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THE ROLE OF K-THEORY IN NON-COMMUTATIVE ALGEBRAIC TOPOLOGY

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ABSTRACT. Since the study of commutative C*-algebras is equivalent to the study of locally compact Hausdorff spaces, one can think of much of algebraic topology as being the study of homotopy-invariant functors from commutative C*-algebras to abelian groups or modules over some ring, and application of these to concrete topological problems. Similarly, one can conceive of "non-commutative algebraic topology" (this term was coined by E. Effros) as the study of homotopy-invariant functors from general C*-algebras to abelian groups or modules over a ring, and applications. Among such functors, K-theory seems to play a distinguished role. Some of the ways K-theory naturally arises in the study of C*-algebras will be discussed, and in particular, the relationship between K-theory and "stable homotopy" will be explored. Curiously, the importance of K-theory in ordinary (commutative) algebraic topology is due in part to its role in the non-commutative setting.

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1. THE CONCEPT OF NON-COMMUTATIVE ALGEBRAIC TOPOLOGY.

Algebraic topology may be defined to be the study of topological problems not directly, but rather by means of reduction to algebraic problems. The reduction is ordinarily accomplished by means of functors from various categories of topological spaces to categories of algebraic objects, such as abelian groups or modules over some ring. There are many such functors, and success often depends on making a choice well-suited to the problem at hand. In other cases, it may be clear how to reduce the topological problem to an algebraic one, and success may depend on developing enough machinery (e.g., exact sequences, spectral sequences, and the like) for manipulating the algebraic objects that arise.

The standard functors of algebraic topology - homotopy, homology, and cohomology - all have one basic feature: they are homotopy-invariant, that is, insensitive to continuous deformation. Thus although they are very useful in helping to classify spaces up to homeomorphism, they really measure something much weaker. Accordingly, they may be viewed as functors defined on homotopy categories of spaces, where the morphisms are homotopy classes of continuous maps.

What, then, is "non-commutative algebraic topology"? I owe this term to Ed Effros, who introduced it [21] a few years ago to describe the application to (not necessarily commutative) C*-algebras of the methods of algebraic topology. It is not hard to see the reason for the name. The category of "pointed compact spaces", familiar to topologists, is the category of compact (Hausdorff) spaces with basepoint, where morphisms are continuous maps mapping basepoint to basepoint. This category is equivalent to that of locally compact spaces and proper maps, since to any pointed compact space $(X,+)$ we may associate the locally compact space $X \setminus \{+\}$. (This may be empty, although X by assumption must contain at least one point.)

Conversely, to any locally compact space Y , we associate the pointed compact space $(Y^+,+)$, where Y^+ is the one-point compactification $Y \cup \{+\}$. But this category is by Gelfand duality $(Y \mapsto C_0(Y))$ equivalent to the opposite of the category of commutative C*-algebras, in which the morphisms are the *-homomorphisms. Once we have a suitable notion of homotopy for C*-algebras (see below), this means that the homotopy category of pointed compact spaces may be contravariantly identified with the homotopy category of commutative C*-algebras. The usual functors of algebraic topology, at least if one takes them to be defined on compact spaces (rather than on, say, simplicial complexes or CW-complexes), may therefore be viewed as homotopy functors from commutative C*-algebras to abelian groups. Non-commutative algebraic topology would involve extending the domain of these functors to the larger homotopy category of all (not necessarily commutative) C*-algebras (see Figure 1).

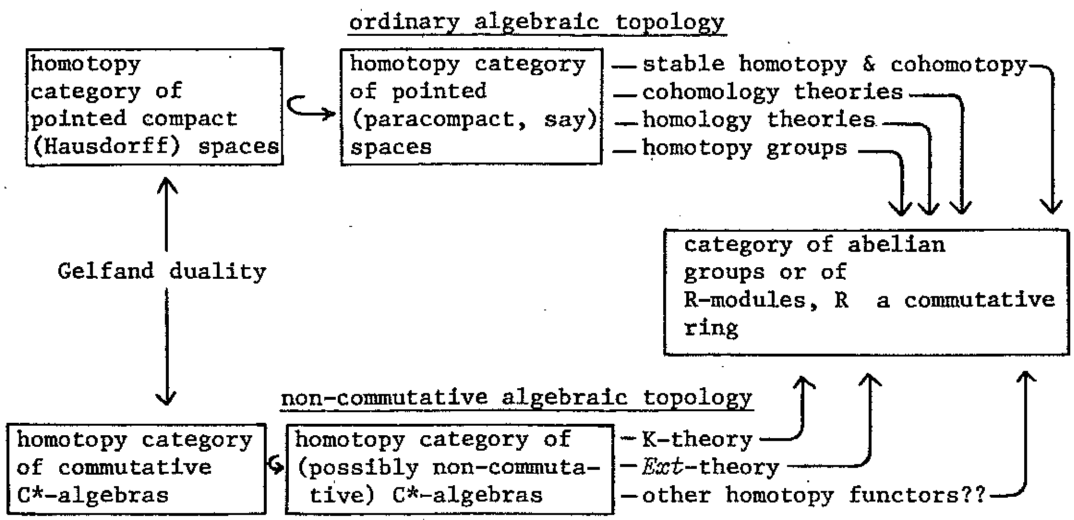

Figure 1. Ordinary vs. non-commutative algebraic topology

Several points must be raised here. For one thing, even if one regards the goal of algebraic topology to be the understanding of the nicest compact spaces, say compact smooth manifolds, one is inevitably led to the study of other infinite-dimensional spaces not in the original category (e.g., loop spaces, Eilenberg-MacLane spaces, classifying spaces of various sorts, etc.) . This makes it important to have well-behaved extensions of the original functors to a somewhat larger category. As an example, Čech cohomology, which is perhaps the most "natural" of all cohomology theories, extends nicely to paracompact spaces. The most serious problem is developing the machinery of non-commutative topology is that there is no obvious choice of a good homotopy category containing both the non-commutative C*-algebras and the analogues of loop spaces and infinite CW complexes. The first thing one would want in such a category would be a left adjoint Ω for the suspension functor Σ (as defined below). (We have reversed left and right from the usual situation of [32, p.185] because of the contravariant correspondence between C*-algebras and spaces.) Perhaps the σ-C*-algebras of Arveson would provide the appropriate category. In any event, we shall not discuss this problem here, and shall instead work only with the homotopy category of C*-algebras. (For certain purposes, we shall also deal with certain full subcategories, such as the category of separable C*-algebras.)

A second point is that it is by no means clear which of the homotopy functors defined on the category of commutative C*-algebras (e.g., cohomology theories on pointed compact spaces) can be extended in a reasonable way to the category of all C*-algebras. Existence of such an extension certainly imposes certain constraints on a homotopy functor which may force it to be of a particularly nice form. Incidentally, it is quite possible that the presence of a large number of "non-commutative spaces" in our category may compensate to some extent for the exclusion of analogues of infinite complexes, so that homotopy functors extendible to the category of non-commutative C*-algebras may come from "non-commutative classifying spaces" even simpler than the usual classifying spaces associated by the Brown Representability Theorem. This will be discussed more fully below.

Finally, we should ask what the goals of non-commutative algebraic topology are. Ultimately, these seem to be of two sorts. On the one hand, one expects non-commutative algebraic topology to be an increasingly important tool in ordinary (commutative) algebraic topology. The clearest evidence for this seems to be the application of the K-homology and *Ext*-functors of Kasparov [28,§§1-7] and Brown-Douglas-Fillmore ([10],[20],[4]) to provide a good setting for the Atiyah-Singer Index Theorem, the use by Connes of non-commutative C*-algebras for the computation of invariants of foliated manifolds [14], and work by Kasparov [28,§8] and Miščenko([33],[34]) relating the non-commutative topology of group C*-algebras to "higher signatures" in differential topology. On the other hand, non-commutative algebraic topology should be an increasingly important tool in the study of C*-algebras for their own sake (see for instance [19]) and in non-commutative harmonic analysis (e.g., [40]). Just as in ordinary topology we cannot understand spaces up to homeomorphism until we at least know something about coarser invariants such as homology groups, so we cannot expect to understand the classification of C*-algebras without some knowledge of their coarsest algebraic invariants.

We conclude this section by summarizing the basic definitions and constructions that will be needed below. These concepts have been collected from various sources and it seems hard to establish priority for any single author; however, key references are [27], [31], and [7]. Much of what we say could be made to work for (non C*-) Banach algebras, but for simplicity we restrict attention to C*-algebras. Thus $\otimes$ generally denotes the C*-algebra tensor product (when it's unique; otherwise one should distinguish between $\otimes_{min}$ and $\otimes_{max}$). Note that for any C*-algebra A, the algebra $C([0,1],A)$ of continuous functions from the unit interval into A may be identified with $C([0,1]) \otimes A$. Also note that we generally do <u>not</u> require C*-algebras to have a unit.

Suppose A and B are C*-algebras and $\alpha,\beta\colon A \to B$ are *-homomorphisms. We say α and β are homotopic if there exists a *-homomorphism $h\colon A \to C([0,1],B)$ (called a homotopy) such that $h(0) = \alpha$ and $h(1) = \beta$. Here, for $0 \leq t \leq 1$, $h(t)$ denotes composition of h with the map "evaluation at t": $C([0,1],B) \to B$. Homotopy is an equivalence relation on *-homomorphisms. A *-homomorphism α is called null-homotopic if it is homotopic to the zero-map $A \to 0 \hookrightarrow B$. A C*-algebra A is called contractible if the identity map $id_A\colon A \to A$ is null-homotopic. Given any C*-algebra A, the cone on A is defined to be the algebra

$$CA = C_0((0,1],A) = \{f \in C([0,1],A)\colon\ f(0) = 0\}.$$

Note that CA is always contractible.

Given a *-homomorphism $\phi\colon A \to B$, we define the associated mapping cone

$$C_\phi = \{(f,a)\colon\ f \in CB,\ a \in A,\ f(1) = \phi(a)\}$$

and mapping cylinder

$$Z_\phi = \{(f,a)\colon\ f \in C([0,1],B),\ a \in A,\ f(1) = \phi(a)\}.$$

Note that C_ϕ is an ideal in Z_ϕ with $Z_\phi/C_\phi \cong B$, and that the map $\psi\colon Z_\phi \to A$ defined by $(f,a) \mapsto a$ is a homotopy equivalence. (In other words, if $\chi(a) =$ (constant function with value $\phi(a),a$), then $\chi\circ\psi$ and $\psi\circ\chi$ are homotopic to the identity maps on Z_ϕ and on A, respectively.) Finally, we define the suspension functor Σ by

$$\Sigma A = \{f \in CA\colon\ f(1) = 0\} \cong C_0(\mathbb{R}) \otimes A$$
$$\Sigma\phi = id_{C_0(\mathbb{R})} \otimes \phi\colon\ C_0(\mathbb{R})\otimes A \to C_0(\mathbb{R})\otimes B.$$

All of the above definitions correspond under Gelfand duality to the usual topological definitions in the event that A and B are commutative. Note also that if $\phi\colon A \to B$ is a surjective *-homomorphism with kernel J, then we have a short exact sequence

$$0 \to J \overset{\alpha}{\to} C_\phi \overset{\beta}{\to} CB \to 0,$$

where $\beta(f,a) = f$ and $\alpha(x) = (0,x)$. In other words, C_ϕ is an extension of a contractible algebra by J, and so may be regarded as "weakly homotopy equivalent" to J (in the sense that α induces isomorphisms $\alpha_*\colon H(J) \to H(C_\phi)$ for all homotopy functors H giving rise to "long exact homology sequences"; examples of such functors will be discussed later). We also note that there is a canonical short exact sequence of C*-algebras

$$0 \to SB \overset{\iota}{\to} C_\phi \overset{\psi}{\to} A \to 0,$$

where $\psi(f,a) = a$ and $\iota(f) = (f,0)$.

2. THE UBIQUITY OF K-THEORY IN NON-COMMUTATIVE ALGEBRAIC TOPOLOGY.

Among the homotopy-invariant functors from C*-algebras to abelian groups, there are two that seem to play an especially distinguished role: K_0 and K_1. Good references for the construction and basic properties of these functors are [46] and [26; n.b. Exercise II.6.14], although the notation in both of these references differs from that which we will be using here. For a C*-algebra A with unit, $K_0(A)$ is the usual K_0-group of algebraic K-theory, the Grothendieck group of the category of finitely generated projective A-modules. $K_1(A)$ is the "Karoubi K_1-group", which may be defined as $\varinjlim GL(n,A)/GL(n,A)_0$, where $GL(n,A)$ is the topological group of invertible $n\times n$ matrices with entries in A, and $GL(n,A)_0$ is its identity component. Note that the Karoubi K_1-group is usually a proper quotient of the "Bass K_1-group" of algebraic K-theory (see [26, Exercise II.6.13] for a comparison). For a C*-algebra without unit, $K_j(A)$ is defined to be $\ker(K_j(A^+) \to K_j(\mathbb{C}))$, where A^+ denotes the algebra obtained by adjoining an identity to A. One may also define $K_j(A)$ to be $K_0(\Sigma^j A)$, $j \geq 0$, then define $K_j(A)$ for negative j using the Bott periodicity theorem. If Y is a locally compact space, then $K_j(C_0(Y)) = K^{-j}(Y)$ is topological (complex) K-theory of Y with compact supports, sometimes denoted $KU^{-j}(Y)$.

The family $K_* = \{K_j\}_{j\in\mathbb{Z}}$ of functors from C*-algebras to abelian groups has the following seven key properties:

(2.1) 1) homotopy invariance: if $\alpha,\beta\colon A \to B$ are homotopic, then $K_j(\alpha) = K_j(\beta)$ for all j. This implies the essentially equivalent property that $K_j(A) = 0$ for all j if A is contractible.

2) compatibility with suspension: $K_j\circ\Sigma^k \cong K_{j+k}$, where $\cong$ denotes natural equivalence of functors, for any integers j,k with $k \geq 0$.

3) exactness: if

$$0 \to J \to A \to B \to 0$$

is a short exact sequence of C*-algebras, K_* induces an associated (doubly infinite) long exact sequence of abelian groups

$$\cdots \to K_{j+1}(B) \xrightarrow{\partial} K_j(J) \to K_j(A) \to K_j(B) \xrightarrow{\partial} K_{j-1}(J) \to \cdots .$$

Furthermore, the boundary maps ∂ are natural.

4) stability: if $\mathcal{K}$ denotes the C*-algebra of compact operators on a separable infinite-dimensional Hilbert space, there are natural isomorphisms $K_j(A) \cong K_j(A\otimes\mathcal{K})$ for all j and for any C*-algebra A. Because of the following property (5), this is equivalent to having natural isomorphisms $K_j(A) \cong K_j(A\otimes M_n)$ for all j and n. (Here M_n denotes $n\times n$ matrices over $\mathbb{C}$.) Note that $\mathcal{K} = \varinjlim M_n$ (C*-algebra inductive limit) and that $\mathcal{K}\otimes M_n \cong \mathcal{K}$ for any n.

5) continuity under limits: the functors K_j commute with C*-algebra inductive limits.

6) Bott periodicity: there are natural equivalences, compatible with the boundary maps ∂ of (3): $K_j \cong K_{j+2}$.

7) normalization: $K_0(\mathbb{C}) \cong \mathbb{Z}$ and $K_1(\mathbb{C}) = 0$. The isomorphism $K_0(\mathbb{C}) \to \mathbb{Z}$ takes the class of a vector space to its dimension.

We remark in passing that although we shall only consider complex C*-algebras and complex K-theory in this article, there is also an appropriate K-theory for real C*-algebras, giving rise on locally compact spaces to real K-theory KO with compact supports. The basic properties of real K-theory are essentially the same except for (6) and (7) - one obtains periodicity of period 8 in place of (6), and (7) is replaced by the formulas $K_j(\mathbb{R}) \cong \mathbb{Z}$ for $j = 0,4$; $\mathbb{Z}/2$ for $j = 1,2$; 0 for $j = 3,5,6,7$.

For purposes of calculation, one often needs only properties (1)-(7) above, and not the original definition of the functors K_j . This makes it natural to ask for an axiomatic treatment. In particular, is there another system of functors satisfying (1)-(7), or do the above properties characterize K-theory uniquely? Although we shall not completely answer this question, we will show in Section 4 that (1)-(7) characterize K-theory if we restrict our category somewhat and interpret the periodicity axiom (6) suitably. With weaker hypotheses ($K_j(\mathbb{C}) \cong \mathbb{Z}$ for j even, 0 for j odd, but no _a priori_ assumption of functorial periodicity), the situation is still unclear. However, we will explain what is known about the problem and suggest how further progress could be made.

Note that of all our axioms, the only one involving non-commutativity in an essential way is (4), stability. If this is dropped, and if one restricts attention to the category of commutative C*-algebras, identified contravariantly with the category of pointed compact spaces, then the axioms say that K^* is a periodic and continuous generalized cohomology theory on compact spaces, with coefficient groups $K^j(pt) \cong \mathbb{Z}$ for j even, 0 for j odd. Because of the continuity axiom, K^* is determined by its restriction to the subcategory of finite CW complexes (cf. [45, p.319]), on which it is given by a CW spectrum E with $\pi_j(E) \cong \mathbb{Z}$ for j even, 0 for j odd. (See [2,§§1,2,6] and [3,Ch.1].) Periodicity says that we also have a homotopy equivalence $E \to E$ of degree 2 . This is "almost", but in fact _not_, enough to specify K^* uniquely, for one could construct another CW spectrum E not of the same homotopy type as the BU spectrum K but still having the same homotopy groups. For instance, one could construct the spectrum whose associated cohomology theory is $E^n(X) = \prod_{k\in\mathbb{Z}} H^{n+2k}(X,\mathbb{Z})$; this spectrum satisfies periodicity but is only rationally homotopy-equivalent to K . Of course, if one had a natural transformation between a given theory and usual K-theory which induced isomorphisms of coefficient groups, then one would actually have a map of spectra $E \to K$ inducing isomorphisms of homotopy groups, hence the associated cohomology theories would coincide, by J.H.C. Whitehead's Theorem [2, Corollary 3.5 and §§6,7].

The stability axiom (4), however, seems to introduce a radically new element. In fact, if one is interested in "homology theories" on the category of separable C*-algebras, (4) is an extremely natural condition, because of the fact [11] that separable C*-algebras A and B are stably isomorphic (i.e., $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$) if and only if they are "strongly Morita equivalent". In turn, strong Morita equivalence (which means identification of the *-representation theories of two C*-algebras by means of an "imprimitivity bimodule") is a very natural notion of equivalence for C*-algebras, frequently useful in the study of C*-algebras arising from harmonic analysis and topological dynamics (see [37],[38] and references quoted there).

It seems therefore that a reasonable attack on the uniqueness problem for homology theories on C*-algebras satisfying the stability axiom (4) should be as follows: (a) first one should study homotopy and stable homotopy in the category of stable ($A \cong A \otimes \mathcal{K}$) C*-algebras, and determine if some analogue of the Brown Representability Theorem ([5],[6],[1]) holds; (b) then one should study the relationship between stable homotopy in this category, which ought to give rise to the "primordial" homology theory satisfying (3) and (4), and K-theory. We have not yet been able to complete this somewhat ambitious program, but several of the necessary steps will be carried out in Sections 3 and 4 of this paper. In particular, we will derive in Section 4 a weak form of a "non-commutative Eilenberg-Steenrod theorem". Without Bott periodicity, our results are at the moment indecisive.

We should mention the one positive result known on axiomatic characterization of K-theory: Karoubi and Villamayor showed in [27] that axioms (1)-(4) above already determine the functors K_n uniquely (and thus force axioms (5) and (6)) provided K_0 is the usual K_0-functor of algebraic K-theory. However, this result in a sense begs the question of whether one could construct an essentially different homology theory on C*-algebras also respecting strong Morita equivalence (i.e., satisfying (4)). The Karoubi-Villamayor theorem does suggest that in some cases, Bott periodicity (axiom (6)) should be automatic, and should not have to be built in from the beginning.

In the remainder of this section, we leave aside for the moment the axiomatic characterization of K-theory, and instead review some of the ways in which K-theory has arisen recently in the study of C*-algebras. As the reader will see, K-theory is involved in problems of many different sorts, in such a way that one is led to suspect that the ubiquity of K-theory is not an accident but rather is due to some fundamental role played by K-theory in non-commutative topology (analogous, perhaps, to the role of singular homology and cohomology in ordinary topology). Anyway, here is a (rather incomplete) list of some applications of K-theory:

(2.2) 1. K-theory has proved very useful in distinguishing non-isomorphic C*-algebras. For instance, K_0 (together with its natural order structure) turns out to be a complete invariant for the classification of the so-called "AF algebras" (inductive limits of finite-dimensional C*-algebras) up to stable isomorphism. This result is essentially due to G. Elliott [24] - see also [25],[23,§3], and [22]. Similarly, K-theory may be used to distinguish (from the AF algebras and among themselves) a related class of simple C*-algebras generated by weighted shifts, introduced by Bunce and Deddens [12]. Or it may be used (even though it is not a complete isomorphism invariant) to help classify the so-called "Cuntz algebras" generated by isometries or associated to Markov chains ([16],[17],[18],[19]). Finally, it has been used to show that the reduced C*-algebras of free groups of different ranks are non-isomorphic [36], a fact which has resisted attempts at a direct proof.

2. Because of the way in which K_0 and K_1 are defined, K-theory is useful in proving facts about projections or unitaries in C*-algebras. For instance, L. Brown has used the K-theory exact sequence to show that if J and B are AF algebras and E is an extension of B by J, i.e., there exists a short exact sequence of C*-algebras

$$0 \to J \to E \to B \to 0 ,$$

then any projection in B lifts to a projection in E ([8],[22]). It follows in fact that E is itself an AF algebra. Connes [13,§V] and Pimsner-Voiculescu [36] have used K-theory calculations to show that certain simple C*-algebras contain no non-trivial projections.

3. K-theory of C*-algebras has provided a setting for several sorts of index theorems for elliptic operators, generalizing the Atiyah-Singer Theorem to various new situations. Important examples are the work of Miščenko-Fomenko on the index of an elliptic operator acting on sections of a bundle of modules over a C*-algebra [35], the work of Connes on the index of an elliptic operator tangential to the leaves of a foliation [14], and the work of Connes-Moscovici on an index theorem for non-compact homogeneous spaces [15].

4. K-theory seems to relate in an interesting way harmonic analysis on a discrete group Γ (as reflected in the K-groups of $C^*(\Gamma)$) and the topology of the classifying space $B\Gamma$. This is an idea of Kasparov [28,§8] and Miščenko ([33],[34]); for a recent survey, see [40].

5. K-theory is closely related to the classification of extensions of C*-algebras, and in particular to the measurement of "stable obstructions to splitting" of extensions. This has been pointed out primarily by the work of Brown-Douglas-Fillmore ([10],[20],[7]) and Kasparov ([29],[30]). We briefly summarize here the construction of Kasparov - for further details, see the original papers or the survey article [39]. Suppose A and B are separable C*-algebras with A nuclear, and one is interested in classifying C*-algebras E which are extensions of A by $B \otimes \mathcal{K}$, i.e., which satisfy a short exact sequence

$$0 \to B\otimes\mathcal{K} \to E \xrightarrow{\pi} A \to 0 \quad . \tag{2.3}$$

This is a problem of some practical interest, since extensions of this sort often arise when one considers C*-algebras of Lie groups [39,§7] or C*-algebras generated by pseudo-differential operators on a manifold ([20,Ch. 6],[4]) or on a foliated manifold [14]. It turns out that "unitary equivalence classes" of extensions of the form (2.3) can be made into a commutative semigroup (this depends on the stability of the ideal $B\otimes\mathcal{K}$), and that the split extensions (those for which there exists a *-homomorphism $\phi\colon A \to E$ with $\pi\circ\phi = \mathrm{id}_A$) form a subsemigroup. The quotient semigroup $\mathcal{E}xt(A,B)$ is then actually a group (this is a deep fact), which when $B = \mathbb{C}$ was denoted $\mathcal{E}xt(A)$ by Brown, Douglas, and Fillmore. It turns out the Kasparov $\mathcal{E}xt$ functor is essentially a mixture of K-theory and the dual contravariant theory. More precisely, there is a natural equivalence $\mathcal{E}xt(\mathbb{C},B) \cong K_1(B)$ ([29, Theorem 5], [41, Theorem 3.1] and [18, 3.2-3.3]), and (with some technical restrictions on A) there is a "universal coefficient theorem" exact sequence ([9], [41 , Theorem 4.2])

$$\begin{aligned} 0 \to \mathrm{Ext}^1_{\mathbb{Z}}(K_0(A),K_0(B)) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(K_1(A),K_1(B)) &\to \mathcal{E}xt(A,B) \\ &\xrightarrow{\gamma} \mathrm{Hom}(K_0(A),K_1(B)) \oplus \mathrm{Hom}(K_1(A),K_0(B)) \to 0 \ . \end{aligned} \tag{2.4}$$

Here the map γ takes the class of the extension (2.3) to the two connecting maps ∂ of the associated long exact K-theory sequence. In essence this says that K-theory measures all the stable obstructions to splitting of C*-algebra extensions of the form (2.3). These obstructions are in turn often related to index invariants.

6. Of course we should remember that non-commutative algebraic topology as defined here includes the algebraic topology of compact spaces as a special case. Thus the applications of K-theory which we have mentioned are all additions to its usual uses in topology, as discussed, say, in [26, Ch.V] and in [2,§11]. These include the Atiyah-Singer Theorem, integrality theorems for characteristic classes, various results on vector fields on spheres and stable homotopy of spheres, etc. (For some of these one should really use

real K-theory.)

We hope the above list gives the reader some idea of the many ways K-theory has arisen in the study of C*-algebras. Our objective in the rest of the paper will be to study enough of the groundwork of non-commutative algebraic topology to indicate why K-theory should play such a distinguished role.

3. FOUNDATIONS OF NON-COMMUTATIVE HOMOTOPY THEORY.

Since ordinary algebraic topology rests on certain principles of homotopy theory, it seems appropriate to begin a study of the non-commutative case with the analogous theorems. We do not claim any great originality for what follows (many of the results are either in [31] or in the folklore), but it seems useful to collect all the basic facts in one place.

We use the definitions of §1 above. In addition, given C*-algebras A and B, we define $[A,B]$ to be the <u>set of homotopy classes</u> of (not necessarily unital) *-homomorphisms $A \to B$. This is a pointed set; the base-point is the class of the zero-map. If A and B are unital, we may also define $[A,B]_+$ to be the set of homotopy classes (via unital homotopies) of unital *-homomorphisms $A \to B$. This set may be empty. Note that $[A,B] \cong [A^+,B^+]_+$. Since (via Gelfand duality) the C*-algebra $C_0(\mathbb{R}^n)$ corresponds to the n-sphere S^n (as a pointed compact space), we are also led to consider the analogues of stable homotopy and homotopy groups. We let

$$\{A,B\} = \varinjlim [\Sigma^k A, \Sigma^k B] \quad ,$$

where $[\Sigma^k A,\Sigma^k B]$ maps to $[\Sigma^{k+1}A,\Sigma^{k+1}B]$ via the suspension functor Σ, and also define (for $q \in \mathbb{Z}$)

$$\{A,B\}_q = \varinjlim [\Sigma^{k+q}A,\Sigma^k B] \ ,$$

$$\pi_q^s(A) = \{\mathbb{C},A\}_q$$

$$\pi_n(A) = [C_0(\mathbb{R}^n),A] \ (n \geq 0) \ .$$

We think of $\pi_n(A)$ and $\pi_q^s(A)$ as <u>homotopy</u> (resp. <u>stable homotopy</u>) <u>groups</u> of A, although for the moment we have no group structure available. Of course, if $A = C_0(X)$ and $B = C_0(Y)$, then

$$[A,B] \cong [C(X^+),C(Y^+)]_+ \cong [Y^+,X^+]_+ \ ,$$

where $[\ ,\ ]_+$ for pointed spaces denotes based homotopy classes,

$$\pi_n(A) \cong [C(S^n),C(X^+)]_+ \cong \pi^n(X^+) \ ,$$

and

$$\pi_q^s(A) \cong \pi_s^q(X^+) \quad .$$

Note that whether or not A is commutative, there is no reason why $\pi_q^s(A)$ should vanish for $q < 0$. (In fact we will see in §4 that often $\pi_q^s(A) \neq 0$ for infinitely many negative values of q .)

The reader is cautioned that in the present context, "stable" has two very different meanings, both of which will come into play in what follows. On the one hand, one may mean suspension stability: thus $\{A,B\}$ is the (suspension-) stable version of $[A,B]$. On the other hand, one may mean matrix stability: a C*-algebra A is (matrix-) stable if $A \cong A\otimes K$, and thus $A \cong A\otimes M_n$ for all n . Both kinds of stability give rise to extra structure on sets of homotopy classes, and fortunately, these structures are mutually compatible:

THEOREM 3.1. Let A, B, and C be C*-algebras. Then

a) if $B \cong B\otimes K$, then $[A,B]$ is a commutative monoid with monoid operation defined by letting $\alpha_1 + \alpha_2$ be the composite

$$A \xrightarrow{(\alpha_1,\alpha_2)} B\oplus B \hookrightarrow B\otimes M_2 \cong B .$$

The null-homotopic maps act as an identity element.

b) if $B \cong C \otimes C_0((0,1))$, then $[A,B]$ is a (not necessarily abelian) group under "loop composition": given $\alpha_1,\alpha_2\colon A \to B$, we define $\alpha_1\cdot\alpha_2\colon\ A \to C_0((0,1),B)$ by

$$(\alpha_1\cdot\alpha_2)(t) = \begin{cases}\alpha_1(2t) & \text{if } 0 \le t \le 1/2\\ \alpha_2(2t-1) & \text{if } 1/2 \le t \le 1\end{cases} .$$

c) if B is a stable suspension, i.e., both (a) and (b) hold, or if B is a double suspension, the two addition operations coincide and $[A,B]$ is an abelian group.

d) if B and C are both suspensions or both stable, then composition $[B,C] \times [A,B] \to [A,C]$ is bilinear, i.e., is a group or semigroup homomorphism in one variable if the other variable is fixed.

Proof. This is proved in essentially the same way as the corresponding theorem in ordinary topology [45, Ch. 1, §§5-6]. For the benefit of the reader, we include some details.

a) The only non-trivial aspects are proving that the zero-map acts as an identity element and that addition is commutative. For the first of these, note that since $B \cong B\otimes K$, the multiplier algebra $M(B)$ contains a copy of $B(H)$ (all bounded operators on the Hilbert space H on which K acts), and so contains a strictly continuous path $\{v_t\}$ of isometries with

$$v_0 = 1 , \quad v_1v_1{}^* = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \text{ (when we identify } M(B) \text{ with } M(B) \otimes M_2) .$$

Then given $\alpha: A \to B$, $a \mapsto v_t\alpha(a)v_t^*$ is a homotopy from α to $\alpha + 0$. For commutativity, note that conjugation by

$$\begin{pmatrix} \cos(\pi t/2) & \sin(\pi t/2) \\ -\sin(\pi t/2) & \cos(\pi t/2) \end{pmatrix} \in M(B) \ , \ 0 \leq t \leq 1$$

defines a homotopy from $\alpha + \beta$ to $\beta + \alpha$.

b) This is quite similar to [45,pp.37-39]. To show that 0 acts as an identity element, note that given $\alpha: A \to B$,

$$g: \ A \to C([0,1]) \otimes C_0((0,1)) \otimes C$$

given by

$$g(a)(s,t) = \begin{cases} 0 & \text{if } 0 \leq 2t \leq 1-s \\ \alpha(a)\ \frac{2t+s-1}{1+s} & \text{if } 1-s \leq 2t \leq 2 \end{cases}$$

defines a homotopy from $0\cdot\alpha$ to α as s goes from 0 to 1. The inverse of α is given by $\tilde{\alpha}$, where $\tilde{\alpha}(t) = \alpha(1-t)$. To define a homotopy g from $\tilde{\alpha}\cdot\alpha$ to 0, set

$$g(a)(s,t) = \begin{cases} \tilde{\alpha}(a)(2t) & \text{if } 0 \leq 2t \leq 1-s \\ \alpha(a)(1-s) & \text{if } 1-s \leq 2t \leq 1+s \\ \alpha(a)(2-2t) & \text{if } 1+s \leq 2t \leq 2 \end{cases},$$

where s ranges from 0 to 1.

c) Follows as usual from [45,Ch.1, §6, Theorem 8]. For instance, suppose B is a stable suspension. To show the addition operation of (b) is commutative and coincides with that of (a), note that we have a string of homotopies

$$\alpha\cdot\beta \sim (\alpha+0)\cdot(0+\beta) = (\alpha\cdot 0+0\cdot\beta) \sim \alpha+\beta \sim \beta+\alpha \sim \beta\cdot\alpha \ .$$

If B is a double suspension, argue as in [45,Ch.1,§6, Corollary 10].

d) To check bilinearity of composition, suppose $\alpha_1,\alpha_2: A \to B$ and $\beta_1,\beta_2: B \to C$ and given. If B and C are suspensions, we clearly have $(\beta_1\cdot\beta_2)\circ\alpha_1 = (\beta_1\circ\alpha_1)\cdot(\beta_2\circ\alpha_1)$ and $\beta_1\circ(\alpha_1\cdot\alpha_2) = (\beta_1\circ\alpha_1)\cdot(\beta_1\circ\alpha_2)$. Similar distributivity holds if B and C are stable. □

COROLLARY 3.2. For any C*-algebras A and B and any $q \in \mathbb{Z}$, $\{A,B\}_q$ is an abelian group. In particular, $\pi_q^s(A)$ is an abelian group. Furthermore, for $n \geq 0$, $\pi_n(A)$ is a commutative monoid if A is stable, is a group if A is a suspension, and is an abelian group if A is a double suspension or a stable suspension. □

For purposes of working in the homotopy category of C*-algebras, it is also useful to know how homotopy classes behave under products, sums, and limits. In what follows, $\oplus$ denotes the direct sum of C*-algebras (also called the c_0-direct sum or restricted product, and defined categorically by

$\bigoplus_{i\in I} A_i = \varinjlim_{} \bigoplus_{i\in F} A_i$, where the limit is taken over all finite subsets F of I, directed by inclusion) and $*$ denotes the free product (or coproduct), defined by the obvious universal property. (For example, $\mathbb{C}*\mathbb{C}$ is the free C*-algebra on two (non-commuting) projections.)

PROPOSITION 3.3. If $\{A_i\}_{i\in I}$ and B are C*-algebras and $A = \ast_{i\in I} A_i$, then $[A,B] = \prod_{i\in I} [A_i,B]$.

Proof. a *-homomorphism $A \to B$ is equivalent to a family of *-homomorphisms $A_i \to B$. □

PROPOSITION 3.4. If A_i, $i = 1,2,\ldots$, and B are C*-algebras with B stable (i.e., $B \cong B\otimes K$), and $A = \bigoplus_{i=1}^{\infty} A_i$, then $[A,B] \cong \prod_{i=1}^{\infty} [A_i,B]$. Thus in the homotopy category of stable C*-algebras, countable C*-direct sums are categorical coproducts (cf. [2, Propositions 3.11 and 3.14]).

Proof. Without any condition on B, there is always an obvious map $\theta\colon [A,B] \to \prod_{i=1}^{\infty} [A_i,B]$, but this is rarely surjective. However, if B is stable and $\alpha_i\colon A_i \to B$ are given, we can form

$$\bigoplus_{i=1}^{\infty} \alpha_i\colon\ A \to \bigoplus_{i=1}^{\infty} B \cong B \otimes c_0 \hookrightarrow B\otimes K \cong B ,$$

so θ is surjective. (It is clear that θ is well-defined on homotopy classes.) On the other hand, given $\alpha\colon A \to B$, let α_i be its restriction to A_i, and form $\bigoplus_{i=1}^{\infty} \alpha_i$ as above. We claim this is homotopic to α, so that θ is invertible. Indeed, let e_i, $i = 1,2,\ldots$ be the minimal central projections of c_0. We choose strictly continuous paths of isometries v_t^i in $M(B) \cong M(B\otimes K)$ such that $v_0^i = 1$ and $v_1^{i*} = 1 \otimes e_i$, then define α_t by

$$\alpha_t(a_i) = v_t^i \alpha(a_i) v_t^{i*} \quad , a_i \in A_i .$$

Then α_t will be a well-defined *-homomorphism $A \to B$ provided we arrange (as we may) that $v_t^i\alpha(A_i)v_t^{i*}v_t^j\alpha(A_j)v_t^{j*} = 0$ for $i \neq j$, and defines a homotopy from α to $\bigoplus_{i=1}^{\infty} \alpha_i$. □

REMARK 3.5. Let A and B be C*-algebras with B stable (i.e., $B \cong B\otimes K$). One can define maps $\sigma\colon [A,B] \to [A\otimes K,B]$ and $\sigma\colon [A\otimes K,B] \to [A,B]$ as follows. Fix a minimal projection e in K and isomorphisms $\beta\colon B \to B\otimes K$, $\alpha\colon A \to A\otimes e \hookrightarrow A\otimes K$. Given $\gamma\colon A \to B$, define $\sigma(\gamma)\colon A\otimes K \to B$ to be $\beta\circ(\gamma\otimes \mathrm{id}_K)$, and given $\gamma\colon A\otimes K \to B$, let $\rho(\gamma)\colon A \to B$ be the composite $\gamma\circ\alpha$. It seems plausible that σ and ρ should be isomorphisms (and inverses of each other when α and β are suitably chosen), but we have been unable to prove this. It is not hard to see σ is injective (with ρ as left inverse if α and β are suitably chosen), but surjectivity of σ seems more difficult.

REMARK 3.6. If A and B are C*-algebras and $B = \bigoplus_{i\in I} B_i$, then there is an obvious map $[A,B] \to \prod_{i\in I} [A,B_i]$ which is a bijection if the index set I is finite. If I is infinite, however, it is usually hard to characterize the image. If A has a unit, then this unit must map to a projection under any *-homomorphism, hence (since for $b = (b_i) \in B$, $\|b_i\| > 1/2$ only for finitely many $i \in I$) the image of A must be contained in a finite sum of B_i's . Thus in this case, $[A,B] \cong \bigoplus_{i\in I} [A,B_i]$, where $\bigoplus$ is the "direct sum" of pointed sets, the space of sequences equal to zero almost everywhere. But in general, the image of $[A,B]$ in $\prod_{i\in I} [A,B_i]$ need not lie in $\bigoplus_{i\in I} [A,B_i]$: consider, for instance, the case where $A = B =$ an infinite direct sum of non-contractible algebras, and look at the identity map $A \to B$.

We are now ready for the analogues of long exact homotopy sequences. Since we have no homotopy extension theorem, the results are similar to those for general spaces rather than the better versions for CW complexes. Recall in what follows that a sequence of pointed sets (not necessarily groups) is called exact if the preimage of zero under any map is the image of the previous map.

PROPOSITION 3.7. Let A,B,C be C*-algebras, ϕ: $A \to B$ a *-homomorphism. Define C_ϕ as in §1, and define α: $C_\phi \to A$ by $(f,a) \mapsto a$. Then

$$[C,C_\phi] \xrightarrow{\alpha_*} [C,A] \xrightarrow{\phi_*} [C,B]$$

is exact (as a sequence of pointed sets).

Proof. We mimic [45,Ch.7, §1, Theorem 3]. Suppose ψ: $C \to A$ is such that $\phi \circ \psi$ is null-homotopic. Recall from §1 that we have a homotopy equivalence χ: $A \to Z_\phi$. We replace ψ by $\chi\circ\psi$: $C \to Z_\phi$ and show that $\chi\circ\psi$ is homotopic to a map with values in $C_\phi \subset Z_\phi$.

By assumption, we have a homotopy

$$h: C \to C([0,1],B)$$

such that $h(0) = \phi\circ\psi$, $h(1) = 0$. Define

$$g: C \to C([0,1],Z_\phi)$$

by $g(0) = \chi\circ\psi$: $C \to Z_\phi$,

$$\pi_1(g(c,t))(t') = \begin{cases} h(c,t-2t'), & 0 \le 2t' \le t \le 1 \\ \phi\circ\psi(c) \in B, & 0 \le t \le 2t' \le 2 \end{cases}$$

$$\pi_2(g(c,t)) = \psi(c) \in A ,$$

where $c \in C$, π_1 and π_2 are the projections of Z_ϕ to $C([0,1],B)$ and to A , and where t is the homotopy variable. This is consistent since $h(c,0) = \phi\circ\psi(c)$, and g is clearly a *-homomorphism. Finally, $g(1)$ takes values in C_ϕ , since $\pi_1(g(c,1))(0) = h(c,1) = 0$. So g is a homotopy from

$\chi\circ\psi$ to a map with values in C_ϕ , as required. □

THEOREM 3.8. Let A,B,C be C*-algebras, ϕ: $A \to B$ a *-homomorphism. Define C_ϕ and α: $C_\phi \to A$ as before. Then the following sequence of pointed sets and/or groups is exact:

$$\cdots \to [C,\Sigma A] \xrightarrow{(\Sigma\phi)_*} [C,\Sigma B] \to [C,C_\phi] \xrightarrow{\alpha_*} [C,A] \xrightarrow{\phi_*} [C,B] .$$

Proof. We repeat the argument of [45,Ch.7,§1,Theorem 9]. We have already proved exactness at $[C,A]$. Applying (3.7) again, we see

$$[C,C_\alpha] \to [C,C_\phi] \xrightarrow{\alpha_*} [C,A]$$

is exact. Now $C_\alpha \cong \{(g,(f,a)): (f,a) \in C_\phi , g \in C([0,1],A) , g(0) = 0 , g(1) = \alpha(f,a) = a\}$, so

$$C_\alpha \cong \{(g,f): f \in CB, g \in CA , f(1) = \phi(g(1))\} .$$

We claim that the map $\Sigma B \to C_\alpha$ given by $f \mapsto (0,f)$ is a homotopy equivalence with homotopy inverse

$$\gamma: (g,f) \to h , \text{ where } h: [0,1] \to B$$

$$\text{and} \quad h(t) = \begin{cases} f(2t) , & 0 \le 2t \le 1 \\ \phi(g(2-2t)) , & 1 \le 2t \le 2 . \end{cases}$$

It is clear that γ is a *-homomorphism $C_\alpha \to \Sigma B$ and that $\Sigma B \to C_\alpha \xrightarrow{\gamma} \Sigma B$ is homotopic to the identity. Going the other way, the composite $C_\alpha \xrightarrow{\gamma} \Sigma B \to C_\alpha$ takes $(g,f) \mapsto (0,h)$, h as in the definition of γ . An explicit homotopy to the identity on C_α is given by

$$k: C_\alpha \to C([0,1],C_\alpha) , \text{ where}$$

$$k(g,f)(t,s) = \begin{cases} (g(ts),f((2-t)s)) , & (2-t)s \le 1 \\ (g(ts),\phi(g(2-(2-t)s))) , & (2-t)^{-1} \le s \le 1 . \end{cases}$$

Here t is the "homotopy variable" and s is the "cone variable". As required, $k(g,f)(0,s) = (0,h(s))$ and $k(g,f)(1,s) = (g(s),f(s))$. This shows C_α and ΣB are homotopy-equivalent and proves exactness at C_ϕ . Now apply (3.7) once more to get exactness of

$$[C,C_\delta] \to [C,\Sigma B] \xrightarrow{\delta} [C,C_\phi] ,$$

where δ: $\Sigma B \to C$ is given by $f \mapsto (f,0)$. We have

$$\begin{aligned} C_\delta &= \{(g,f): g \in C_0((0,1],C_\phi),f \in \Sigma B, g(1) = (f,0)\} \\ &\cong \{(g,f,h): f \in \Sigma B,h \in \Sigma A, g \in C_0((0,1]\times(0,1],B) , \\ &\qquad g(t,1) = \phi(h(t)), g(1,s) = f(s)\} \\ &\cong \{(g,h): g \in C([0,1]\times[0,1],B), h \in \Sigma A, g(t,1) = \phi(h(t)) , \\ &\qquad g \text{ vanishes on } (\{0\}\times[0,1])\cup([0,1]\times\{0\})\cup\{(1,1)\}\} . \end{aligned}$$

Now one can see (since graphically,

 and 

are homotopy-equivalent) that C_δ is homotopy-equivalent to

$$\{(g,h)\colon g \in C([0,1],\Sigma B),\ h \in \Sigma A\ ,\ g(1) = \Sigma\phi(h)\} = Z_{\Sigma\phi}\ ,$$

which is homotopy-equivalent to ΣA . This proves exactness at ΣB . For the rest of the sequence, just repeat the whole argument for $\Sigma\phi$. □

COROLLARY 3.9. Let A,B,C be C*-algebras, $\phi\colon A \to B$ a *-homomorphism. Then there is a long exact sequence of abelian groups

$$\ldots \to \{C,B\}_{q-1} \to \{C,C_\phi\}_q \to \{C,A\}_q \xrightarrow{\phi_*} \{C,B\}_q \to \{C,C_\phi\}_{q+1} \to \ \ldots\ .$$

Proof. For k sufficiently large, the following is exact:

$$\ldots \to [\Sigma^{k+q}C,\Sigma^{k+1}B] \to [\Sigma^{k+q}C,\Sigma^{k}C_\phi] \to [\Sigma^{k+q}C,\Sigma^{k}A] \xrightarrow{(\Sigma^k\phi)_*} [\Sigma^{k+q}C,\Sigma^{k}B] \to [\Sigma^{k+q}C,\Sigma^{k-1}C_\phi] \to \ldots\ ,$$

and everything is natural with respect to suspensions. Also, all sets of homotopy classes are abelian groups, by Corollary 3.2. Now pass to the limit as $k \to \infty$. □

PROPOSITION 3.10. Let A,B,C be C*-algebras, $\phi\colon A \to B$ and $\psi\colon B \to C$ *-homomorphisms, and $i\colon f \mapsto (f,0)$ the inclusion $\Sigma B \to C_\phi$. Then if $\psi\circ\phi$ is null-homotopic, $\Sigma\psi$ is homotopic to a composite $\Sigma B \overset{i}{\hookrightarrow} C_\phi \to \Sigma C$.

Proof. This is the non-commutative analogue of [45,Ch.8, Exercise E5]. Recall from Theorem 3.1(b) that $\Sigma\psi\colon \Sigma B \to \Sigma C$ is homotopic to η , where

$$\eta(f)(t) = \begin{cases} \psi(f(2t)), & 0 \le t \le 1/2 \\ 0\ , & 1/2 \le t \le 1\ . \end{cases}$$

By assumption, we may choose a homotopy

$$g\colon\ A \to C([0,1]) \otimes C$$

such that $g(0) = \psi\circ\phi$ and $g(1) = 0$. Define $\sigma\colon C_\phi \to \Sigma C$ by

$$\sigma(f,a)(t) = \begin{cases} \psi(f(2t))\ , & 0 \le t \le 1/2 \\ g(2t-1)(a), & 1/2 \le t \le 1\ . \end{cases}$$

This is consistent since for $(f,a) \in C_\phi$, $f(1) = \phi(a)$, hence $\psi(f(1)) = \psi\circ\phi(a) = g(0)(a)$. Then σ extends η , as required. □

THEOREM 3.11. Let A,B,C be C*-algebras, $\phi\colon A \to B$ a *-homomorphism. Then there is a long exact sequence of abelian groups:

$$\ldots \to \{C_\phi,C\}_{q-1} \to \{B,C\}_q \xrightarrow{\phi^*} \{A,C\}_q \to \{C_\phi,C\}_q \to \{B,C\}_{q+1} \to \ldots\ .$$

Proof. By Proposition 3.10,

$$\{C_\phi, C\}_{q-1} \xrightarrow{i^*} \{B,C\}_q \xrightarrow{\phi^*} \{A,C\}_q$$

is exact. But we may also apply the same argument to $\alpha\colon C_\phi \to A$, recalling from the proof of Theorem 3.8 that C_α is homotopy-equivalent to ΣB. This shows that

$$\{\Sigma B, C\}_{q-1} \to \{A,C\}_q \xrightarrow{\alpha^*} \{C_\phi, C\}_q$$

$$\text{or} \quad \{B,C\}_q \xrightarrow{\phi^*} \{A,C\}_q \xrightarrow{\alpha^*} \{C_\phi, C\}_q$$

is exact. Again apply Proposition 3.10 to $\delta\colon \Sigma B \to C_\phi$, recalling from the proof of Theorem 3.8 that C_δ is homotopy-equivalent to ΣA. This shows

$$\{\Sigma A, C\}_{q-1} \to \{C_\phi, C\}_q \xrightarrow{\delta^*} \{\Sigma B, C\}_q$$

$$\text{or} \quad \{A,C\}_q \to \{C_\phi, C\}_q \to \{B,C\}_{q+1}$$

is exact. The theorem follows by "splicing". □

With Corollary 3.9 and Theorem 3.11, we have recaptured in the non-commutative case all the usual exact sequences of homotopy theory. In fact, (3.9) and (3.11) show that if we fix a C*-algebra C and let $H_q(A) = \{C,A\}_{-q}$, $H^q(A) = \{A,C\}_q$, then $\{H_q\}_{q\in\mathbb{Z}}$ and $\{H^q\}_{q\in\mathbb{Z}}$ are essentially a homology and a cohomology theory on C*-algebras, respectively. The "stability property" of K-theory could be build in if we instead let $H_q(A) = \{C, A\otimes K\}_{-q}$. However, we do not quite have exactness in the best possible form, since we do not know that if $\phi\colon A \to B$ is surjective with kernel J, then $H_*(J) \cong H_*(C_\phi)$. In the next section, we shall discuss the relationship between K-theory and these "generalized homology theories". This relationship could be made clearer if we had a good version of the Brown Representability Theorem ([5],[6],[1]) valid for homotopy functors on C*-algebras. The methods of [6] and [1] seem promising, since they involve only category-theoretic constructions, most of which seem to work for C*-algebras. A stumbling block at the moment, however, is a lack of a good substitute for the subcategory of finite CW complexes. We hope to return to this subject in a future paper.

4. STABLE HOMOTOPY AND K-THEORY.

Our intention in this section is to explain how some of the notions of "non-commutative homotopy theory" from Section 3 can be applied to explain the "ubiquity of K-theory" which we discussed in Section 2.2. The key observation will be that K-theory is a "representable" homology functor - in fact, the associated "non-commutative classifying space" is just the algebra K of compact operators.

THEOREM 4.1. Let A be any C*-algebra. Then there is a natural isomorphism $K_1(A) \cong \pi_1(A\otimes K)$ (compare the corresponding formula [2,§6] for the homology groups associated to a spectrum: $E_n(X) = \pi_n(X \wedge E)$).

Proof. First suppose A has a unit. Then

$$\pi_1(A\otimes K) = [C_0(\mathbb{R}), A\otimes K] \cong [C(S^1), (A\otimes K)^+]_+ \quad .$$

However, $C(S^1)$ is isomorphic to the group C*-algebra of $\mathbb{Z}$, i.e., is the free C*-algebra on one unitary generator z (the identity map $S^1 \to S^1 \hookrightarrow \mathbb{C}$). So a unital *-homomorphism $\phi\colon C(S^1) \to (A\otimes K)^+$ is equivalent to a choice of a unitary element $\phi(z)$ in $(A\otimes K)^+$. Hence $\pi_1(A\otimes K)$ may be identified with the group of path components of the unitary group of $(A\otimes K)^+$, which coincides with $\varinjlim GL(n,A)/GL(n,A)_0 \cong K_1(A)$.

Now suppose A has no unit, and A^+ is its "unitalization". We know $K_1(A) \cong K_1(A^+) \cong U((A^+\otimes K)^+)$, where $U(\cdot)$ denotes "unitary group of". So the problem is to show that the groups of path components of $U((A\otimes K)^+)$ and of $U((A^+\otimes K)^+)$ coincide. But we have a split extension of C*-algebras

$$0 \to A \otimes K \to A^+ \otimes K \to K \to 0 \quad .$$

Thus $U((A\otimes K)^+)$ is a normal subgroup of $U((A^+\otimes K)^+)$, and the quotient must be an open subgroup of

$$U(K)_+ = \{1 + x \in U(K^+) :\ x \in K\} \quad .$$

However, $U(K)_+$ is path-connected, so the quotient is all of $U(K)_+$. To complete the argument, we consider the exact homotopy sequence (for ordinary homotopy of spaces, not homotopy of C*-algebras) of the Serre fibration

$$U((A\otimes K)^+) \to U((A^+\otimes K)^+) \to U(K)_+ :$$

$$\pi_1(U((A^+\otimes K)^+)) \xrightarrow{\beta} \mathbb{Z} \to \pi_0(U((A\otimes K)^+)) \xrightarrow{\alpha} \pi_0(U((A^+\otimes K)^+)) \to 0 \quad .$$

(It is well-known that $\pi_1(U(K)_+) \cong \mathbb{Z}$ and that $\pi_0(U(K)_+) = 0$.) The map β is surjective, since we have a splitting map $K \to A^+\otimes K$; hence α is an isomorphism, as desired. □

COROLLARY 4.2. For any C*-algebra A, $K_0(A) \cong \pi_1(A\otimes C_0(\mathbb{R})\otimes K)$.

Proof. By Bott periodicity, $K_0(A) \cong K_1(\Sigma A)$. □

Now that we have a characterization of the K-groups of a C*-algebra in terms of non-commutative homotopy theory, we are ready to prove a "non-commutative Eilenberg-Steenrod uniqueness theorem", characterizing (on a certain subcategory of the category of separable C*-algebras) K-theory as the unique homology theory satisfying axioms (1)-(7) of (2.1). However, it will be necessary to assume rather precise forms of the stability and Bott periodicity axioms ((4) and (6) on the list). Namely, we replace (4) and (6) by the following:

(4*) If A is any C*-algebra and e is a minimal projection in K, then the map $A \to A \otimes K$ defined by $a \mapsto a \otimes e$ induces isomorphisms of homology groups in every degree.

(6*) Let $\lambda \in [\mathbb{C}, C(S^2) \otimes K]$ be the homotopy class of rank-one projections in $C(S^2) \otimes K$ corresponding to the Hopf line bundle on S^2 (for an explicit formula, see [46,p.162]), and let $\tau \in [\mathbb{C}, C(S^2) \otimes K]$ be the class of trivial rank-one projections corresponding to the trivial line bundle on S^2. Thus $\lambda - \tau$ (the formal difference) is the Bott element representing the standard generator of $\tilde{K}^0(S^2) \cong K^0(\mathbb{R}^2)$. The periodicity map

$$\pi\colon \; K_j(A) \to K_{j+2}(A)$$

is defined as follows. Given $a \in K_j(A)$, form $(id_A \otimes \lambda)_*(a) \in K_j(A \otimes C(S^2) \otimes K)$ and similarly form $(id_A \otimes \tau)_*(a)$. The difference in $K_j(A \otimes C(S^2) \otimes K)$ lies in the kernel of the natural map to $K_j(A \otimes \mathbb{C} \otimes K)$, hence, since the extension

$$0 \to A \otimes C_0(\mathbb{R}^2) \otimes K \to A \otimes C(S^2) \otimes K \to A \otimes \mathbb{C} \otimes K \to 0$$

is split, may be identified with an element of $K_j(A \otimes C_0(\mathbb{R}^2) \otimes K) \cong K_j(A \otimes C_0(\mathbb{R}^2)) = K_j(\Sigma^2 A) \cong K_{j+2}(A)$. We assume that π is an isomorphism for all $j \in \mathbb{Z}$ and for all C*-algebras A.

<u>THEOREM 4.3</u>. Let C be the smallest category of separable C*-algebras containing the type I separable C*-algebras and closed under extensions, countable direct limits, and stable isomorphism. Let $\{H_j\}_{j \in \mathbb{Z}}$ be a family of functors from C to abelian groups satisfying axioms (1)-(3) (homotopy-invariance, compatibility with suspension, exactness), (4*) (strong stability), (5) (continuity under limits), (6*) (strong Bott periodicity), and (7) (normalization). Then there is a natural equivalence $\nu\colon \; K_* \xrightarrow{\cong} H_*$.

<u>Proof</u>. We have $H_1(C_0(\mathbb{R})) = H_1(\Sigma\mathbb{C}) \cong H_2(\mathbb{C}) \cong \mathbb{Z}$. So fix a generator ζ for $H_1(C_0(\mathbb{R}))$. We define our natural transformation

$$\nu\colon \; K_1(A) \cong [C_0(\mathbb{R}), A \otimes K] \to H_1(A)$$

by $\nu([\phi]) = \phi_*(\zeta) \in H_1(A \otimes K) \cong H_1(A)$, for any *-homomorphism $\phi\colon \; C_0(\mathbb{R}) \to A \otimes K$. To check naturality, suppose we have a *-homomorphism $\alpha\colon \; A \to B$. Without loss of generality, we may assume A and B are stable. Then given $\phi\colon \; C_0(\mathbb{R}) \to A$ representing a class $[\phi]$ in $K_1(A)$, $\alpha_*([\phi]) = [\alpha \circ \phi]$ and $\nu(\alpha_*([\phi])) = (\alpha \circ \phi)_*(\zeta) = \alpha_*(\phi_*(\zeta)) = \alpha_*(\nu([\phi]))$, as required. Note also that ν commutes with the boundary maps of (3), since these are induced by the mapping cone construction of Proposition 3.7.

Next we check that ν is an isomorphism on spheres, i.e., when $A = C_0(\mathbb{R}^n)$. If n is even, $K_1(A)$ and $H_1(A)$ both vanish, so this is trivial. Therefore we may assume $n = 2k+1$, $k \geq 0$. If $k = 0$, ν is an isomorphism by construction, for $K_1(C_0(\mathbb{R}))$ is generated by the image in $[C_0(\mathbb{R}), C_0(\mathbb{R})\otimes\mathcal{K}]$ of the class of the identity map in $[C_0(\mathbb{R}), C_0(\mathbb{R})] \cong [C(S^1), C(S^1)]_+ \cong \mathcal{U}(C(S^1))/\mathcal{U}(C(S^1))_0 \cong \mathbb{Z}$, and this class is taken by ν to ζ. Now consider the case $k \geq 0$. Because of the fact that we have assumed our periodicity maps π are actually implemented by $(\mathrm{id}\otimes\lambda)_* - (\mathrm{id}\otimes\tau)_*$, we have a commutative diagram

$$\begin{array}{ccc} K_1(C_0(\mathbb{R})) & \xrightarrow{\ \nu\ } & H_1(C_0(\mathbb{R})) \\ \downarrow{\scriptstyle \pi^k} & & \downarrow{\scriptstyle \pi^k} \\ K_1(C_0(\mathbb{R}^{2k+1})) & \xrightarrow{\ \nu\ } & H_1(C_0(\mathbb{R}^{2k+1})) \end{array} .$$

Since the top map and the side maps are isomorphisms, so is the bottom map.

We now know (using the fact that $K_{1+j}(A) \cong K_1(\Sigma^j A)$, $H_{1+j} \cong H_1(\Sigma^j A)$, with both isomorphisms natural) that $\nu = \nu_1$ defines natural transformations $\nu_j\colon K_j \to H_j$ for $j \geq 1$, and that these transformations are compatible with suspension and periodicity. Using the periodicity maps in reverse, we have ν_j defined for all j. Now restrict attention to the full subcategory $\mathcal{A} \subset \mathcal{C}$ of separable commutative C*-algebras (the opposite of the category of pointed compact metric spaces). We have a natural transformation of continuous cohomology theories inducing isomorphisms of coefficient groups, hence by the argument discussed above in Section 2, ν is a natural equivalence on $\mathcal{A}$. (Recall this goes via J. H. C. Whitehead's Theorem or the Atiyah-Hirzebruch spectral sequence.)

To extend the result to the larger category $\mathcal{C}$, we use a method introduced in [42], [43], and [41]. Namely, let $\mathcal{D}$ be the class of objects A of $\mathcal{C}$ for which ν induces isomorphisms $K_j(A) \to H_j(A)$ in every degree. We know that $\mathcal{D}$ contains $\mathcal{A}$. $\mathcal{D}$ is closed under extensions, for given an extension

$$0 \to J \to E \to A \to 0$$

of C*-algebras with $A, J \in \mathcal{D}$, we get by naturality of ν and the exactness axioms a commutative diagram of abelian groups with exact rows:

$$\begin{array}{ccccccccccc} \to & K_{j+1}(A) & \xrightarrow{\partial} & K_j(J) & \to & K_j(E) & \to & K_j(A) & \xrightarrow{\partial} & K_{j-1}(J) & \to \cdots \\ & \downarrow{\scriptstyle \nu_A} & & \downarrow{\scriptstyle \nu_J} & & \downarrow{\scriptstyle \nu_E} & & \downarrow{\scriptstyle \nu_A} & & \downarrow{\scriptstyle \nu_J} & \\ \to & H_{j+1}(A) & \xrightarrow{\partial} & H_j(J) & \to & H_j(E) & \to & H_j(A) & \to & H_{j-1}(J) & \to \cdots \end{array} .$$

Since ν_A and ν_J are isomorphisms, ν_E is an isomorphism by the 5-lemma. Because of axiom (4*), $\mathcal{D}$ is closed under stable isomorphism. Thus $\mathcal{D}$ contains what were called solvable algebras in [43], i.e., type I separable C*-algebra with a finite composition series in which the composition factors are all of the form $B \otimes M_n$ or $B \otimes \mathcal{K}$, $B \in \mathcal{A}$. But we claim $\mathcal{D}$ is also closed under countable inductive limits, for if $A = \varinjlim A_n$ with $A_n \in \mathcal{D}$, then using (5) we have

$$K_j(A) \cong \varinjlim K_j(A_n) \xrightarrow[\nu]{\cong} \varinjlim H_j(A_n) \cong H_j(A) \quad .$$

As in [43] and [41], we conclude that $\mathcal{D} = \mathcal{C}$. This relies on the fact that each type I C*-algebra has a countable (possibly transfinite) composition series with composition factors of continuous trace, together with the fact due to Dixmier and Douady (for a quick summary, see [39, §6] and references quoted there) that a separable continuous-trace algebra is stably isomorphic to one locally of the form $C_0(Y) \otimes \mathcal{K}$. □

Theorem 4.3 was the main goal of this paper, but Theorem 4.1 still suggests a substantial loose end: how is $K_*(A)$ related (if at all) to $\pi_n(A\otimes\mathcal{K})$ for $n \geq 1$ or to $\pi_q^s(A\otimes\mathcal{K})$ for $q \in \mathbb{Z}$? In particular, is $K_1(A) \cong \pi_1^s(A\otimes\mathcal{K})$? If so, we could conclude that K-theory "is" stable stable homotopy, where now the "stable" is to be used in both senses of §3. We do not yet know the complete answer to this question, but we will see the answer is yes when A is abelian. One might speculate that a complete calculation of $\pi_*^s(A\otimes\mathcal{K})$ might settle the question raised in §2 about whether K-theory "gives rise to" all matrix-stable homology theories on C*-algebras (without the a priori assumption of Bott periodicity). Either it should turn out that $A \to \pi_*^s(A\otimes\mathcal{K})$ itself gives a new homology functor with perhaps new and unforseen applications in non-commutative topology, or else one should probably be able to weaken the periodicity assumption in Theorem 4.3.

To analyze our problem, we begin with a simple consequence of Theorem 4.1.

PROPOSITION 4.4. For any C*-algebra A and any $q \leq 1$, there is a split monomorphism of abelian groups

$$J\colon\ K_{2-q}(A) \to \pi_q^s(A\otimes\mathcal{K}) \quad .$$

Moreover, the map J is natural with respect to *-homomorphisms of C*-algebras.

Proof. It is enough to prove this for $q = 1$, since $\pi_{1-k}^s(A\otimes\mathcal{K}) \cong \pi_1^s(\Sigma^k A\otimes\mathcal{K})$. But for any $n \geq 0$, we have the following commutative diagram:

$$\begin{array}{ccc} [C_0(\mathbb{R}),\ A\otimes\mathcal{K}] & \longrightarrow & K_1(A) \\ & {\scriptstyle\Sigma^n}\searrow \quad \nearrow & \\ & [C_0(\mathbb{R}^{n+1}),\ \Sigma^n A\otimes\mathcal{K}] & \end{array}$$

Here the horizontal arrow is the isomorphism of Theorem 4.1 and the map $[C_0(\mathbb{R}^{n+1}), \Sigma^n A\otimes\mathcal{K}] \to K_1(A)$ (essentially the ν of Theorem 4.3) takes the homotopy class of $\phi\colon C_0(\mathbb{R}^{n+1}) \to \Sigma^n A\otimes\mathcal{K}$ to the induced map in $\mathrm{Hom}(K_*(C_0(\mathbb{R}^{n+1})),$

$K_*(\Sigma^n A\otimes K)) \cong \mathrm{Hom}(K^*(\mathbb{R}^{n+1}), K_*(\Sigma^n A)) \cong K_1(A)$. Now pass to the limit as $n \to \infty$. □

Remark 4.5. We have called the map J in Proposition 4.4 to suggest that it should be viewed as the analogue (in non-commutative matrix-stable homotopy) of the J-invariant in ordinary stable homotopy [26,Ch.V,§5]. The analogy is perhaps far-fetched, but we subscribe to the philosophy that K-theory should give the "effectively computable" part of stable stable homotopy, just as it gives the more accessible part of ordinary stable homotopy. However, in our case (unlike the usual case), it appears that J is surjective.

Now let us consider how $\pi_*^s(A\otimes K)$ could actually be computed. We begin by studying $\pi_n(B)$; we will of course be interested in the case where n is large and B is stable and a multiple suspension. Note that for any C*-algebra B , the set of *-homomorphisms $C_0(\mathbb{R}^n) \to B$ may be identified with the set of unital *-homomorphisms $C(S^n) \to B^+$, which in turn may be identified with

$$V_n(B) = \{(a_0,\dots,a_n):\ a_j = a_j^* \in B^+ \text{ and } a_j a_\ell = a_\ell a_j \text{ for } j,\ell = 0,\dots,n\ ;\ \sum_{j=0}^{n} a_j^2 = 1\} .$$

Thus $\pi_n(B)$ is just $\pi_0(V_n(B))$, the set of path components of $V_n(B)$. Of special interest is $V_n(K)$, because if $A = C_0(Y)$ is commutative and $Y^+ = Y \cup \{\infty\}$ is the one-point compactification of $\hat{A}$, then

$$V_n(A\otimes K) = \{f\colon Y^+ \to V_n(K) \text{ continuous, } \sigma\circ f \text{ constant-valued, } f(\infty) \in V_n(0) = S^n\} .$$

It is useful to consider the approximations $\pi_n(A\otimes M_k)$ to $\pi_n(A\otimes K)$; we find that

$$\pi_n(C_0(Y)\otimes M_k) \cong [Y^+, V_{n,k}]_+ ,$$

where $[\ ,\ ]_+$ denotes the set of based homotopy classes and

$$V_{n,k} = \{(a_0,\dots,a_n):\ a_j = a_j^* \in M_k \text{ and } a_j a_\ell = a_\ell a_j \text{ for } j,\ell = 0,\dots,n\ ;\ \sum_{j=0}^{n} a_j^2 = 1\} . \tag{4.6}$$

(The choice of a base-point in $V_{n,k}$ is not particularly important - one might choose $(1,0,\dots,0)$.) So the computation of stable stable homotopy in the case of commutative C*-algebras basically comes down to the problem of determining the homotopy type of the spaces $V_{n,k}$. Note that each $V_{n,k}$ is a finite cell complex and a real algebraic variety, although not necessarily a smooth manifold.

For fixed k, the spaces $V_{n,k}$ constitute a spectrum V_k with structure maps $\Sigma V_{n-1,k} \to V_{n,k}$ defined by identifying $\Sigma V_{n-1,k}$ with the space of $(n+1)$-tuples $(a_0,\ldots,a_n) \in V_{n,k}$ for which a_0 is a scalar matrix. (Since a_0 is hermitian, it must then lie in $[-1,1]$. If $a_0 = \pm 1$, then $a_j = 0$ for all $j > 0$. Otherwise, the space of possibilities for $(a_1,\ldots,a_n)$ with a_0 fixed is homeomorphic to $V_{n-1,k}$.) In fact, it is easy to see that

$$\pi_q^s(C_0(Y) \otimes M_k) \cong [\Sigma^\infty Y^+, V_k]_{-q} \tag{4.7}$$

(in the notation of [26,Ch.1]). Stable stable homotopy will be gotten by passing to the limit as $k \to \infty$, because of the following result:

PROPOSITION 4.8. For any $A \in \mathcal{A}$, the natural map

$$\lim_{k\to\infty} \pi_n(A \otimes M_k) \to \pi_n(A \otimes K)$$

is an isomorphism.

Proof. Let $A = C_0(Y)$ and consider a continuous map $f \in V_n(A \otimes K) \subseteq C(Y^+, V_n(K))$. Given $\varepsilon > 0$, we can (using density of $\bigcup M_k$ in K and compactness of Y^+) choose k so that the image of f lies within ε of $V_n(M_k)$. Assuming $\varepsilon < \frac{1}{2}$, by continuously changing the eigenvalues in the entries of f (recall $f(y)$ is an $(n+1)$-tuple of commuting self-adjoint operators of the form scalar + compact, for each $y \in Y^+$) we may deform f back into $V_n(M_k)$. □

The spectra V_k and $V = \varinjlim V_k$ were first introduced in [44,§1], except that Segal worked with real C*-algebras, which are actually harder to deal with. Proposition 4.8 is essentially [44,Proposition 1.2]. Segal's main result on this subject, reformulated in our context, is the following.

THEOREM 4.9. $V = bu$, the spectrum of connective complex K-theory, as defined in [2,§6]. Equivalently, $\lim_{k\to\infty} V_{n,k}$ is $(n-1)$-connected, and $\Omega^{n-1}(\lim_{k\to\infty} V_{n,k})$ is homotopy-equivalent to $U(\infty) = \lim_{k\to\infty} U(k)$.

Proof. See [44,§1], except replace $\mathbb{R}$ by $\mathbb{C}$ and BO by BU. Another very similar proof was kindly communicated to the author by D. McDuff; the idea is to construct a quasifibration with contractible total space, with fiber $\lim_{k\to\infty} V_{n-1,k}$ and with base $\lim_{k\to\infty} V_{n,k}$ (cf. [44,Proposition 1.5]). Then it is enough to observe that $V_{1,k}$ may be identified by $U(k)$ via the map $(a_0,a_1) \mapsto a_0 + ia_1$, so that $\lim_{k\to\infty} V_{1,k} = U(\infty)$. □

COROLLARY 4.10. If A is a commutative C*-algebra, the map J of Proposition 4.4 is an isomorphism

$$K_{2-q}(A) \to \pi_q^s(A \otimes K) \ , \ q \leq 1 \ .$$

Furthermore, if $A = C_0(Y)$ with Y^+ a finite CW complex, then for any q,

$$\pi_q^s(A \otimes K) \cong \tilde{bu}^q(Y^+)$$

and in particular, $\pi_q^s(A \otimes K)$ vanishes for $q > \dim(Y^+)$.

Proof. We have seen that $\pi_q^s(C_0(Y) \otimes K) \cong [\Sigma^\infty Y^+, bu]_{-q}$, which by definition is $\tilde{bu}^q(Y^+)$ when Y^+ is a finite complex. Even for any locally compact Y, we still have $K_1(A) \cong [Y^+, U(\infty)]_+$, $K_0(A) \cong [Y^+, \mathbb{Z} \times BU]_+$, and surjectivity of the J-invariant for $q \leq 1$ follows since

$$\begin{aligned}\pi_q^s(A \otimes K) &\cong \lim_{n\to\infty}[\Sigma^n Y^+, \lim_{k\to\infty} V_{n+q,k}]_+ \\ &\cong [\Sigma^r Y^+, \Omega^{n-r}(\lim_{k\to\infty} V_{n+q,k})]_+ \cong [\Sigma^r Y^+, U(\infty)]\end{aligned}$$

for suitable $r \geq 0$. If Y^+ is a finite complex of dimension d , then since $\lim_{k\to\infty} V_{d+1,k}$ is d-connected, $\pi_q^s(A \otimes K) = 0$ for $q \geq d+1$. □

It seems probable that surjectivity of the J-invariant for $A \in \mathcal{A}$ should imply the same for $A \in \mathcal{C}$, by the argument of Theorem 4.3, but we have not been able to prove this because of a technical difficulty: the exactness property of π_*^s , Corollary 3.9, involves mapping cones, and we have not been able to show that given an extension of C*-algebras

$$0 \to J \to A \xrightarrow{\phi} B \to 0 \ ,$$

the inclusion $J \to C_\phi$ induces isomorphisms of homotopy groups. Also we are not certain if π_* (and hence π_*^s) always commutes with C*-algebra inductive limits; this is at least true for π_1 and for abelian algebras.

5. AFTERWORD.

The present paper follows the outline of the talk given by the author in San Francisco, but has been much improved on the basis of helpful suggestions from several sources. I would like to thank several members of the audience at the Special Session for helping to clarify the problem of axiomatic treatment of K-theory, discussed above in §2, and in particular to thank Chuck Weibel and Ed Effros for referring me to the work of Karoubi and Villamayor. I would especially like to thank Dusa McDuff for explaining in a recent letter how to prove Theorem 4.9, which in the original talk was only a vague conjecture, and for providing the very apt reference to her work and the work of G. Segal on configuration spaces. Finally, I have learned that some of the ideas discussed

here have also been developed independently by Claude Schochet and Joachim Cuntz.

Quite a few open problems remain. For instance, §4 above shows how to represent K-theory in terms of homotopy; presumably one could do this same for the dual theory, i.e., the *Ext*-theory of Brown, Douglas, and Fillmore or the K-homology theory of Kasparov. The natural conjecture, discussed in part in [44], is that for suitable separable nuclear C*-algebras A ,

$$\{A, Q\otimes K\}_q \cong Ext^q(A) \quad ,$$

where Q denotes the Calkin algebra. More generally, one should have

$$\{A, Q(B)\otimes K\}_q \cong Ext^q(A,B)$$

for the Kasparov K-functor, where $Q(B)$ denotes the outer multiplier algebra of $B\otimes K$. Perhaps $Q(B) \otimes K$ may also be replaced by $Q(B)$ here.

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