow K_j(A) \xrightarrow{\phi_*-1} K_j(A) \longrightarrow K_j(\mathbb{Z}\ltimes_\alpha A) \longrightarrow K_{j-1}(A) \longrightarrow \cdots$$

as desired.

Note that, although the Pimsner-Voiculescu theorem was used previously in this paper, there are some algebras for which the Künneth spectral sequence can be established by other methods, so that for suitable pairs (A,ϕ) this is indeed an independent proof.

If we argue similarly using Takai duality for $\mathbb{Z}^r$- and $\mathbb{T}^r$-actions, Theorem 6.1 leads to Kasparov's generalization [Ka3, §7, Theorem 2] of the Pimsner-Voiculescu sequence in the form of a spectral sequence

$$H_p(\mathbb{Z}^r;\ K_q(A)) \implies K_{p+q}(\mathbb{Z}^r \ltimes_\alpha A),$$

since

$$\begin{aligned} \mathrm{Tor}_p^{R(\mathbb{T}^r)}(\mathbb{Z},-) &\cong \mathrm{Tor}_p^{\mathbb{Z}[\mathbb{Z}^r]}(\mathbb{Z},-) \\ &\cong H_p(\mathbb{Z}^r,-) \end{aligned}$$

by [CE, Ch. X].

SECTION 7: THE UNIVERSAL COEFFICIENT SPECTRAL SEQUENCE

SPECIAL CASES

Given a compact group G and a pair (A,B) of G-algebras satisfying the usual technical hypotheses, there is a natural map

$$\gamma = \gamma(A,B) \; : \; KK_G^*(A,B) \longrightarrow \mathrm{Hom}_{R(G)}(K_*^G(A),\; K_*^G(B))$$

given as the adjoint of the Kasparov pairing $\otimes_A$. Alternatively, one may by [Ka2,§7] identify an element of $KK_1^G(A,B)$ with the equivalence class of an extension of G-algebras

$$0 \longrightarrow B\otimes\mathcal{K} \longrightarrow E \longrightarrow A \longrightarrow 0$$

and assign to an element the pair of connecting homomorphisms in the six-term exact sequence in equivariant K-theory associated to the extension. The map γ is an edge homomorphism in the universal coefficient spectral sequence. In particular, if $K_*^G(B)$ is injective as an R(G)-module, then $\gamma(A,B)$ should be an isomorphism for all suitable A. In this section we prove that this is indeed the case. As in the case of the Künneth Theorem, we prove this first for tori and then pass to general Hodgkin groups using Theorem 3.7. This requires an additional algebraic fact (Theorem 7.5) which seems to be new: if G is a Hodgkin group with maximal torus T, then R(T) is "self-dual" over R(G).

<u>PROPOSITION</u> <u>7.1</u> Assume that $K_*^G(B)$ is an injective R(G)-module. Then $KK_G^*(-,B)$ and $\mathrm{Hom}_{R(G)}(K_*^G(-),\; K_*^G(B))$ are additive cohomology theories (on the category of G-algebras), and $\gamma(-,B)$ is a natural transformation of theories

$$\gamma(-,B)\colon KK_G^*(-,B) \longrightarrow \mathrm{Hom}_{R(G)}(K_*^G(-),\; K_*^G(B)).$$

<u>Proof</u>: Each theory is clearly homotopy invariant. The theory $KK_G^*(-,B)$ satisfies the exactness axiom in the first variable, by

[Ka2], and it is additive (that is, it yields a $\lim^1$ sequence, as in [Sc3, RS2]). The theory $K_*^G(-)$ satisfies the exactness and additivity axioms as usual. If $K_*^G(B)$ is R(G)-injective, then $\mathrm{Hom}_{R(G)}(-, K_*^G(B))$ is an exact functor, thus $\mathrm{Hom}(K_*^G(-), K_*^G(B))$ satisfies the exactness and additivity axioms. □

<u>PROPOSITION</u> <u>7.2.</u> Suppose that $K_*^G(B)$ is R(G)-injective.

a) If J is an invariant ideal of the G-algebra A and if two of the maps $\gamma(J,B)$, $\gamma(A,B)$, $\gamma(A/J,B)$ are isomorphisms, then so is the third map.

b) If $A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow \ldots$ is a countable direct system of G-algebras with limit A and if $\gamma(A_i,B)$ is an isomorphism for each i, then $\gamma(A,B)$ is an isomorphism.

c) If $\gamma(A,B)$ is an isomorphism, then so is $\gamma(A \otimes C_o(\mathbb{R}^n),B)$.

d) If $\gamma(A,B)$ is an isomorphism and if A is G-stably equivalent to A', then $\gamma(A',B)$ is an isomorphism.

e) If $\gamma(A,B)$ is an isomorphism and if A is exterior equivalent to A', then $\gamma(A',B)$ is an isomorphism.

f) If $\gamma(A,B)$ is an isomorphism and if $A \rtimes \Gamma$ denotes the crossed product by the group $\Gamma = \mathbb{R}$ or $\mathbb{Z}$ whose action commutes with the G-action, then $\gamma(A \rtimes \Gamma,B)$ is an isomorphism.

<u>PROOF:</u> Parts a) and b) depend upon Proposition 7.1. Part c) holds by Bott periodicity. For part d), we note that both sides are invariant under G-stable isomorphism. Part e) follows from Proposition 3.1. For part f) in the case of the group $\mathbb{R}$, one uses Connes' Thom isomorphism and its generalization by Fack and Skandalis [FS] to KK. Part (f) in the case $\Gamma = \mathbb{Z}$ follows from the case of $\mathbb{R}$ and from (a) and (d), since given an action α of $\mathbb{Z}$ on a G-algebra A (commuting with the G-action), $A \rtimes_\alpha \mathbb{Z}$ and $T_\alpha A \rtimes \mathbb{R}$ are G-stably isomorphic, where $T_\alpha A$ is the mapping torus of α (cf. [Co], p. 48). □

<u>PROPOSITION</u> <u>7.3.</u> Suppose that $K_*^G(B)$ is R(G)-injective and that

$\gamma(A,B)$ is an isomorphism for all A in some A_G-fundamental (resp., C_G-fundamental) family F. Then $\gamma(A,B)$ is an isomorphism for all separable abelian (resp., Type I) G-algebras A.

PROOF: This is an immediate consequence of Proposition 7.2. □

PROPOSITION 7.4. Suppose that $T = \mathbb{T}^n$ is a torus, H is a closed subgroup, $A = C(T/H)$ with the evident T-action, and $K_*^T(B)$ is R(T)-injective. Then $\gamma(A,B)$ is an isomorphism.

PROOF: We shall use the special case of the Hodgkin spectral sequence (6.2)

$$E^2 = \mathrm{Tor}_*^{R(T)}(R(H), K_*^T(B)) \implies KK_T^*(C(T/H),B).$$

We compare the functor $\mathrm{Tor}_*^{R(T)}(R(H),-)$ with $\mathrm{Ext}_{R(T)}^*(R(H),-)$. First suppose that n=1. If $H = \mathbb{T}$ there is nothing to prove, so we may assume that $H = \mathbb{Z}_k$, embedded in $\mathbb{T}$ in the standard manner. We have

$$R(\mathbb{T}) = \mathbb{Z}[t,t^{-1}] \qquad R(H) = R(\mathbb{T})/(t^k-1),$$

so we obtain the free resolution

$$0 \longrightarrow R(\mathbb{T}) \xrightarrow{(t^k-1)} R(\mathbb{T}) \longrightarrow R(H) \longrightarrow 0.$$

From this we see that for any $R(\mathbb{T})$-module M,

$$R(H) \otimes_{R(\mathbb{T})} M \cong \mathrm{coker}(\text{mult by } t^k-1 \text{ on } M).$$

$$\mathrm{Tor}_1^{R(\mathbb{T})}(R(H),M) \cong \ker(\text{mult by } t^k-1 \text{ on } M)$$

and dually,

$$\mathrm{Hom}_{R(\mathbb{T})}(R(H),M) \cong \mathrm{Tor}_1^{R(\mathbb{T})}(R(H),M),$$

and

$$\mathrm{Ext}^1_{R(\mathbb{T})}(R(H),M) \cong R(H)\otimes_{R(\mathbb{T})}M.$$

Using the injectivity assumption on $M = K_*^T(B)$, we see that the Hodgkin spectral sequence has $E_p^2 = 0$ for $p\neq 1$, and the isomorphism

$$KK_{\mathbb{T}}^*(\mathbb{C},C(\mathbb{T}/H)\otimes B) \xrightarrow{\cong} \mathrm{Tor}_1^{R(\mathbb{T})}(R(H),K_*^{\mathbb{T}}(B))$$

may be identified under the Poincaré duality pairing with the natural map

$$\gamma(C(\mathbb{T}/H),B) : KK_{\mathbb{T}}^*(C(\mathbb{T}/H),B) \longrightarrow \mathrm{Hom}_{R(\mathbb{T})}(R(H),K_*^{\mathbb{T}}(B))$$

which completes the case of the circle. If $n>1$ then as pointed out in [Sn], we may assume that H is standard, i.e., the pair (T,H) is a product of similar pairs associated to one-dimensional tori. This makes it possible to reduce to the one-dimensional case. □

<u>THEOREM</u> <u>7.5.</u> Let G be a compact Lie group satisfying the Hodgkin condition, and let T be a maximal torus in G. Then

a) the Poincaré duality isomorphism

$$\delta: KK^G(C(G/T),\mathbb{C}) \longrightarrow KK^G(\mathbb{C},C(G/T))$$

is an isomorphism of R(T)-modules (where the R(T)-module structure on the left-hand side will be explained below);

b) for any R(G)-module M,

$$\mathrm{Hom}_{R(G)}(R(T),M) \cong R(T)\otimes_{R(G)}M$$

as R(T)-modules;

c) if B is a G-algebra such that $K_*^G(B)$ is R(G)-injective, then $K_*^T(B)$ is R(T)-injective.

<u>PROOF:</u> a) We recall the Poincaré duality map used in the proof of Theorem 3.7. Let

$$\Delta\colon G/T \longrightarrow G/T \times G/T$$

be the diagonal embedding, and let

$$\Delta^* \in \mathrm{Hom}_G(C(G/T)\otimes C(G/T),C(G/T)) \longrightarrow KK^G(C(G/T)\otimes C(G/T),C(G/T))$$

be the induced element. Then

$$[D_L] = ([L]\otimes_{C(G/T)}\Delta^*)\otimes_{C(G/T)}[D] \in KK^G(C(G/T),\mathbb{C})$$

is the class of the Dolbeault complex over C(G/T) with values in the homogeneous holomorphic line bundle L, and the Poincaré duality map δ satisfies

$$\delta([D_L]) = [L].$$

Recall that R(T) is R(G)-free on the Steinberg basis $\{e_w\}$, that $KK^G(C(G/T),\mathbb{C})$ is R(G)-free on the (dual) basis $\{a_w\}$, and that the Kasparov pairing

$$\otimes_{C(G/T)}\colon KK^G(\mathbb{C},C(G/T)) \times KK^G(C(G/T),\mathbb{C}) \longrightarrow R(G)$$

induces a natural isomorphism (a priori of R(G)-modules)

$$KK^G(C(G/T),\mathbb{C}) \cong \mathrm{Hom}_{R(G)}(R(T),R(G)). \tag{7.6}$$

To complete the proof of part a), then, it suffices to prove that (7.6) is an isomorphism of R(T)-modules. The ring $R(T) \cong KK^G(\mathbb{C},C(G/T))$ operates naturally on $KK^G(C(G/T),\mathbb{C})$ by the formula

$$y\cdot x = y\otimes_{C(G/T)}\Delta^*\otimes_{C(G/T)}x,$$

for $y \in KK^G(\mathbb{C},C(G/T))$ and $x \in KK^G(C(G/T),\mathbb{C})$.

Let $[L_1]$, $[L_2] \in \hat{T}$ (identified with the corresponding induced line bundles over G/T), and let

$$d^* \in KK^G(C(G/T)\otimes C(G/T)\otimes C(G/T),C(G/T))$$

be the iterated diagonal map, defined similarly to Δ^*. Then we have

$$[L_1]\cdot[D_{L_2}] = [L_1]\otimes_{C(G/T)}\Delta^*\otimes_{C(G/T)}([L_2]\otimes_{C(G/T)}\Delta^*\otimes_{C(G/T)}[D])$$

$$= ([L_1]\otimes_{\mathbb{C}}[L_2])\otimes_{C(G/T)\otimes C(G/T)}d^*\otimes_{C(G/T)}[D]$$

$$= (([L_1]\otimes_{\mathbb{C}}[L_2])\otimes_{C(G/T)\otimes C(G/T)}\Delta^*)\otimes_{C(G/T)}\Delta^*\otimes_{C(G/T)}[D]$$

$$= [L_1\otimes L_2]\otimes_{C(G/T)}\Delta^*\otimes_{C(G/T)}[D]$$

$$= [D_{L_1\otimes L_2}].$$

Hence,

$$\delta([L_1]\cdot[D_{L_2}]) = \delta([D_{L_1\otimes L_2}])$$

$$= [L_1\otimes L_2] = [L_1]\cdot[L_2]$$

$$= [L_1]\cdot\delta([D_{L_2}]),$$

so δ is R(T)-linear. This completes a) and shows that (7.6) is an isomorphism of R(T)-modules.

b) Since R(T) is finitely generated and free over R(G) of rank the order of the Weyl group, we have

$$\mathrm{Hom}_{R(G)}(R(T),M) \cong R(T)\otimes_{R(G)}M \cong \oplus_{w\in W}M$$

as R(G)-modules, and also

$$\mathrm{Hom}_{R(G)}(R(T),M) \cong \mathrm{Hom}_{R(G)}(R(T),R(G))\otimes_{R(G)}M$$

as R(T)-modules. Thus we may assume that M = R(G), and then part b) reduces to (7.6).

c) By Theorem 3.7, we know

$$K_*^T(B) \cong R(T)\otimes_{R(G)}K_*^G(B)$$

as R(T)-modules, and part b) implies that

$$K_*^T(B) \cong \mathrm{Hom}_{R(G)}(R(T), K_*^G(B))$$

as $R(T)$-modules. But for any ring extension $R \subset S$ and any injective R-module M, $\mathrm{Hom}_R(S,M)$ is S-injective by [CE, Ch. II, Proposition 6.1a]. □

We complete this section with its main result.

THEOREM 7.7. Let G be a compact connected Lie group satisfying the Hodgkin condition and let B be any G-algebra such that $K_*^G(B)$ is $R(G)$-injective. Then for each $A \in \tilde{\mathcal{B}}_G$, $\gamma(A,B)$ is an isomorphism.

PROOF: If G is a torus, this follows by combining Theorem 2.8 and Propositions 7.2, 7.3, and 7.4. If G is a general Hodgkin group with maximal torus T, then by Theorem 7.5c, $K_*^T(B)$ is $R(T)$-injective. Hence, by the result for tori,

$$\gamma^T(A,B)\colon KK_*^T(A,B) \longrightarrow \mathrm{Hom}_{R(T)}(K_*^T(A), K_*^T(B))$$

is an isomorphism. Now consider the commutative diagram

$$\begin{array}{ccc} KK_*^G(A,B) & \xrightarrow{\gamma^G(A,B)} & \mathrm{Hom}_{R(G)}(K_*^G(A), K_*^G(B)) \\ \Big\downarrow r & & \Big\downarrow r' \\ KK_*^T(A,B) & \xrightarrow{\gamma^T(A,B)} & \mathrm{Hom}_{R(T)}(K_*^T(A), K_*^T(B)). \end{array}$$

Since r is injective, in fact

$$KK_*^T(A,B) \cong R(T) \otimes_{R(G)} r(KK_*^G(A,B))$$

by Theorem 3.7(ii), we see that $\gamma^G(A,B)$ is injective. On the other hand,

$$\mathrm{Hom}_{R(T)}(K_*^T(A), K_*^T(B)) \cong \mathrm{Hom}_{R(T)}(K_*^T(A), \mathrm{Hom}_{R(G)}(R(T), K_*^G(B)))$$

by (3.7)(i) and (7.5)

$$\cong \mathrm{Hom}_{R(G)}(K_*^T(A), K_*^G(B)) \qquad \text{by adjoint associativity}$$

$$\cong \mathrm{Hom}_{R(G)}(R(T)\otimes_{R(G)} K_*^G(A), K_*^G(B)) \qquad \text{by (3.7)}$$

$$\cong R(T)\otimes_{R(G)} \mathrm{Hom}_{R(G)}(K_*^G(A), K_*^G(B))$$

since R(T) is finitely generated free over R(T), and by faithful flatness of $R(G) \subset R(T)$, surjectivity of $\gamma^T(A,B)$ implies surjectivity of $\gamma^G(A,B)$. □

SECTION 8: GEOMETRIC INJECTIVE RESOLUTIONS

In this section we shall construct a geometric injective resolution for a separable G-algebra B. The usual algebraic construction of injectives would take us out of the separable category, so we must proceed in a more delicate fashion. First we prove the algebraic results that we shall utilize. Next we show that any countably generated graded R(G)-module may be realized as $K_*^G(A)$ for a suitable G-algebra A, provided that the group G is compact Lie and satisfies the Hodgkin condition. Then the resolutions themselves are constructed.

If R is a commutative ring and M any R-module, we denote by E(M) the "R-injective hull" of M, i.e., the smallest injective R-module containing M. This is unique up to isomorphism as explained in [Ma].

PROPOSITION 8.1. Let R be a commutative Noetherian ring and let M be a finitely generated R-module. Then M can be embedded in a finite direct sum of modules of the form $E(R/p)$, $p \in \mathrm{Ass}(M) \subseteq \mathrm{Spec}(R)$.

PROOF: By [Mat, Theorem 10], M has a finite composition series

$$0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M$$

with $M_i/M_{i-1} \cong R/p_i$, $p_i \in \mathrm{Ass}(M) \subseteq \mathrm{Spec}(R)$. One proves the proposition by induction on n. If n=1 we are obviously done. Otherwise, assume that $M_{n-1} \subseteq E_1$, where E_1 is a finite direct sum of modules $E(R/p_i)$, and consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_{n-1} & \longrightarrow & M & \longrightarrow & M/M_{n-1} & \longrightarrow & 0 \\ & & \downarrow & & & & & & \\ & & E_1 & & & & & & \end{array}$$

Since E_1 is injective, the inclusion $M_{n-1} \longrightarrow E_1$ extends to a map $\phi: M \longrightarrow E_1$. Consider also the composite

$$\xi : M \longrightarrow M/M_{n-1} \cong R/p_n \subset E(R/p_n).$$

Clearly, $\phi\oplus\xi : M \longrightarrow E_1 \oplus E(R/p_n)$ is an injection of M into a finite direct sum of modules $E(R/p_i)$. (Of course, E(M) might be a proper submodule of this.) □

PROPOSITION 8.2. Let R be a countable commutative Noetherian ring and let $p \in \mathrm{Spec}(R)$. Then $E(R/p)$ is countably generated as an R-module.

PROOF: The argument of [Ma, Theorem 3.11] shows that $E(R/p)$ is countably generated over the local ring R_p. But if R itself is countable then R_p is countably generated as an R-module, as one needs only countably many denominators. □

PROPOSITION 8.3. Let G be a compact Lie group and let M be a countably generated R(G)-module. Then there exists an injective countably generated R(G)-module E and an injection of R(G)-modules $M \longrightarrow E$.

PROOF: By [Se1, Cor 3.3], R(G) is a Noetherian ring, and of course R(G) is countable since $\hat{G}$ is countable. Since M is countably generated, it is a countable injective limit of finitely generated submodules M_i. By [Ma, Prop. 1.2], $\varinjlim E(M_i)$ is injective and it obviously contains a copy of M. So it's enough to show that if M is finitely generated over R(G), then E(M) is countably generated. This is immediate from Propositions 8.1 and 8.2. □

REMARK 8.4. One can give an alternative proof of Proposition 8.3 by using the method of [CE, Ch.I, Theorem 3.3], and noting that if R is a countable Noetherian ring then a countable direct limit of countably generated modules will work. The present proof, however, has the advantage of displaying explicitly the sorts of injective modules that are actually required.

The proof of the next proposition works for any connected Lie group for which R(G) has finite global dimension. It's possible the statement is true for any compact Lie group, but, if so, a completely different argument would be needed.

<u>PROPOSITION</u> <u>8.5.</u> Let G be a compact Lie group satisfying the Hodgkin condition and let M_* be any countably generated $\mathbb{Z}_2$-graded R(G)-module. Then there exists a G-algebra A in $\mathcal{B}_G$ with

$$K_*^G(A) \cong M_*$$

as graded R(G)-modules.

<u>PROOF:</u> By [Ho, Prop. 8.3], R(G) has finite global dimension. Thus one can construct a finite projective resolution

$$0 \longrightarrow (F_n)_* \xrightarrow{\xi_n} (F_{n-1})_* \xrightarrow{\xi_{n-1}} \cdots \xrightarrow{\xi_2} (F_1)_* \xrightarrow{\xi_1} M_* \longrightarrow 0$$

by countably generated $\mathbb{Z}/2$-graded R(G)-modules. The usual construction will actually make $F_1, \ldots, F_{n-1}$ countably generated and free, and F_n countably generated and projective. By using "Eilenberg's Lemma" [Ba], we can make F_n free as well, merely by adding on the same countably generated free module to both F_n and F_{n-1}.

We now proceed as follows. Suppose we have constructed C^*-algebras $A_1, A_2, \ldots, A_n \in \mathcal{B}_G$ with $K_*^G(A_j) \cong (F_j)_*$, and also (if n>1) maps $\phi_j : A_j \longrightarrow A_{j-1}$, $2 \leq j \leq n$, such that $(\phi_j)_* = \xi_j$ and $\phi_{j-1}\phi_j = 0$ (the latter condition for j=3,...,n). We begin by forming the mapping cone $C\phi_n$. From the short exact mapping cone sequence

$$0 \longrightarrow SA_{n-1} \longrightarrow C\phi_n \longrightarrow A_n \longrightarrow 0$$

we obtain the exact triangle of $\mathbb{Z}/2$-graded R(G)-modules

$$\begin{array}{ccc} (F_n)_* & \xrightarrow{\xi_n} & (F_{n-1})_* \\ & \nwarrow \quad \swarrow & \\ & K_*^G(C\phi_n) & \end{array},$$

from which we see that $K_*^G(C\phi_n) \cong \operatorname{coker}(\xi_n)$. If n=2, we are done (take $A = SC\phi_2$) and of course if n=1 we were done before when we constructed A_1. Otherwise, construct a map $\tilde{\phi}_{n-1} : C\phi_n \longrightarrow SA_{n-2}$ by

$$\tilde{\phi}_{n-1}(f,a)(t) = \phi_{n-1}(f(t)).$$

(Recall that $C\phi_n = \{(f,a): f \in C_o((0,1], A_{n-1}),\ a \in A_n,\ f(1) = \phi_n(a)\}$, so that $\phi_{n-1}(f(1)) = \phi_{n-1}\phi_n(a) = 0$ as required.) Then we clearly have $(\phi_{n-1})_* = \xi_{n-1}$ (interpreted as a monomorphism $\operatorname{coker}(\xi_n) \longrightarrow F_{n-2}$), except for a shift of degree. Forming the mapping cone $C\phi_{n-1}$ gives a short exact sequence

$$0 \longrightarrow S^2A_{n-2} \longrightarrow C\tilde{\phi}_{n-1} \longrightarrow C\phi_n \longrightarrow 0$$

and the exact triangle

$$\begin{array}{ccc} \operatorname{Coker}(\xi_n) & \xrightarrow{\ \xi_{n-1}\ } & (F_{n-2})_* \\ & \nwarrow \quad \swarrow & \\ & K_*^G(C\tilde{\phi}_{n-1}) & \end{array}$$

so that $K_*^G(C\tilde{\phi}_{n-1}) \cong \operatorname{coker}(\xi_{n-1})$. If $n=3$ we are done. Otherwise, repeat the construction to get $\tilde{\phi}_{n-2}$ with $K_{*+1}^G(C\tilde{\phi}_{n-2}) = \operatorname{coker}(\xi_{n-2})$, etc. The process eventually stops.

It remains to construct $A_1,\ldots,A_n$ and the ϕ's. For this it is enough to treat the case where $(F_j)_0 = 0$ for all j, since we can eventually take the direct sum of an algebra with vanishing K_0^G and of one with vanishing K_1^G (constructed as the suspension of one with vanishing K_0^G). We may begin by taking for A_n a (c_o-) direct sum of copies of $C_o(\mathbb{R})$ with trivial G-action, one summand for each element in an R(G)-basis of F_n. Then take for A_{n-1} a similar direct sum of copies of $C_o(\mathbb{R}) \otimes \mathcal{K}$. We may choose $\phi_n : A_n \longrightarrow A_{n-1}$ with $(\phi_n)_* = \xi_n$ by the proof of the geometric resolution Proposition 4.1 needed in the proof of the Künneth theorem: each basis element of $K_1^G(A_n)$ is sent to an element of $K_1^G(A_{n-1})$ represented by a map $C_o(\mathbb{R}) \longrightarrow A_{n-1}^G$, and this map will serve as the appropriate component of ξ_n. The $A_{n-2},\ldots,A_1$ may also be chosen to be direct sums of copies of $C_o(\mathbb{R}) \otimes \mathcal{K}$, and the same construction will give the maps $\phi_{n-1},\ldots,\phi_2$. The only problem is to insure that the composition of any two successive ϕ's is zero. We could guarantee this by being sufficiently careful, but it is easier to note that since $(\phi_{j-1})_*(\phi_j)_* = \xi_{j-1}\xi_j = 0$, it must be that $\phi_{j-1}\phi_j$ is null-homotopic. (This follows from [Ro2, Theorem

4.1]. For $j<n$, one needs also [Ro2, Remark 3.5] along with the observation that we can take our $\phi_j : A_j \longrightarrow A_{j-1}$ to be in the image of what is there called σ with respect to an abelian subalgebra of A_j.) This is good enough for our purposes, because if $h_j(t)$, $0 \leqslant t \leqslant 1$, is a homotopy from $h_j(0) = \phi_{j-1}\phi_j$ to $h_j(1) = 0$, then we can define $\tilde{\phi}_{n-1}$ instead by

$$\tilde{\phi}_{n-1}(f,a)(t) = \begin{cases} \phi_{n-1}(f(2t)), & 0 \leqslant t \leqslant 1/2 \\ h_n(2t-1)(a), & 1/2 \leqslant t \leqslant 1, \end{cases}$$

and the induced map on K-theory will be the same as before. □

THEOREM 8.6. Let G be a compact Lie group satisfying the Hodgkin condition, and let B be a G-algebra. Then there exists a G-algebra $D \in \mathcal{B}_G$ with $K_*^G(D)$ R(G)-injective and a map of C^*-algebras from some suspension of $B \otimes \mathcal{K}$ into D which induces an inclusion

$$K_*^G(B) \longrightarrow K_*^G(D).$$

PROOF: The R(G)-module $K_*^G(B)$ is countably generated, so by Proposition 8.3 there is an embedding $K_*^G(B) \subset I_*$ with I_* countably generated and R(G)-injective. By Proposition 8.5, there is a G-algebra E with $K_*^G(E) \cong I_*$. Fix an embedding

$$\phi : K_*^G(B) \subset K_*^G(E)$$

and let Δ be the diagonal embedding

$$\Delta : K_*^G(B) \longrightarrow K_*^G(B) \oplus K_*^G(E) \qquad \Delta(x) = (x, -\phi(x)).$$

The image M_* of Δ is a submodule of $K_*^G(B \oplus E)$. We will construct a C^*-algebra $D \in \mathcal{B}_G$ and for some n a map $S^n(B \oplus E) \otimes \mathcal{K} \longrightarrow D$ such that the induced map in K_*^G theory is the quotient map

$$K_*^G(B \oplus E) \longrightarrow K_*^G(B \oplus E)/M_* \cong I_*.$$

Composing with the natural embedding $B \longrightarrow B \oplus E$, we obtain a map $B \longrightarrow D$ which induces ϕ on K_*^G.

To construct D, we proceed somewhat as in the proof of Proposition 8.5. First choose a finite resolution of M_* by countably generated free R(G)-modules,

$$0 \longrightarrow (F_n)_* \xrightarrow{\xi_n} \cdots \longrightarrow (F_1)_* \xrightarrow{\xi_1} M_* \longrightarrow 0.$$

As before, we may choose an algebra $A_1 \in \mathcal{A}_G$ (abelian with trivial G-action) and a map

$$\mu_1 : SA_1 \longrightarrow S(B\oplus E)\otimes \mathcal{K}$$

such that $(\mu_1)_* = \xi_1$ (composed with the inclusion $M_* \subset K_*^G(B\oplus E)$). Form the mapping cone $C\mu_1$. We obtain a short exact sequence

$$0 \longrightarrow S^2((B\oplus E)\otimes \mathcal{K}) \xrightarrow{\nu_1} C\mu_1 \longrightarrow SA_1 \longrightarrow 0$$

inducing the exact triangle

$$\begin{array}{ccc} (F_1)_* & \xrightarrow{\xi_1} & K_*^G(B\oplus E) \\ & \nwarrow \qquad \swarrow {\scriptstyle (\nu_1)_*} & \\ & K_*^G(C\mu_1), & \end{array}$$

where the image of ξ_1 is M_*. Thus we obtain a short exact sequence

$$0 \longrightarrow K_*^G(B\oplus E)/M_* \xrightarrow{(\nu_1)_*} K_*^G(C\mu_1) \longrightarrow \ker \xi_1 \longrightarrow 0.$$

Hence if n=1 we are done, with $D = C\mu_1$. In any event, since $K_*^G(B\oplus E)/M_*$ is injective, the above short exact sequence splits. Choose a splitting map $\ker \xi_1 \longrightarrow K_*^G(C\mu_1)$ and compose it with $\xi_2 : (F_2)_* \longrightarrow \ker \xi_1$ to obtain a map $\tilde{\xi}_2 : (F_2)_* \longrightarrow K_*^G(C\mu_1)$. Once again, we may choose an abelian algebra A_2 with trivial G-action and a map $\mu_2 : SA_2 \longrightarrow SC\mu_1 \otimes \mathcal{K}$ such that $K_*^G(A_2) = (F_2)_*$ and $(\mu_2)_* = \tilde{\xi}_2$. Form the mapping cone $C\mu_2$, the associated mapping cone sequence

$$0 \longrightarrow S^2C\mu_1\otimes \mathcal{K} \xrightarrow{\nu_2} C\mu_2 \longrightarrow SA_2 \longrightarrow 0$$

and resulting exact triangle

$$\begin{array}{ccc} (F_2)_* & \xrightarrow{\tilde{\xi}_2} & K_*^G(C\mu_1) \\ & \nwarrow \quad \swarrow (\nu_2)_* & \\ & K_*^G(C\mu_2). & \end{array}$$

This time the image of $\tilde{\xi}_2$ is the complement to $K_*^G(B\oplus E)/M_*$, so there is a short exact sequence

$$0 \longrightarrow K_*^G(B\oplus E)/M_* \xrightarrow{(\nu_2\nu_1)_*} K_*^G(C\mu_2) \longrightarrow \ker \xi_2 \longrightarrow 0.$$

Continue this process to arrive at

$$K_*^G(B\oplus E)/M_* \cong K_*^G(C\mu_n).$$

The quotient map $K_*^G(B\oplus E) \longrightarrow K_*^G(C\mu_n)$ is induced by the composition $\nu_n\nu_{n-1}\cdots\nu_1$ (with appropriate adjustment for suspensions and tensoring with K). □

We proceed to the construction of the geometric injective resolutions. Suppose that G is a compact Lie group satisfying the Hodgkin condition and that B is a G-algebra. We assume without loss of generality that B and all other G-algebras constructed are stable. Theorem 8.6 implies that there is a G-algebra I_1 with $K_*^G(I_1)$ R(G)-injective and a G-map

$$\phi_1 : S^{r_0}B \longrightarrow I_1$$

for some suspension $S^{r_0}B$ of B (and we take r_0 to be even) which induces an inclusion $K_*^G(B) \longrightarrow K_*^G(I_1)$. The mapping cone sequence (with $W_1 = C\phi_1$)

$$0 \longrightarrow SI_1 \longrightarrow W_1 \longrightarrow S^{r_0}B \longrightarrow 0$$

has associated to it the short exact sequence of R(G)-modules

$$0 \longrightarrow K_j^G(B) \longrightarrow K_j^G(I_1) \longrightarrow K_{j-1}^G(W_1) \longrightarrow 0$$

which is the beginning of an injective resolution of $K_*^G(B)$.

Repeat the process commencing with a K_*^G- injection $S^{r_1}W_1 \longrightarrow I_2$ with r_1 odd to obtain

$$0 \longrightarrow SI_2 \longrightarrow W_2 \longrightarrow S^{r_1}W_1 \longrightarrow 0$$

and associated short exact sequence

$$0 \longrightarrow K_{j-1}^G(W_1) \longrightarrow K_j^G(I_2) \longrightarrow K_{j-1}^G(W_2) \longrightarrow 0.$$

The K^G-theory exact sequences splice together to yield the exact sequence

$$0 \longrightarrow K_j^G(B) \longrightarrow K_j^G(I_1) \longrightarrow K_j^G(I_2) \longrightarrow K_{j-1}^G(W_2) \longrightarrow 0.$$

Let us fix some k which is greater than the injective dimension of R(G) and carry out the above procedure k times, with the K_*^G- injections

$$\phi_j : S^{r_{j-1}}W_{j-1} \longrightarrow I_j \quad (r_j \text{ odd})$$

and associated mapping cone sequences (with $W_j = C\phi_j$)

$$0 \longrightarrow SI_j \longrightarrow W_j \xrightarrow{p_j} S^{r_{j-1}}W_{j-1} \longrightarrow 0.$$

Then there is a sequence of surjective G-maps

$$W_k \xrightarrow{\tilde{p}_k} S^{r_{k-1}}W_{k-1} \longrightarrow \cdots \xrightarrow{\tilde{p}_2} S^{r_1+\cdots+r_{k-1}}W_1 \xrightarrow{\tilde{p}_1} S^{r_0+\cdots+r_{k-1}}B.$$

Let

$$s_j = \tilde{p}_j \circ \cdots \circ \tilde{p}_k : W_k \longrightarrow S^{r_{j-1}+\cdots+r_{k-1}}W_{j-1}$$

and

$$s_1 = \tilde{p}_1 \circ \cdots \circ \tilde{p}_k : W_k \longrightarrow S^{r_0+\cdots+r_{k-1}} B$$

with associated short exact sequences

$$0 \longrightarrow \ker s_j \xrightarrow{t_j} W_k \xrightarrow{s_j} S^{r_{j-1}+\cdots+r_{k-1}} W_{j-1} \longrightarrow 0$$

and

$$0 \longrightarrow \ker s_1 \xrightarrow{t_1} W_k \xrightarrow{s_1} S^{r_0+\cdots+r_{k-1}} B \longrightarrow 0.$$

Then there is an ascending sequence of ideals

$$\ker s_k \subseteq \ker s_{k-1} \subseteq \cdots \subseteq \ker s_2 \subseteq \ker s_1$$

and corresponding ascending sequence of ideals

$$Ct_k \subseteq Ct_{k-1} \subseteq \cdots \subseteq Ct_2 \subseteq Ct_1.$$

The G-algebra Ct_1 is equivariantly weakly equivalent to a suspension of B. More precisely, there is a short exact sequence

$$0 \longrightarrow \mathrm{Cone}(\ker s_1) \longrightarrow Ct_1 \xrightarrow{\pi} S^{r_0+\cdots+r_{k-1}+1} B \longrightarrow 0$$

so that the map π induces an isomorphism of K_*^G- groups. Finally, we identify the successive quotients in the filtration. For each j there is a natural exact diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{cone}(\ker s_{j+1}) & \longrightarrow & Ct_{j+1} & \longrightarrow & S^{1+r_j+\cdots+r_{k-1}} W_j & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow{\scriptstyle d_j} & & \big\downarrow{\scriptstyle S^{1+r_j+\cdots+r_{k-1}} p_j} & & \\
0 & \longrightarrow & \mathrm{cone}(\ker s_j) & \longrightarrow & Ct_j & \longrightarrow & S^{1+r_{j-1}+\cdots+r_{k-1}} W_{j-1} & \longrightarrow & 0.
\end{array}$$

Thus, up to equivariant weak equivalence,

$$\begin{aligned} Ct_j/Ct_{j+1} &\simeq S^{r_j+\cdots+r_{k-1}}(\ker p_j) \\ &= S^{1+r_j+\cdots+r_{k-1}} I_j. \end{aligned}$$

In particular, $K_i^G(Ct_j/Ct_{j+1}) \cong K_{i+k-j+1}^G(I_j)$ is R(G)- injective. Set $B_j = Ct_{k-j}$ and $B_0 = \{0\}$. Then there is an increasing sequence of ideals

$$0 = B_0 \subseteq B_1 \subseteq \ldots \subseteq B_k$$

with

$$K_i^G(B_j/B_{j-1}) \cong K_{i+j-1}^G(I_{k-j})$$

and a weak equivalence of B_k with a suspension of B. We have established the following theorem.

<u>THEOREM</u> <u>8.8.</u> Let G be a compact Lie group satisfying the Hodgkin condition and let B be a G-algebra. Then there is a G-algebra B_k which is equivariantly weakly equivalent to a suspension of B and an ascending sequence of G-invariant ideals

$$0 = B_0 \subseteq B_1 \subseteq \ldots \subseteq B_k$$

with $K_*^G(B_j/B_{j-1})$ R(G)-injective for each j, and such that the natural maps induce an injective resolution of $K_*^G(B)$ of the form

$$0 \longrightarrow K_*^G(B) \longrightarrow K_*^G(B_k/B_{k-1}) \longrightarrow K_*^G(B_{k-1}/B_{k-2}) \longrightarrow \ldots \longrightarrow K_*^G(B_1/B_0) \longrightarrow 0. \qquad \square$$

SECTION 9: CONSTRUCTION OF THE UNIVERSAL COEFFICIENT SPECTRAL SEQUENCE

This section is devoted to the construction of the Universal Coefficient spectral sequence. Recall that this is a spectral sequence which converges to $KK_*^G(A,B)$ and has

$$E_2^{p,*} = \mathrm{Ext}^p_{R(G)}(K_*^G(A), K_*^G(B)).$$

To be precise about the internal degree, elements of $\mathrm{Ext}^p_{R(G)}(K_r^G(A), K_s^G(B))$ are given internal degree r+s and total degree p+r+s. Generally, we suppress degrees for simplicity of the presentation. Note though that in the universal coefficient spectral sequence the differential d_r changes total degree by one (mod 2) and raises homological degree by r.

In Section 7 we studied the special situation which obtains when one assumes that $K_*^G(B)$ is R(G)-injective. Then the spectral sequence vanishes for homological degree p>0 and the universal coefficient spectral sequence reduces to the assertion that the map

$$\gamma(A,B)\colon\ KK_*^G(A,B) \longrightarrow \mathrm{Hom}_{R(G)}(K_*^G(A), K_*^G(B))$$

is an isomorphism. The following theorem makes the ascent to the general case.

<u>THEOREM</u> <u>9.1.</u> Suppose that G is a compact Lie group which satisfies the Hodgkin condition. Fix some G-algebra A. Suppose that the map $\gamma(A,B)$ is an isomorphism for all G-algebras B provided that $K_*^G(B)$ is R(G)-injective. Then the universal coefficient spectral sequence holds for (A,B) for all G-algebras B.

<u>PROOF:</u> Fix some G-algebra B and apply Theorem 8.8. Then there is some G-algebra B_r which is equivariantly weakly equivalent to (some suspension of) B and a sequence of G-invariant ideals

$$0 = B_0 \subset B_1 \subset \dots \subset B_r$$

with $K_*^G(B_j/B_{j-1})$ R(G)-injective for each j and such that the natural maps induce an injective resolution of $K_*^G(B)$ of the form

$$0 \longrightarrow K_*^G(B) \longrightarrow K_*^G(B_r/B_{r-1}) \longrightarrow K_*^G(B_{r-1}/B_{r-2}) \longrightarrow \dots \longrightarrow K_*^G(B_1/B_0) \longrightarrow 0. \quad (*)$$

We next apply the homology theory $KK_*^G(A,-)$ to this situation as in [Sc1] to obtain a spectral sequence which converges strongly to $KK_*^G(A,B)$ (adjusting for suspensions), and with

$$E_1^{p,*} \cong KK_*^G(A, B_p/B_{p-1})$$

$$\cong \mathrm{Hom}_{R(G)}(K_*^G(A), K_*^G(B_p/B_{p-1}))$$

since each $K_*^G(B_p/B_{p-1})$ is R(G)-injective. We may identify E_2 easily if we observe that (*) is an R(G)-injective resolution of $K_*^G(B)$. Then, essentially by definition,

$$E_2^{p,*} = \mathrm{Ext}^p_{R(G)}(K_*^G(A), K_*^G(B))$$

as desired. □

<u>THEOREM</u> <u>9.2.</u> (Universal Coefficient Spectral Sequence) Let G be a compact Lie group which satisfies the Hodgkin condition. For $A \in \tilde{\mathcal{B}}_G$ and B a G-algebra, there is a spectral sequence of R(G)-modules which strongly converges to $KK_*^G(A,B)$ with

$$E_2^{p,*} \cong \mathrm{Ext}^p_{R(G)}(K_*^G(A), K_*^G(B)).$$

The spectral sequence has the canonical grading, so that $\mathrm{Ext}^p_{R(G)}(K_s^G(A), K_t^G(B))$ has homological degree p and total degree p+s+t (mod 2). The edge homomorphism

$$KK_*^G(A,B) \longrightarrow E_2^{0,*} \cong \mathrm{Hom}_{R(G)}(K_*^G(A), K_*^G(B))$$

is the map γ. The spectral sequence is natural with respect to pairs (A,B) in the category. If G has rank r then $E_2^{p,q} = 0$ for $p>r+1$ and $E_{r+2} = E_\infty$.

PROOF: Theorem 7.7 tells us that $\gamma(A,B)$ is an isomorphism for $A \in \tilde{B}_G$ and $K_*^G(B)$ $R(G)$-injective, and Theorem 9.1 completes the argument. The identification of the edge homomorphism is immediate from the construction in the proof of (9.1). □

REMARK 9.3. Let G be a Hodgkin group with maximal torus T. There is a very close relationship between the spectral sequences

$$E_r^{p,*}(G) \Longrightarrow KK_*^G(A,B)$$

and

$$E_r^{p,*}(T) \Longrightarrow KK_*^T(A,B).$$

Theorem 3.7 (ii) says that

$$KK_*^T(A,B) \cong R(T)\otimes_{R(G)}KK_*^G(A,B).$$

An argument from homological algebra, similar to the argument at the end of Section 7 and using the fact that

$$\mathrm{Ext}^p_{R(T)}(M,\mathrm{Hom}_{R(G)}(R(T),N)) \cong \mathrm{Ext}^p_{R(G)}(M,N)$$

(which holds since $R(T)$ is a free $R(G)$-module), yields

$$\mathrm{Ext}^p_{R(T)}(K_*^T(A),K_*^T(B)) \cong R(T)\otimes_{R(G)}\mathrm{Ext}^p_{R(G)}(K_*^G(A),K_*^G(B)).$$

This suggests that the two spectral sequences should be related, that for each r,

$$E_r^{p,*}(T) \cong R(T)\otimes_{R(G)}E_r^{p,*}(G)$$

as differential graded $R(T)$-modules. In fact this is true and, moreover, it is possible to construct the spectral sequence $E(G)$ in this manner, by first arguing for the torus via Theorem 9.2, then arguing via faithful flatness. This is not an easier road of development, but it does make one point clearly: all differentials in the spectral sequence for $KK_*^G(A,B)$ arise in the spectral sequence for $KK_*^T(A,B)$.

REMARK 9.4. The spectral sequence is natural in various ways. In the A-variable, the spectral sequence is natural for each E_r; in the B-variable naturality commences with r=2. More generally, an element $\lambda \in KK^G(A,A')$ induces a morphism of spectral sequences $\lambda^*:E_r(A',B) \longrightarrow E_r(A,B)$ by the Kasparov pairings. Similarly, an element $\lambda \in KK^G(B,B')$ induces a morphism of spectral sequences $\lambda_*:E_r(A,B) \longrightarrow E_r(A,B')$.

REMARK 9.5. In general the Künneth and Universal Coefficient spectral sequences do not collapse- that is, the differentials d_r (r>1) are generally non-trivial. These differentials correspond to higher-order operations. It would be of some interest to identify these operations.

Note further that since $Ext^*_{R(G)}$ does not commute with localization in general (with respect to a collection P of prime ideals in R(G)), the naive analogue of Theorem 6.3 for the Universal Coefficient spectral sequence is actually false. However, one does have the following result, which in the special case G = $\mathbb{T}$, A = C(M) (M a closed G-manifold), and B = $\mathbb{C}$, appears as [IP, Part I, Theorem 5.6].

THEOREM 9.6. Let G, A, and B be as in Theorem 9.2 and assume in addition that $K^G_*(A)$ is finitely generated as an R(G)-module. Let P be a collection of prime ideals in R(G) and let $R = R(G)_P$. Then there is a spectral sequence which strongly converges to $KK^G_*(A,B)_P$ with

$$E_2^{p,*} = Ext^p_R(K^G_*(A)_P, K^G_*(B)_P).$$

In particular, if R is a principal ideal domain, there is a short exact sequence

$$0 \longrightarrow Ext^1_R(K^G_*(A)_P, K^G_*(B)_P) \longrightarrow KK^G_*(A,B)_P \longrightarrow Hom_R(K^G_*(A)_P, K^G_*(B)_P) \longrightarrow 0.$$

PROOF: Apply the localization functor to the spectral sequence of (9.2). Since R(G) is Noetherian and $K^G_*(A)$ is finitely generated,

$$\mathrm{Ext}^p_{R(G)}(K^G_*(A),K^G_*(B))_P \cong \mathrm{Ext}^p_R(K^G_*(A)_P,K^G_*(B)_P),$$

by [CE, Ch. VI, Exercise 11 and Ch. VII, Exercise 10]. □

SECTION 10: APPLICATIONS- KK^G EQUIVALENCE

In our previous work [RS2] we showed that if A and B are C^*-algebras in $\mathcal{C} = \mathcal{C}_{\{1\}}$ with $K_*(A) \cong K_*(B)$ then A is KK-equivalent to B. In this section we consider equivariant generalizations. Generally speaking, the map

$$\gamma = \gamma(A,B): KK_*^G(A,B) \longrightarrow \mathrm{Hom}_{R(G)}(K_*^G(A), K_*^G(B))$$

is not surjective, and so a particular isomorphism $f: K_*^G(A) \cong K_*^G(B)$ may not be in the image of γ. Even if $f = \gamma(x)$ for some $x \in KK_0^G(A,B)$ it is not clear that there must be a KK^G-inverse.

We study the problem by means of the Universal Coefficient spectral sequence, where γ appears as the edge homomorphism

$$KK_*^G(A,B) \longrightarrow E_2^{0,*} \cong \mathrm{Hom}_{R(G)}(K_*^G(A), K_*^G(B))$$

which we also denote by $\gamma(A,B)$. The second problem mentioned above is dispatched by the following theorem.

THEOREM 10.1. Let G be a compact Lie group satisfying the Hodgkin condition and let A and B be G-algebras in $\tilde{\mathcal{B}}_G$. Suppose that there is an isomorphism of graded R(G)-modules

$$f: K_*^G(A) \cong K_*^G(B).$$

and that there exists $x \in KK^G(A,B)$ with $\gamma(x) = f$. Then x is a KK^G-equivalence.

PROOF: We claim that x has both a left and a right inverse. For instance, to show that x has a left inverse, consider the map of spectral sequences induced by $(-)\otimes_A x$:

$$\begin{array}{ccc} \mathrm{Ext}^*_{R(G)}(K^G_*(B),K^G_*(A)) & \Longrightarrow & KK^G_*(B,A) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Ext}^*_{R(G)}(K^G_*(B),K^G_*(B)) & \Longrightarrow & KK^G_*(B,B) \end{array}$$

This commutes by functoriality of the product. Since the map of spectral sequences is an isomorphism at E_2 (since x induces the isomorphism f), we deduce that

$$(-)\otimes_A x : KK^G_0(B,A) \longrightarrow KK^G_0(B,B)$$

is an isomorphism. In particular, there exists an element $y \in KK^G_0(B,A)$ such that $y\otimes_A x = 1_B$. The argument on the other side is similar. □

Thus our attention is focused upon conditions which imply that the edge homomorphism is surjective. A simple hypothesis is that the spectral sequence itself collapse: $E_2 = E_\infty$. So we search for conditions which imply that the spectral sequence collapses. Here is a simple but common situation where this happens.

<u>Lemma</u> <u>10.2.</u> Suppose that G is a Hodgkin group and that A and B are G-algebras in $\tilde{\mathcal{B}}_G$ with

$$K^G_0(A) \cong K^G_0(B)$$

and

$$K^G_1(A) = K^G_1(B) = 0.$$

Then all even differentials d_{2r} vanish.

<u>PROOF:</u> This is an immediate consequence of the grading of the differentials. □

Sometimes one has control on the homological or injective dimension of the modules involved.

THEOREM 10.3. Suppose that G is a Hodgkin group, A and B are G-algebras in $\tilde{\mathcal{B}}_G$ with $K_*^G(A) \cong K_*^G(B) \cong M$, and suppose that M has homological or injective dimension ≤ 1. Then A and B are KK^G-equivalent (and the equivalence covers the given isomorphism).

PROOF: The dimension assumption implies that $E_2^{p,*} = 0$ for $p>1$, so $E_2 = E_\infty$ and the result follows from (10.1). □

Consider the special case where the group G is the circle. Since the rank of $\mathbb{T}$ is 1, we know that $E_3 = E_\infty$ for any (A,B) in the category. The only possible non-trivial differential is

$$d_2 : E_2^{0,q} \longrightarrow E_2^{2,q-1}.$$

If $K_*^{\mathbb{T}}(A) \cong K_*^{\mathbb{T}}(B)$ is concentrated in one degree then one of these two groups is zero, and hence $E_2 = E_\infty$. We have established the following proposition.

PROPOSITION 10.4. Suppose that A and B are $\mathbb{T}$-algebras in $\mathcal{C}_{\mathbb{T}}$ with $K_*^{\mathbb{T}}(A) \cong K_*^{\mathbb{T}}(B)$ concentrated in one degree. Then A and B are $KK_{\mathbb{T}}$-equivalent.

For example, any $\mathbb{T}$-algebra in the category with $K_*^{\mathbb{T}}(A) \cong K_*^{\mathbb{T}}(C(\mathbb{T}))$ is $KK_{\mathbb{T}}$-equivalent to $C(\mathbb{T})$. More generally, we have

PROPOSITION 10.5. Suppose that G is a Hodgkin group, $A,B \in \tilde{\mathcal{B}}_G$, $K_*^G(A) \cong K_*^G(B)$ is concentrated in one degree, and the homological or injective dimension of $K_0^G(A)$ is ≤ 2. (This is automatic if G has rank 1.) Then A is KK^G-equivalent to B.

PROOF: The dimension assumption implies that $E_3 = E_\infty$, so we need only consider d_2: it vanishes by (10.2). □

If two G-algebras A and B are KK^G-equivalent, then $K_*^G(A) \cong K_*^G(B)$ as R(G)-modules. In the non-equivariant setting the converse of this statement holds (by [RS2]) for algebras in an appropriate category. That is, if two C^*-algebras have isomorphic K-groups, then they are KK-equivalent. The following example shows that the general equivariant converse is false, so that the hypotheses of Propositions 10.3 and 10.5 are in fact necessary.

Example 10.6. We shall construct commutative $\mathbb{T}$-algebras A and B with the following properties:

$$K_*^{\mathbb{T}}(A) \cong K_*^{\mathbb{T}}(B) \text{ as } R(\mathbb{T})\text{-modules}$$

$$K_*(A) \neq K_*(B)$$

so that A and B can't be KK-equivalent (let alone $KK_{\mathbb{T}}$-equivalent).

We take $A = C(\mathrm{Spin}^c(4))$ with the free $\mathbb{T}$-action which naturally arises from the identification

$$\mathrm{Spin}^c(4) = \mathbb{T} \times_{\mathbb{Z}_2} \mathrm{Spin}(4).$$

There is a natural fibration

$$\mathbb{T} \longrightarrow \mathrm{Spin}^c(4) \longrightarrow SO(4)$$

which is the usual (non-split) extension of compact Lie groups. Then

$$K_*^{\mathbb{T}}(A) \cong K^*(SO(4)) \quad (\text{as a } \mathbb{Z}\text{-module})$$

$$\cong \Lambda_R(\xi,\eta)/(\Theta-1)\eta \quad \text{by [Ho, Proposition 12.4]}$$

where $R = \mathbb{Z}[\Theta]/(\Theta^2-1, 2(\Theta-1)) = \mathbb{Z}\oplus\mathbb{Z}_2(\Theta-1)$. Thus

$$K_0^{\mathbb{T}}(A) = \mathbb{Z}\oplus\mathbb{Z}_2(\Theta-1)\oplus\mathbb{Z}\xi\eta$$

and

$$K_1^{\mathbb{T}}(A) \cong \mathbb{Z}\xi\oplus\mathbb{Z}_2(\Theta-1)\xi\oplus\mathbb{Z}\eta$$

as abelian groups, and the $R(\mathbb{T})$-module action is determined by the fact that if we write $R(\mathbb{T}) \cong \mathbb{Z}[t,t^{-1}]$, then t acts by tensoring vector bundles on the base by the line bundle associated to the circle bundle above; in other words, t acts by multiplication by Θ. Since $\mathrm{Spin}^c(4)$ is a Hodgkin group, it follows by [Ho, Theorem 11] that $K_*(A)$ is an exterior algebra over $\mathbb{Z}$, and in particular it is free abelian.

The construction of B is more involved. Let $Y = S^1 \times (\mathbb{R}P^2 \vee S^2)$. Then

$$H^*(Y) = \mathbb{Z}[x_1, x_2, y_2]/(2x_2, x_2^2, x_2y_2, x_1^2, y_2^2),$$

where the generators (in degrees 1,2,2 respectively) correspond to the usual generators of the cohomology of S^1, $\mathbb{R}P^2$, and S^2 respectively. Let F be the total space of the principal $\mathbb{T}$-bundle with base space Y and characteristic class $x_2 \in H^2(Y;\mathbb{Z})$, and let $B = C(F)$. Then

$$K_*^{\mathbb{T}}(B) \cong K^*(S^1) \otimes K^*(\mathbb{R}P^2 \vee S^2)$$

$$\cong H^*(Y)$$

as abelian groups. The $R(\mathbb{T})$-module structure on $K_*^{\mathbb{T}}(B)$ is determined by the fact that t acts by tensoring vector bundles on Y by the line bundle associated to the circle bundle $F \longrightarrow Y$, so that $t-1$ acts by multiplication by x_2. To compute $K_*(B) \cong K^*(F)$, we note that $F = S^1 \times W$, where $p: W \longrightarrow (\mathbb{R}P^2 \vee S^1)$ is the circle bundle defined by $x_2 \in H^2(\mathbb{R}P^2;\mathbb{Z}) = \mathbb{Z}_2$. So by the Gysin sequence

$$0 = H^3(\mathbb{R}P^2 \vee S^2) \longrightarrow H^3(W) \xrightarrow{p_!} H^2(\mathbb{R}P^2 \vee S^2) \cong \mathbb{Z} \oplus \mathbb{Z}_2 \longrightarrow 0$$

we deduce that $H^3(W)$ contains 2-torsion. Since W is a finite 3-complex, $H^3(W)$ is a direct summand of $K^1(W)$, so $K^1(W)$ has 2-torsion. This implies that $K_1(B) \cong K^1(S^1 \times W)$ has 2-torsion.

Comparing the two algebras A and B, we see that they are not KK-equivalent, since $K_*(A)$ is torsion-free, whereas $K_*(B)$ has 2-torsion. However, $K_*^{\mathbb{T}}(A) \cong K_*^{\mathbb{T}}(B)$ as graded $R(\mathbb{T})$-modules: the isomorphism is obtained by mapping

$$1 \longrightarrow 1 \qquad (\Theta-1) \longrightarrow x_2 \qquad \xi\eta \longrightarrow y_2$$

$$\xi \longrightarrow x_1 \qquad (\Theta-1)\xi \longrightarrow x_1x_2 \qquad \eta \longrightarrow x_1y_2.$$

This completes the example.

REMARK 10.7. Recall that we showed in [RS2, Corollary 7.5] that if A is any C^*-algebra in $\mathcal{C}$ ($= \mathcal{C}_G$ for $G = \{1\}$), then A is KK-equivalent to a commutative algebra of the form $C^0 \oplus C^1$, where $K_i(C^j) = 0$ unless $i=j$. From this we deduced the splitting of the Künneth Theorem and the Universal Coefficient Theorem exact sequences. Accordingly, one might wonder if the corresponding statement is true in the equivariant case. Example 10.6 shows that this is not always true, for if it were true that any $A \in \mathcal{A}_G$ were KK^G-equivalent to $C^0 \oplus C^1$, where C^0 and C^1 had their K_*^G-groups concentrated in one degree, then it would follow from Proposition 10.4 that $K_*^{\mathbb{T}}(A)$ determines A up to $KK_{\mathbb{T}}$-equivalence, which (by 10.6) is not always the case. However, we can prove an interesting "splitting theorem" under the hypotheses of Theorem 10.3. We believe this to be new even for $A \in \mathcal{A}_G$, or for that matter, even for $A = C(X)$, where X is a finite G-complex.

THEOREM 10.8. Let G be a compact Lie group satisfying the Hodgkin condition, and let $A \in \tilde{\mathcal{B}}_G$ with $K_*^G(A)$ having homological or injective dimension ≤ 1 (as an R(G)-module). Then

a) A is KK^G-equivalent to a G-algebra in $\mathcal{B}_G$ of the form $C^0 \oplus C^1$, where $K_i^G(C^j) = 0$ unless $i=j$.

b) If B is any G-algebra with $K_1^G(B) = 0$, then there are <u>split</u> exact sequences of the form

$$0 \longrightarrow K_i^G(A) \otimes_{R(G)} K_0^G(B) \longrightarrow K_i^G(A \otimes B) \longrightarrow \operatorname{Tor}_1^{R(G)}(K_{i-1}^G(A), K_0^G(B)) \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{Ext}^1_{R(G)}(K_{i-1}^G(A), K_0^G(B)) \longrightarrow KK_G^i(A,B) \longrightarrow$$
$$\longrightarrow \operatorname{Hom}_{R(G)}(K_i^G(A), K_0^G(B)) \longrightarrow 0.$$

c) In particular, there are <u>split</u> exact sequences

$$0 \longrightarrow K_0^G(A)\otimes_{R(G)}\mathbb{Z} \longrightarrow K_0(A) \longrightarrow \mathrm{Tor}_1^{R(G)}(K_1^G(A),\mathbb{Z}) \longrightarrow 0$$

and

$$0 \to \mathrm{Ext}^1_{R(G)}(K_1^G(A),R(G)) \to K_G^0(A) \to \mathrm{Hom}_{R(G)}(K_0^G(A),R(G)) \to 0.$$

The splittings of these sequences are not natural.

<u>PROOF:</u> By Proposition 8.5 (applied twice), there exist G-algebras C^0, $C^1 \in \mathcal{B}_G$ with

$$K_0^G(C^0) = K_0^G(A), \qquad K_1^G(C^0) = 0,$$

$$K_1^G(C^1) = K_1^G(A), \qquad K_0^G(C^1) = 0.$$

Then by Proposition 10.3, A and $C = C^0 \oplus C^1$ are KK^G-equivalent. Hence for any G-algebra B,

$$K_0^G(A\otimes B) \cong K_0^G(C^0\otimes B)\oplus K_0^G(C^1\otimes B)$$

and

$$KK^G(A,B) \cong KK^G(C^0,B)\oplus KK^G(C^1,B).$$

If A has homological or injective dimension ≤ 1, then the various universal coefficient and Künneth spectral sequences for the pairs (A,B), (C^0,B), (C^1,B) all collapse to short exact sequences. Then, if $K_1^G(B) = 0$, the short exact sequences for (C^0,B) and for (C^1,B) degenerate to isomorphisms (because of the grading), and the result follows. Part c) arises from the special cases

$$K_0(A) \cong K_0^G(A\otimes C(G)), \quad B = C(G), \quad K_0^G(B) = \mathbb{Z}, \quad K_1^G(B) = 0$$

and

$$K_G^0(A) = KK^G(A,\mathbb{C}), \quad B = \mathbb{C}, \; K_0^G(B) = R(G), \quad K_1^G(B) = 0. \qquad \square$$

We conclude this section by considering the following examples.

Example 10.9. Suppose that G acts freely on a compact space X. Then I(G) (the augmentation ideal of R(G)) acts nilpotently on $K_G^*(X)$. If in addition G is of rank 1 ($\mathbb{T}$ or SU(2)) then we obtain a short exact sequence

$$0 \to \mathrm{Ext}^2_{R(G)}(K_G^*(X),R(G)) \to K^G_*(X) \to \mathrm{Ext}^1_{R(G)}(K^{*+1}(X),R(G)) \to 0$$

Does this sequence split? Not in general. For an example of non-splitting, set $G = \mathbb{T}$, $X = \mathrm{Spin}^c(4)$, so $X/G = SO(4)$. Write

$$R = \mathbb{Z}[t,t^{-1}]/(t^2-1,2(t-1)).$$

Then $K_G^0(X) \cong K_G^1(X) \cong R\oplus\mathbb{Z}$. A free R(G)-resolution of R is given by the sequence

$$0 \to R(G) \xrightarrow{(-2,t+1)} R(G)\oplus R(G) \xrightarrow{(t^2-1,2(t-1))} R(G) \to R \to 0.$$

Apply $\mathrm{Hom}_{R(G)}(-,R(G))$ to obtain the complex

$$0 \to R(G) \xrightarrow{(t^2-1,2(t-1))} R(G)\oplus R(G) \xrightarrow{(-2,t+1)} R(G),$$

and then take homology. One finds that $\mathrm{Ext}^0_{R(G)}(R,R(G)) = 0$,

$$\mathrm{Ext}^1_{R(G)}(R,R(G)) = \{((t+1)h,2h) : h \in \mathbb{Z}[t,t^{-1}]\}/\{(t^2-1)h,\ 2(t-1)h\}$$

$$\cong R(G)/(t-1) = \mathbb{Z}$$

and that

$$\mathrm{Ext}^2_{R(G)}(R,R(G)) = R(G)/(2,t+1) = \mathbb{Z}_2.$$

Recall also that

$$\mathrm{Ext}^j_{R(G)}(\mathbb{Z},R(G)) = \begin{cases} \mathbb{Z} & j = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus the short exact sequence above becomes

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow K_0^G(X) \longrightarrow \mathbb{Z}\oplus\mathbb{Z} \longrightarrow 0.$$

(Here $\mathbb{Z}_2$ and $\mathbb{Z}$ have t acting by the identity.) This sequence splits as abelian groups but <u>not</u> as R(G)-modules, since by Poincaré duality,

$$K_0^G(X) \cong K_G^1(X) \cong R\oplus\mathbb{Z},$$

on which t acts non-trivially.

We remark that this sequence <u>does</u> split in the case of the other $\mathbb{T}$-algebra in Example 10.6.

<u>Example</u> <u>10.10.</u> Suppose that $G = \mathbb{T}$ or SU(2) and $A \in \mathcal{C}_G$ satisfies $K_1^G(A) = 0$. Then A is determined up to KK^G-equivalence by $K_0^G(A)$. In particular, there is a short exact sequence

$$0 \longrightarrow \mathrm{Ext}^2_{R(G)}(K_0^G(A),R(G)) \longrightarrow K_G^0(A) \xrightarrow{\gamma} \mathrm{Hom}_{R(G)}(K_0^G(A),R(G)) \longrightarrow 0.$$

We do not know whether this sequence always splits.

SECTION 11: APPLICATIONS- MOD P K-THEORY

In this section we discuss some applications of our results to the "equivariant algebraic topology" of C^*-algebras. The homology theory

$$K_*^G(-;\mathbb{Z}_n) \cong K_*^G(-\otimes N)$$

is introduced and is shown to be independent of $N \in \tilde{\mathcal{B}}_G$ (for G in the Hodgkin class, of course). The basic properties of equivariant K-theory mod n are developed. The ring of R(G)-linear operations is determined; it turns out to be isomorphic to $KK_*^G(Cn,Cn)$ (where Cn is the mapping cone of the canonical degree n map of the circle), which is isomorphic to the exterior algebra over R(G) on the Bockstein element. Finally, we classify admissible multiplications; they are in bijection with $R(G)\otimes\mathbb{Z}_n$ and (if n is odd) exactly one of these is graded commutative.

Let us fix an integer n (in almost all applications this will be a prime p) and let Cn be the mapping cone of the canonical self-map of $C_o(\mathbb{R})$ of degree n, with trivial G-structure. For any group G and for any G-algebra D, one defines

$$K_j^G(D;\mathbb{Z}_n) \equiv K_j^G(D\otimes Cn).$$

In fact, the definition of "mod n K-theory" has little to do with the algebra Cn, for we have

<u>THEOREM</u> <u>11.1.</u> Suppose that G is a compact Lie group satisfying the Hodgkin condition. If N is <u>any</u> G-algebra in $\tilde{\mathcal{B}}_G$ with

$$K_0^G(N) = R(G)\otimes_{\mathbb{Z}}\mathbb{Z}_n \quad \text{and} \quad K_1^G(N) = 0,$$

then N is KK^G-equivalent to Cn, and hence there is a natural equivalence of G-homology theories

$$K_*^G(-;\mathbb{Z}_n) \cong K_*^G(-\otimes N).$$

PROOF: The homological dimension of $R(G)\otimes_{\mathbb{Z}}\mathbb{Z}_n$ is one, since there is a projective resolution of it of the form

$$0 \longrightarrow R(G) \xrightarrow{n} R(G) \longrightarrow R(G)\otimes_{\mathbb{Z}}\mathbb{Z}_n \longrightarrow 0$$

and then Proposition (10.5) implies the result. □

In particular, we see that the Cuntz algebra $\mathcal{O}_{n+1}$ is KK^G-equivalent to Cn.

The elementary properties of $K_*^G(\ ;\mathbb{Z}_n)$ are easy to develop. The mapping cone sequence of the degree n map yields a long exact sequence of the form

$$\longrightarrow K_j^G(A) \xrightarrow{n} K_j^G(A) \xrightarrow{\rho_n} K_j^G(A;\mathbb{Z}_n) \xrightarrow{\beta_n} K_{j-1}^G(A) \longrightarrow \tag{11.2}$$

where "n" is multiplication by n, ρ_n is the reduction map, and β_n is the Bockstein map. We let $\beta = \rho_n\beta_n$; then β is a self-homology operation for $K_*^G(-;\mathbb{Z}_n)$ and $\beta^2 = 0$.

It is clear that $K_*^G(-;\mathbb{Z}_n)$ is homotopy invariant, additive, and satisfies an exactness axiom in the equivariant category. A formal argument yields the short exact sequence

$$0 \longrightarrow K_j^G(A)\otimes_{\mathbb{Z}}\mathbb{Z}_n \longrightarrow K_j^G(A;\mathbb{Z}_n) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(K_{j-1}^G(A),\mathbb{Z}_n) \longrightarrow 0 \tag{11.3}$$

courtesy of the identification

$$\mathrm{Tor}_p^{\mathbb{Z}}(K_j^G(A),\mathbb{Z}_n) \cong \mathrm{Tor}_p^{R(G)}(K_j^G(A),R(G)\otimes_{\mathbb{Z}}\mathbb{Z}_n). \tag{11.4}$$

Note that we can prove splitting of the exact sequence (11.3) for certain G-algebras A using Theorem 10.8.

Next we require the analog of the Cuntz representability theorem.

PROPOSITION 11.5. Let G be a compact group, let D be a G-algebra, and suppose given $z \in K_0^G(D)$ with $nz = 0$. Then there exists a G-map

$$f: \mathcal{O}_{n+1} \longrightarrow (D\otimes\mathcal{K})^+\otimes\mathcal{O}_\infty$$

such that $f_*[1] = z$. (The group G acts trivially on $\mathcal{O}_{n+1}$ and $\mathcal{O}_\infty$.)

PROOF: Without loss of generality we may assume that D is stable. Then $D\rtimes G$ is G-stably isomorphic to the fixed point algebra D^G. So it suffices to construct a map $\mathcal{O}_{n+1} \longrightarrow (D^+)^G\otimes\mathcal{O}_\infty$, and this is done in [Sc4, Section 6] or [Cu, Section 6.6]. The map is a G-map, since G acts trivially! □

We move to consideration of homology operations on the theory $K_*^G(\ ;\mathbb{Z}_n)$. We restrict attention to R(G)-linear operations. (If one simply requires linearity then there are more operations, but these operations do not pay sufficient attention to the G-structure.) Any element of $KK_*^G(Cn,Cn)$ gives a homology operation by Kasparov product, so the first order of business is to compute this ring.

PROPOSITION 11.6. Let G be a compact Lie group satisfying the Hodgkin condition. Then the $\mathbb{Z}_2$-graded ring $KK_*^G(Cn,Cn)$ is a free $R(G)\otimes_{\mathbb{Z}}\mathbb{Z}_n$-module of rank 2 with generators 1_{Cn} of degree 0 and β_{Cn}, the Bockstein element, of degree 1, with multiplication determined by the relation

$$\beta_{Cn}^2 = 0.$$

PROOF: The universal coefficient spectral sequence in this situation reduces to the usual short exact sequence, and the additive results follow immediately. The element β_{Cn} is the generator of $KK_1^G(Cn,Cn)$ which corresponds to the extension of R(G)-modules

$$0 \longrightarrow R(G)\otimes_{\mathbb{Z}}\mathbb{Z}_n \longrightarrow R(G)\otimes_{\mathbb{Z}}\mathbb{Z}_{n^2} \longrightarrow R(G)\otimes_{\mathbb{Z}}\mathbb{Z}_n \longrightarrow 0$$

However, β_{Cn} induces the zero-map $K_*^G(Cn)\longrightarrow K_{*+1}^G(Cn)$, so β_{Cn}^2 induces the zero map $K_*^G(Cn) \longrightarrow K_*^G(Cn)$. This says that the edge homomorphism $\gamma(Cn,Cn)$ annihilates β_{Cn}. The map $\gamma(Cn,Cn)$ is an isomorphism in this degree, so $\beta_{Cn}^2 = 0$.□

We wish to determine all of the R(G)-linear homology operations on $K_*^G(\ ;\mathbb{Z}_n)$. As in the non-equivariant case [RS2, Section 8], any homology operation commutes with the Bott periodicity map, so we may take our theory to be $\mathbb{Z}_2$-graded.

<u>THEOREM</u> <u>11.7.</u> Let G be a compact Lie group satisfying the Hodgkin condition. Then the $\mathbb{Z}_2$-graded ring of R(G)-linear (self-) homology operations for $K_*^G(\ ;\mathbb{Z}_n)$ (on the category of G-algebras) is a free $R(G)\otimes_{\mathbb{Z}}\mathbb{Z}_n$-module of rank 2 with generators the identity map (of degree 0) and the Bockstein operation β_n (of degree 1). As a $\mathbb{Z}_2$-graded ring over $R(G)\otimes_{\mathbb{Z}}\mathbb{Z}_n$, it is the exterior algebra over $R(G)\otimes_{\mathbb{Z}}\mathbb{Z}_n$ on β_n.

<u>PROOF:</u> In light of Proposition 11.6, we must show that there are no other homology operations other than those which come from $KK_*^G(Cn,Cn)$.

Suppose that θ is a homology operation and that D is a G-algebra. It suffices to show that the action of θ on $K_*^G(D;\mathbb{Z}_n)$ is determined by the action of θ on $K_*^G(\mathcal{O}_{n+1};\mathbb{Z}_n)$, since $\mathcal{O}_{n+1}$ is KK^G-equivalent to Cn. Write the generators of $K_*^G(\mathcal{O}_{n+1};\mathbb{Z}_n)$as g in degree 0 (the reduction of the integral class $1_{\mathcal{O}_{n+1}}$), and h in degree 1, where $\beta(h) = g$. Thus if the operation θ on $K_*^G(\mathcal{O}_{n+1};\mathbb{Z}_n)$ is degree preserving, it must be given by multiplication by some $r \in R(G)$ in degree 0 and by some $s \in R(G)$ in degree 1. It will follow (from the representation argument to follow) that on arbitrary G-algebras, θ is given by $s + (r-s)e$, where e is a projection onto the kernel of β. However, one knows that the Künneth exact sequence has no *natural* splitting, i.e., there is no natural choice for e. Hence r=s and θ is multiplication by some element of R(G). A similar argument applies to show that θ is a multiple of β if it is degree-reversing.

It remains to show that θ is determined by its action on $K_*^G(\mathcal{O}_{n+1};\mathbb{Z}_n)$. Since θ is compatible with suspensions, it is enough to consider an arbitrary G-algebra D and an element $x \in$

$K_0^G(D;\mathbf{Z}_n)$ and to compute $\theta(x)$ in terms of θ for $\mathcal{O}_{n+1}$. If $\beta(x) = 0$ then x is the reduction mod n of a class $u \in K_0^G(D)$, by (11.2). Then u corresponds to a fixed projection in $D^+ \otimes \mathcal{K}$, hence, viewing u as $f_*\langle 1\rangle$, where $f: \mathbb{C} \longrightarrow D^+ \otimes \mathcal{K}$ and $\langle 1\rangle$ is the standard generator of $K_0^G(\mathbb{C})$, we have

$$x = (f\otimes \mathrm{id})_*(\langle 1\rangle \otimes g),$$

where $f\otimes \mathrm{id}: \mathbb{C}\otimes\mathcal{O}_{n+1} \longrightarrow D^+\otimes\mathcal{K}\otimes\mathcal{O}_{n+1}$. (Here we're identifying $K_*^G(D;\mathbf{Z}_n)$ with a summand in $K_*^G(D^+\otimes\mathcal{K};\mathbf{Z}_n)$ as usual.) Thus, by naturality of θ,

$$\theta(x) = (f\otimes \mathrm{id})_*\theta(\widehat{\langle 1\rangle}),$$

where $\widehat{\langle 1\rangle}$ is now the standard generator of $K_*^G(\mathbb{C};\mathbf{Z}_n)$. Since the unital inclusion of $\mathbb{C}$ in $\mathcal{O}_{n+1}$ induces an isomorphism on $K_0^G(\ ;\mathbf{Z}_n)$, we see that $\theta(x)$ is determined by the restriction of θ to $K_0^G(\mathcal{O}_{n+1};\mathbf{Z}_n)$.

Now even if $\beta(x) \neq 0$, $\beta\beta(x) = 0$, and so $\beta(x)$ is the reduction of some class $w \in K_1^G(D)$ with $nw = 0$. By (the suspension of) Proposition 11.5, there is an equivariant map

$$\phi: \mathcal{O}_{n+1} \longrightarrow (SD\otimes\mathcal{K})^+\otimes\mathcal{O}_\infty$$

with $\phi_*(h) = w$. Recall also that in $K_*^G(\mathcal{O}_{n+1};\mathbf{Z}_n)$, $\beta(h) = g$. Then

$$\beta(x) = \phi_*(g) = \phi_*(\beta(h))$$

and so $x-\phi_*(h)$ is the reduction of an integral class, hence was already dealt with above. Thus

$$\theta(x) = \phi_*(\theta(h)) + \theta(x-\phi_*(h))$$

is determined by θ restricted to $K_*^G(\mathcal{O}_{n+1};\mathbf{Z}_n)$. We have shown in the process that $\mathcal{O}_{n+1}$ is a sort of universal object for $K_*^G(\ ;\mathbf{Z}_n)$, just as it is in the non-equivariant situation. □

We proceed next to the consideration of admissible multiplications on the theory $K_*^G(-;\mathbf{Z}_n)$. The non-equivariant definition generalizes in an obvious way. Thus, an <u>admissible multiplication</u> is an R(G)-bilinear natural transformation

$$\mu : K_i^G(A;\mathbb{Z}_n) \times K_j^G(B;\mathbb{Z}_n) \longrightarrow K_{i+j}^G(A\otimes B;\mathbb{Z}_n)$$

which satisfies the following obvious conditions:

1) μ is associative.

2) μ commutes (in the graded sense) with suspension in each variable.

3) μ should be the obvious multiplication when one or the other of the classes to be multiplied is the reduction of an integral class; that is, in the image of the map

$$K_*^G(D) \longrightarrow K_*^G(D)\otimes\mathbb{Z}_n \longrightarrow K_*^G(D;\mathbb{Z}_n).$$

4) The Bockstein map β is a graded derivation.

In the non-equivariant setting, there are exactly n admissible multiplications, each of which arises from a KK-element. If n is odd then exactly one multiplication is commutative; if n is even then no multiplication is commutative. The situation in the equivariant setting is quite parallel, so much so that we only sketch the proof of the following theorem, referring the reader to [RS2, Theorem 8.9] for detail.

<u>THEOREM</u> <u>11.8.</u> Suppose that G is a compact group which satisfies the Hodgkin condition, and that n>1 is some integer. The admissible multiplications on $K_*^G(-;\mathbb{Z}_n)$ are in one-to-one correspondence with those elements of $KK_*^G(Cn,Cn)$ whose image under γ in the group

$$\mathrm{Hom}_{R(G)}(K_*^G(Cn)\otimes_{R(G)\otimes\mathbb{Z}_n}K_*^G(Cn),K_*^G(Cn)) \cong R(G)\otimes_{\mathbb{Z}}\mathbb{Z}_n$$

is exactly the usual multiplication map $1_n = 1_{R(G)\otimes\mathbb{Z}_n}$. Such multiplications are in bijection with $R(G)\otimes\mathbb{Z}_n$, corresponding to the set $\gamma^{-1}(1_n)$. When n is odd, exactly one of these is commutative. When n = 2, none are commutative. The multiplication which corresponds to a KK^G element λ is given by the composition (involving Kasparov products)

$$K_i^G(D\otimes Cn)\otimes_{R(G)}K_j^G(E\otimes Cn) \xrightarrow{(-)\otimes_{\mathbb{C}}(-)} K_{i+j}^G(D\otimes Cn\otimes E\otimes Cn) \xrightarrow{\sigma_{2,3}}$$

$$\longrightarrow K_{i+j}^G(D\otimes E\otimes Cn\otimes Cn) \xrightarrow{(-)\otimes_{Cn\otimes Cn}\lambda} K_{i+j}^G(D\otimes E\otimes Cn)$$

where $\sigma_{2,3}$ is induced by the "flip" automorphism.

<u>PROOF:</u> The proof is quite similar to to proof of the analogous non-equivariant proposition [RS2, Theorem 8.9], so we give only a brief outline. By naturality and associativity of the Kasparov product, any element $\lambda \in KK^G(Cn\otimes Cn,Cn)$ will give rise to a natural R(G)-bilinear associative multiplication μ_λ of the correct bidegree, by the above construction. It is also clear that if μ_λ is to be the usual multiplication in the case D = E = $\mathbb{C}$, it must satisfy $\gamma(\mu_\lambda) = 1_n$. We must determine $\gamma^{-1}(1_n)$.

The Künneth spectral sequence degenerates for the pair (Cn,Cn) to yield

$$K_j^G(Cn\otimes Cn) \cong R(G)\otimes\mathbb{Z}_n \equiv M \qquad j = 0,1$$

and then the Universal Coefficient spectral sequence degenerates to the short exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_{R(G)}(M,M) \longrightarrow KK^G(Cn\otimes Cn,Cn) \longrightarrow \mathrm{Hom}_{R(G)}(M\otimes M,M) \longrightarrow 0$$

which is the extension

$$0 \longrightarrow M \longrightarrow KK^G(Cn\otimes Cn,Cn) \xrightarrow{\gamma} M \longrightarrow 0.$$

Thus $\gamma^{-1}(1_n) \cong M$, as claimed. (The fact that no single element of $\gamma^{-1}(1_n)$ is immediately preferred comes from the lack of a natural splitting of the UCT sequence.)

A direct check (parallel to the non-equivariant argument [RS2, Theorem 8.9]) implies that each element λ of $\gamma^{-1}(1_n)$ gives rise to an admissible multiplication μ_λ and that the map $\lambda \longrightarrow \mu_\lambda$ is one-to-one. Furthermore, the representability Proposition 11.5 implies that each admissible multiplication arises in this manner.

It remains to discuss commutativity. An admissible multiplication μ_λ is (graded) commutative if and only if λ is invariant under the automorphism σ of $KK^G(Cn \otimes Cn, Cn)$ which is induced by the flip interchanging the two factors in $Cn \otimes Cn$. The map σ is trivial on $K_0^G(Cn \otimes Cn)$ and acts as -1 on $K_1^G(Cn \otimes Cn)$. Consider the universal coefficient sequence

$$0 \longrightarrow \mathrm{Ext}^1_{R(G)}(K_1^G(Cn \otimes Cn), K_0^G(Cn)) \longrightarrow KK^G(Cn \otimes Cn, Cn) \longrightarrow$$

$$\mathrm{Hom}_{R(G)}(K_0^G(Cn \otimes Cn), K_0^G(Cn)) \longrightarrow 0.$$

One must distinguish now between the cases n even and odd. If n is odd then σ has two distinct eigenvalues on $KK^G(Cn \otimes Cn, Cn)$ and since 2 is a unit in M, we have a direct sum splitting

$$KK^G(Cn \otimes Cn, Cn) \cong \mathrm{Ext}^1_{R(G)}(K_1^G(Cn \otimes Cn), K_0^G(Cn)) \oplus KK^G(Cn \otimes Cn, Cn)^\sigma.$$

The admissible multiplications all have the same component in the fixed-point set, so μ_λ is commutative if and only if λ has projection 0 in $\mathrm{Ext}^1_{R(G)}(K_1^G(Cn \otimes Cn), K_0^G(Cn))$, which happens for exactly one λ.

If n=2 then, just as in the non-equivariant case, σ acts by a unipotent matrix and leaves no admissible multiplication fixed. □

REFERENCES

Ad1 J.F. Adams, Lectures on generalised cohomology, in Category Theory, Homology Theory and their Applications III, Lecture Notes in Math., vol. 99, Springer, 1969, pp. 1-138.

Ad2 J.F. Adams, Stable Homotopy and Generalised Homology (Part III), U. of Chicago Press, Chicago, 1979, pp. 123-373.

At M.F. Atiyah, Bott periodicity and the index of elliptic operators, Quart. J. Math. (Oxford) 19 (1968), 113-140.

BK L. Baggett and A. Kleppner, Multiplier representations of abelian groups, J. Funct. Anal. 14 (1973), 299-324.

Ba H. Bass, Big projective modules are free, Illinois J. Math 7(1963), 24-31.

Bo M. Bökstedt, Universal coefficient theorems for equivariant K- and KO-theory, Preprint, Aarhus University (1981).

Bra O. Bratteli, Fixedpoint algebras versus crossed products, Proc. Symp. Pure Math. 38 (1982), Part 1, 357-359.

Br L.G. Brown, Operator algebras and algebraic K-theory, Bull. Amer. Math. Soc. 81 (1975), 1119-1121.

CE H. Cartan and S. Eilenberg, Homological Algebra, Princeton U. Press, Princeton, 1956.

Co A. Connes, An analogue of the Thom isomorphism for crossed products of a C^*-algebra by an action of $\mathbb{R}$, Adv. Math. 39 (1981), 31-55.

CS A. Connes and G. Skandalis, Théorème de l'indice pour les feuilletages, C.R. Acad. Sci. Paris, Ser. I 292 (1981), 871-876.

Di J. Dixmier, Les C^*-algèbres et leur représentations, Gauthier-Villars, Paris, 1969.

FS T. Fack and G. Skandalis, Connes' analogue of the Thom isomorphism for the Kasparov groups, Invent. Math. 64 (1981), 7-14.

Gl J. Glimm, Locally compact transformation groups, Trans. Amer. Math. Soc. 101 (1961), 124-138.

Gr1 P. Green, The structure of imprimativity algebras, J. Funct. Anal. 36 (1979), 88-104.

Gr2 P. Green, Equivariant K-theory and crossed product C^*-algebras, in Operator Algebras and Applications, ed. by R. Kadison, Proc. Symp. Pure Math., vol. 38, Amer. Math. Soc., 1982, Part I, pp. 337-338.

Gr3 A. Grothendieck, Le groups de Brauer, I: Algèbres d'Azumaya et interprétations diverses, exposé no. 290, Séminaire Bourbaki, 1964/65, Benjamin, New York.

Ho L. Hodgkin, The equivariant Künneth Theorem in K-theory, in Topics in K-Theory, Lecture Notes in Math., vol. 496, Springer, 1975, pp. 1-101.

IP W. Iberkleid and T. Petrie, Smooth S^1 Manifolds, Lecture Notes in Math., vol. 557, Springer, 1976.

Ju1 P. Julg, K-théorie equivariante et produits croisés, C.R. Acad. Sci. Paris, Sér A 292, 629-632.

Ju2 P. Julg, Induction holomorphe pour le produit croisé d'une C^*-algèbre par un groupe de Lie compact, C.R. Acad. Sci. Paris, Sér. I 294 (1982), 193-196.

Ka1 G.G. Kasparov, Hilbert C^*-modules: Theorems of Stinespring and Voiculescu, J. Operator Theory 4 (1980), 133-150.

Ka2 G.G. Kasparov, The operator K-functor and extensions of C^*-algebras, Izv. Akad. Nauk SSSR 44 (1980), 571-636 = Math. USSR Izv. 16 (1981), 513-572.

Ka3 G.G. Kasparov, K-theory, group C^*-algebras, and higher signatures (conspectus), preprint, Chernogolovka, 1981.

Mad I. Madsen and M. Rothenberg, Periodic maps of spheres of odd order, Chapter I: equivariant cobordism and K-theory, Preprint, Aarhus University, 1984.

Ma E. Matlis, Injective modules over Noetherian rings, Pacific J. Math. 8 (1958), 511-528.

Mat H. Matsumura, Commutative Algebra, 2nd Ed., Benjamin/Cummings, 1980.

May J.P. May, Equivariant homotopy and cohomology theory, in Symposium on Algebraic Topology in Honor of José Adem, Contemp. Math., Vol. 12, Amer. Math. Soc., Providence, 1982, pp. 209-217.

Mc J. McLeod, The Künneth formula in equivariant K-theory, in Algebraic Topology, Waterloo, Ontario, 1978, Lecture Notes in Math., vol. 741, Springer, 1979, pp. 316-333.

MZ D. Montgomery and L. Zippin, Topological Transformation Groups, Wiley-Interscience, 1955.

Mo C.C. Moore, Group extensions and cohomology for locally compact groups, Trans. Amer. Math. Soc. 221 (1976), 1-33.

MR C.C. Moore and J. Rosenberg, Groups with T_1 primitive ideal spaces, J. Funct. Anal. 20 (1976), 204-224.

Pa V. Paulsen, A covariant version of Ext, Michigan Math. J. 29 (1982), 131-142.

Pe T. Petrie, Smooth S^1 actions on homotopy complex projective spaces and related topics, Bull. Amer. Math. Soc. 78 (1972), 105-153.

Ph N.C. Phillips, K-theoretic Freeness of Actions of Finite Groups on C^*-algebras, Lecture Notes in Math., Springer-Verlag, New York, to appear.

PV M. Pimsner and D. Voiculescu, K-groups of reduced crossed products by free groups, J. Operator Theory 8 (1982), 95-130.

Pi H. Pittie, Homogeneous vector bundles on homogeneous spaces, Topology 11 (1972), 199-204.

Pu L. Pukanszky, Unitary representations of solvable Lie groups, Ann. Sci. Ecole Norm. Sup. 4 (1971), 457-608.

RR I. Raeburn and J. Rosenberg, Crossed products of continuous-trace C^*-algebras by smooth actions, MSRI preprint, 1985.

Ro1 J. Rosenberg, Appendix to O. Bratteli's paper on "Crossed products of UHF algebras", Duke Math. J. 46 (1979), 25-26.

Ro2 J. Rosenberg, The role of K-theory in non-commutative algebraic topology, in Operator Algebras and K-theory, ed. by R.G. Douglas and C. Schochet, Contemp. Math. vol. 10, Amer. Math. Soc., 1982, pp. 155-182.

Ro3 J. Rosenberg, Some results on cohomology with Borel cochains, with applications to group actions on operator algebras, in Proc. 9th Internat. Conf. on Operator Theory, Timosoara and Herculane, G. Arsene and D. Voiculescu, eds., to appear.

RS1 J. Rosenberg and C. Schochet, The classification of extensions of C^*-algebras, Bull. Amer. Math. Soc. (N.S.) 4 (1981), 105-109.

RS2 J. Rosenberg and C. Schochet, The Künneth Theorem and the Universal Coefficient Theorem for Kasparov's generalized K-functor, MSRI preprint, 1984.

Sc1 C. Schochet, Topological methods for C^*-algebras I: spectral sequences, Pacific J. Math. 96 (1981), 193-211.

Sc2 C. Schochet, Topological methods for C^*-algebras II: geometric resolutions and the Künneth formula, Pacific J. Math. 98 (1982), 443-458.

Sc3 C. Schochet, Topological methods for C^*-algebras III: axiomatic homology, Pacific J. Math. 114 (1984), 399-445.

Sc4 C. Schochet, Topological methods for C^*-algebras IV: mod p homology, Pacific J. Math. 114 (1984), 447-468.

Se1 G. Segal, The representation ring of a compact Lie group, Publ. Math. Inst. Hautes Etudes Sci. 34 (1968), 113-128.

Se2 G. Segal, Equivariant K-theory, Publ. Math. Inst. Hautes Etudes Sci. 34 (1968), 129-151.

Sn V.P. Snaith, On the Künneth formula spectral sequence in equivariant K-theory, Proc. Camb. Phil. Soc. 72 (1972), 167-177.

St R. Steinberg, On a theorem of Pittie, Topology 14 (1975), 173-177.

Wa A Wassermann, Equivariant K-Theory II: the Hodgkin spectral sequence in Kasparov theory, in preparation.

Wi D. Wigner, Algebraic cohomology of topological groups, Trans. Amer. Math. Soc. 178 (1973), 83-93.

Jonathan Rosenberg
Mathematics Department, University of Maryland
College Park, MD 20742

Claude Schochet
Mathematics Department, Wayne State University
Detroit, MI 48202

both authors, 1984-5
Mathematical Sciences Research Institute
1000 Centennial Drive
Berkeley, CA 94720

This page intentionally left blank

General instructions to authors for
PREPARING REPRODUCTION COPY FOR MEMOIRS

For more detailed instructions send for AMS booklet, "A Guide for Authors of Memoirs." Write to Editorial Offices, American Mathematical Society, P. O. Box 6248, Providence, R. I. 02940.

MEMOIRS are printed by photo-offset from camera copy fully prepared by the author. This means that, except for a reduction in size of 20 to 30%, the finished book will look exactly like the copy submitted. Thus the author will want to use a good quality typewriter with a new, medium-inked black ribbon, and submit clean copy on the appropriate model paper.

Model Paper, provided at no cost by the AMS, is paper marked with blue lines that confine the copy to the appropriate size. Author should specify, when ordering, whether typewriter to be used has PICA-size (10 characters to the inch) or ELITE-size type (12 characters to the inch).

Line Spacing – For best appearance, and economy, a typewriter equipped with a half-space ratchet – 12 notches to the inch – should be used. (This may be purchased and attached at small cost.) Three notches make the desired spacing, which is equivalent to 1-1/2 ordinary single spaces. Where copy has a great many subscripts and superscripts, however, double spacing should be used.

Special Characters may be filled in carefully freehand, using dense black ink, or INSTANT ("rub-on") LETTERING may be used. AMS has a sheet of several hundred most-used symbols and letters which may be purchased for $5.

Diagrams may be drawn in black ink either directly on the model sheet, or on a separate sheet and pasted with rubber cement into spaces left for them in the text. Ballpoint pen is *not* acceptable.

Page Headings (Running Heads) should be centered, in CAPITAL LETTERS (preferably), at the top of the page – just above the blue line and touching it.

LEFT-hand, EVEN-numbered pages should be headed with the AUTHOR'S NAME;

RIGHT-hand, ODD-numbered pages should be headed with the TITLE of the paper (in shortened form if necessary).

Exceptions: PAGE 1 and any other page that carries a display title require NO RUNNING HEADS.

Page Numbers should be at the top of the page, on the same line with the running heads.

LEFT-hand, EVEN numbers – flush with left margin;

RIGHT-hand, ODD numbers – flush with right margin.

Exceptions: PAGE 1 and any other page that carries a display title should have page number, centered below the text, on blue line provided.

FRONT MATTER PAGES should be numbered with Roman numerals (lower case), positioned below text in same manner as described above.

MEMOIRS FORMAT

It is suggested that the material be arranged in pages as indicated below.
Note: Starred items (*) are requirements of publication.

Front Matter (first pages in book, preceding main body of text).

Page i – *Title, *Author's name.

Page iii – Table of contents.

Page iv – *Abstract (at least 1 sentence and at most 300 words).

**1980 Mathematics Subject Classification (1985 Revision).* This classification represents the primary and secondary subjects of the paper, and the scheme can be found in Annual Subject Indexes of MATHEMATICAL REVIEWS beginning in 1984.

Key words and phrases, if desired. (A list which covers the content of the paper adequately enough to be useful for an information retrieval system.)

Page v, etc. – Preface, introduction, or any other matter not belonging in body of text.

Page 1 – Chapter Title (dropped 1 inch from top line, and centered).

Beginning of Text.

Footnotes: *Received by the editor date.
Support information – grants, credits, etc.

Last Page (at bottom) – Author's affiliation.

ABCDEFGHIJ–89876