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An Equivariant Novikov Conjecture

Dedicated to Alexander Grothendieck

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Abstract. We discuss and formulate the correct equivariant generalization of the strong Novikov conjecture. This will be the statement that certain G-equivariant higher signatures (living in suitable equivariant K-groups) are invariant under G-maps of manifolds which, nonequivariantly, are homotopy equivalences preserving orientation. We prove this conjecture for manifolds modeled on a complete Riemannian manifold of nonpositive curvature on which G (a compact Lie group) acts by isometries. We also use the theory of harmonic maps to construct (in some cases) G-maps into such model spaces.

Key words. Novikov conjecture, equivariant K-theory, G-pseudoequivalence, C^*-algebra, fundamental groupoid, KK-theory

1. Formulation of the Problem and of the Main Results

Let M be a connected, closed, oriented manifold. (For the moment, we take our manifolds to be smooth, though later we shall generalize some results to the topological and PL categories.) The Novikov conjecture asserts that the *higher signature* of M, i.e., $f_*(\mathbb{L}(M) \cap [M]) \in H_*(B\pi_1(M); \mathbb{Q})$, is invariant under orientation-preserving homotopy-equivalences $h: M' \to M$. Here, $f: M \to B\pi_1(M)$ is a classifying map for the universal cover of M and $\mathbb{L}$ is the total Hirzebruch L-class. More precisely, the conjecture asserts that

$$f_*(\mathbb{L}(M) \cap [M]) = (f \circ h)_* \quad (\mathbb{L}(M') \cap [M']).$$

This conjecture is known in a large number of cases (see, for instance, [4, 10, 19, 16, and 44]). For purposes of the proofs, it is usually best to fix a group π and drop the requirement that this be the fundamental group of M. The *Novikov Conjecture for*

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π is then the assertion that for any diagram

$$\begin{array}{ccc} M & \searrow^{f} & \\ {\scriptstyle h}\uparrow & & B\pi \\ M' & \nearrow_{g} & \end{array}$$

commuting up to homotopy, with h an orientation-preserving homotopy equivalence, we have

$$f_*(\mathbb{L}(M) \cap [M]) = g_*(\mathbb{L}(M') \cap [M']).$$

Our intention in this paper is to give a suitable generalization of the above to the case where a compact Lie group G acts (smoothly) on M. When π and G are trivial, $B\pi$ is (homotopy equivalent to) a point and $f_*(\mathbb{L}(M) \cap [M])$ is by the Hirzebruch signature formula just the signature of M, an obvious homotopy invariant. The equivariant generalization of this for a G-manifold is the G-signature of M, the virtual representation of G (in $R(G)$, the representation ring of G) obtained from the action of G on middle cohomology.

As pointed out originally by Petrie (see [29]), the G-signature of a G-manifold is preserved, not only under (orientation preserving) G-equivariant homotopy equivalences, but also under *G-pseudoequivalences*, or G-maps which *nonequivariantly* are homotopy equivalences. (Technical point: a G-pseudoequivalence as defined may not have a G-equivariant homotopy inverse (example: the map $EG \to pt$). Thus, the relation of existence of a G-pseudoequivalence is not symmetric. We will occasionally be interested in the equivalence relation it generates.)

Thus, we see that an equivariant Novikov conjecture should be a statement about G-pseudoequivalence invariance of 'higher G-signatures'. However, there is an additional complication which must be taken into account. Namely, the fundamental group of a space depends on a choice of basepoint, and if M is a G-manifold, there may be no G-fixed basepoint (or the fixed set M^G may be disconnected). Therefore, we are forced to deal with the fundamental group*oid* $\pi(M)$ of M, which is canonically defined and carries a natural action of G. (In the notation of [24], this is $\pi^{\tau}(M)$.) In fact, for each closed subgroup H of G, we are forced to contend with the fundamental groupoid of the fixed set M^H. In the appendix [24] to this paper, J. P. May constructs a G-space $B\pi(M)$ which plays in the equivariant category the same role played by $B\pi$ in the nonequivariant case. The G-space $B\pi(M)$ has the property that for any closed subgroup H of G, all components of the fixed set $(B\pi(M))^H$ are aspherical. Furthermore, as explained in [24], there is a natural G-'map' $f\colon M \to B\pi(M)$ inducing isomorphisms on π_0 of all fixed sets and on π_1 of all components of fixed sets, and such a map is unique up to G-homotopy. A similar construction is given in [21], using different techniques, but there appear to be technical difficulties with the method of [21] when G is not discrete.

Let D be the signature operator on M in the sense of Atiyah–Singer [1], computed with respect to some G-invariant Riemannian metric. Via the formalism of Kasparov (see, for example, [19] or [2, §17.1.2 and §24.1]), D defines a class

$\Delta(M) = [D] \in K_*^G(M)$ (living in even or odd degree, depending on the parity of the dimension of M) which is independent of the choice of a Riemannian metric. When $\dim M$ is even, the G-index of the G-invariant Fredholm operator D (acting on suitable Sobolev spaces) is the G-signature of M. In other words, we have

$$G\text{-sign}(M) = c_*([D]) \in K_0^G(pt) = R(G),$$

where c collapses M to a point. If M is $G-1$-connected, i.e., all fixed sets M^H are connected and 1-connected, then $B\pi(M)$ is G-contractible, so $c: M \to pt$ 'is the map' $f: M \to B\pi(M)$, up to G-homotopy. Thus, it is reasonable to make the following definition.

1.1. DEFINITION. The *higher G-signature* of M is

$$f_*(\Delta(M)) = f_*([D]) \in K_*^G(B\pi(M)).$$

(Here for an infinite G-CW complex such as $B\pi(M)$, equivariant K-homology is defined to be what Kasparov calls RK_*^G in [19], i.e., the inductive limit $\varinjlim K_*^G(X)$, where X runs over the directed set of finite G-subcomplexes.)

Arguing by analogy from the nonequivariant use, one might be tempted to guess that the higher G-signature of M is a G-pseudoequivalence invariant. However, this is usually false, as one can see from the following simple examples.

1.2. EXAMPLES. (a) Even when G is trivial, the higher signature $f_*([D]) \in K_*(B\pi(M))$ is known not to be homotopy-invariant for many cases where $\pi_1(M)$ has torsion. For instance, in the case of lens spaces with fundamental group

$$\pi = \mathbb{Z}_p, \qquad K_1(B\mathbb{Z}_p) = \lim_{r\to\infty} \mathbb{Z}_{p^r} \cong \mathbb{Z}\left[\frac{1}{p}\right]\Big/\mathbb{Z}$$

(sometimes denoted $\mathbb{Z}_{p^\infty}$) and $f_*([D])$ is a function of the mod-p Pontryagin classes, which are not homotopy invariants for lens spaces (see [28, §3]).

(b) If M is 1-connected and G acts freely on M, then as a G-space, $B\pi(M)$ is G-homotopy-equivalent to EG. Once again, the higher G-signature is not a G-homotopy invariant. For instance, let $h: N^{2n} \to \mathbb{CP}^n$ be an orientation-preserving homotopy equivalence, where N is a fake complex projective space and h does not preserve rational Pontryagin classes. The canonical S^1-bundle over $\mathbb{CP}^n$ pulls back to N, yielding a G-homotopy equivalence

$$\tilde{h}: \Sigma^{2n+1} \to S^{2n+1},$$

where G is S^1 acting freely and Σ is a homotopy sphere. Note that S^{2n+1} is a G-skeleton of $EG = S^\infty$ and $K_1^G(S^{2n+1}) \cong R(G)/I^{n+1}$ injects in $K_1^G(EG) \cong R(G)[1/I]/R(G)$, where $R(G) \cong \mathbb{Z}[t, t^{-1}]$ and $I = (t-1) \subset R(G)$ is the augmentation ideal.

The classifying map $f: \Sigma \to B\pi(\Sigma)$ is just the composite of $\tilde{h}$ with the inclusion

$\iota: S^{2n+1} \hookrightarrow EG$, so $f_*([D_\Sigma])$ is determined by $\tilde{h}_*([D_\Sigma])$, which rationally corresponds (under the Chern character) to

$$2^n h_*(\mathscr{L}(N) \cap [N]) \in H_*(\mathbb{CP}^n; \mathbb{Q})$$

($\mathscr{L}$ is the Atiyah–Singer L-class, which differs from the Hirzebruch L-class $\mathbb{L}$ only by certain powers of 2). Since one can recover all the rational Pontryagin classes of N from this, we clearly do *not* get $\iota_*([D_S])$, even rationally.

Examples 1.2a and 1.2b show that when $B\pi(M)$ does not have the G-homotopy type of a finite G-CW-complex, it is unreasonable to expect the higher G-signature to be a G-homotopy invariant, let alone a G-pseudoequivalence invariant. In fact, asserting the latter does not even make sense, since if $h: M \to M'$ is a G-pseudoequivalence, the higher G-signatures of M and M' live in $K_*^G(B\pi(M))$ and $K_*^G(B\pi(M'))$, respectively, which are usually quite different as $R(G)$-modules. Nevertheless, H induces by [24] a functorial map

$$h_*: K_*^G(B\pi(M)) \to K_*^G(B\pi(M')),$$

so it is natural to try to compare the image of the higher signature of M with the higher signature of M'. As we shall see later, these often agree; still more often, they will agree after localizing at various prime ideals of $R(G)$. For example, in 1.2a, we must invert p, the order of the torsion in the fundamental group; and 1.2b, we must invert the augmentation ideal I.

1.3. CONJECTURE (equivariant Novikov conjecture). Let $h: M \to M'$ be an orientation-preserving G-pseudoequivalence of connected, closed, oriented G-manifolds, and consider the associated commutative diagram

$$\begin{array}{ccc} K_*^G(M) & \xrightarrow{(f_M)_*} & K_*^G(B\pi(M)) \\ \downarrow{\scriptstyle h_*} & & \downarrow{\scriptstyle h_*} \\ K_*^G(M') & \xrightarrow{(f_{M'})_*} & K_*^G(B\pi(M')). \end{array}$$

(a) If $K_*^G(B\pi(M'))$ is finitely generated over $R(G)$, then the higher G-signatures agree, i.e.,

$$h_* \circ (f_M)_*([D_M]) = (f_{M'})_*([D_{M'}]).$$

(b) More generally, if for some collection $\mathscr{S}$ of prime ideals of $R(G)$, the localization $K_*^G(B\pi(M'))_{\mathscr{S}}$ is finitely generated over $R(G)_{\mathscr{S}}$, then the same statement holds after localizing.

(c) Still more generally, assume one has a G-space X (say a G-GW complex) and a commutative diagram of G-maps

$$\begin{array}{ccc} M & \searrow^{\varphi} & \\ \uparrow{\scriptstyle h} & & X. \\ M' & \nearrow_{\psi} & \end{array}$$

Then if $K_*^G(B\pi(X))$ is finitely generated over $R(G)$,

$$(f_X)_* \circ \varphi_*([D_M]) = (f_X)_* \circ \psi_*([D_{M'}])$$

in $K_*^G(B\pi(X))$, and similarly (if one only assumes finite generation after localizing as in (b)) after localizing. □

While it may be that the finite generation hypothesis can be weakened somewhat, some such restriction is needed because of the examples in 1.2. This does not seem serious, since the best results about the nonequivariant Novikov conjecture apply only to groups π for which $B\pi$ is rationally equivalent to a finite complex or, in fact, if one wants torsion information, to the case where $B\pi$ can be taken to be a finite complex.

The main results of this paper are the following:

1.4. THEOREM (see 5.1 below). *Suppose X is a complete (not necessarily compact) Riemannian manifold of nonpositive curvature on which G acts by isometries, with $K_G^*(X)$ finitely generated, and*

$$\begin{array}{ccc} M & \searrow^{\varphi} & \\ \uparrow h & & X \\ M' & \nearrow_{\psi} & \end{array}$$

is a commutative diagram of G-maps, with h an orientation-preserving G-pseudo-equivalence of closed oriented G-manifolds. Then $\varphi_*([D_M]) = \psi_*([D_{M'}])$ in $K_*^G(X)$.

It is important to note that Theorem 1.4 is one case of Conjecture 1.3(c). This follows immediately from:

1.5. PROPOSITION ('Equivariant Cartan–Hadamard theorem'). *Suppose X is a complete Riemannian manifold of nonpositive curvature on which a compact group G acts by isometries. Then $X \to B\pi(X)$ is a G-equivalence, i.e., X is 'G-equivariantly aspherical'.*

Proof: For each subgroup H of G, X^H is totally geodesic in G, since H sends geodesics to geodesics. Hence, X^H is also geodesically complete with nonpositive curvature. By the *non*equivariant Cartan–Hadamard theorem, X^H is thus a disjoint union of aspherical manifolds. This being true for all H, X is equivariantly aspherical. □

Thus, when X is as in Theorem 1.4, we may replace $B\pi(X)$ by X. However, Theorem 1.4 is applicable only when one can prove the existence of G-maps into such a G-manifold X. Our Section 4 is devoted to such existence theorems.

We discussed the case where G acts trivially on X in [37], Proposition 1.2, and derived the corresponding version of the equivariant Novikov conjecture in Theorem 3.8 of the same paper. To handle more general group actions for geometrically interesting fundamental groups, it is useful to invoke the machinery of harmonic maps, as developed in [9, 40, and 7]. Typical results include the following.

THEOREM (4.1 below). *Suppose a compact Lie group G acts on a closed manifold M, and we are given an isomorphism* $H^1(M; \mathbb{Z}) \to \mathbb{Z}^n$.

Then there exists a smooth G-map $\varphi: M \to T^n$, *where* T^n *is a flat torus on which G acts by isometries, such that* φ^* *is an isomorphism on* $H^1(; \mathbb{Z})$.

THEOREM (4.5 below). *Suppose a finite group G acts on a closed manifold M, and we are given an isomorphism* $\pi_1(M, x_0) \to \pi_1(S, y_0)$ *for some* $x_0 \in M$, $y_0 \in S$, *S a closed surface of genus* $g > 1$. *(Once again, we do not require anything about* M^G.*) Then there exists a smooth G-map* $\varphi: M \to S$ *for some hyperbolic structure on S and action of G on S by isometries, such that* φ_* *is conjugate to the given map of fundamental groups.*

2. C^*-Algebras of Fundamental Groupoids

The main technical tool in our analysis will be to extend the method of Kasparov [19], which heavily uses the C^*-algebras of fundamental *groups*, to the case of fundamental *groupoids*. Though our applications will be to manifolds, for technical reasons, it will be useful to work with a slightly larger category of spaces. We collect together in this section the facts that we will need about C^*-algebras of fundamental groupoids.

2.1. DEFINITION. Let G be a compact Lie group and X a G-space. We shall call X *admissible* if the following conditions are satisfied.

(a) X is connected, locally compact, and second-countable.
(b) X has a canonical G-invariant measure class of full support.
(c) X is locally path-connected and semi-locally simply connected.

Note that (c) is required so that standard covering space theory applies to X.

2.2. EXAMPLES (a) The main example to keep in mind is that where X is a smooth connected G-manifold; the canonical measure class is, of course, the class of Lebesgue measure in any local coordinate chart.

(b) More generally, one can allow X to be a Lipschitz manifold with a Lipschitz G-action. There is still a canonical invariant measure class, since Lipschitz homeomorphisms of $\mathbb{R}^n$ preserve the class of Lebesgue measure (by Rademacher's Theorem [45, p. 272]).

(c) Alternatively, one can suppose merely that X is a connected, locally finite G-CW complex of 'uniform dimension' $n = \dim X$. (Thus, we suppose that cells of (nonequivariant) dimension n are dense.) The canonical measure class comes from the equivariant cells $(G/H) \times \mathbb{R}^d$ of maximal dimension (i.e., with $\dim(G/H) + d = \dim X$), where it is given by the product of invariant measure on G/H and Lebesgue measure on $\mathbb{R}^d$.

(d) Note that if X is an admissible G-space and Y is a compact G-invariant subspace of measure zero in X, then X/Y (the G-space obtained by collapsing Y to a point) is also admissible provided 2.1(c) still holds (which is true if Y is at all 'reasonable'), since measure theoretically, X and X/Y are the same. Some condition on Y is needed for this, as one can see from the example $X = [0, 1]$, $Y = \{0, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}$, but it is enough, for instance, to assume (X, Y) has the relative homotopy type of a finite CW-pair.

2.3. DEFINITION. Let X be an admissible G-space in the sense of 2.1. Recall that the *fundamental groupoid* $\pi(X)$ of X is the topological groupoid with unit space X and with morphisms the homotopy classes (rel. boundary) of paths $I \to X$. If we fix a universal covering $\tilde{X}$ of X and let π_1 be the corresponding group of covering transformations, then $X \cong \tilde{X}/\pi_1$ and $\pi(X) \cong (\tilde{X} \times \tilde{X})/\pi_1$. If we choose a measure μ in the canonical measure class of X, we may pull it back to a π_1-invariant measure on $\tilde{X}$ and, thus, in an obvious way get a Haar system λ on $(\tilde{X} \times \tilde{X})/\pi_1 \cong \pi(X)$ in the sense of [33]. We define $C^*(\pi(X))$ and $C^*_r(\pi(X))$ to be the corresponding full and reduced groupoid C^*-algebras as defined in [33]. These do not depend (up to $*$-isomorphism) on the choice of μ within our canonical measure class. See also [36, §2] for an exposition.

Note that though G may not act on $\tilde{X}$, G acts naturally on $\pi(X)$. Furthermore, by assumption, G is compact and preserves the class of μ. Thus (by averaging over Haar measure on G) we may choose μ to be G-invariant, and then G acts on the convolution algebra $(C_c(\pi(X)), \lambda)$, hence on $C^*(\pi(X))$ and on $C^*_r(\pi(X))$.

2.4. *Remark.* For all future purposes in this paper, it will not matter whether we use $C^*(\pi(X))$ or $C^*_r(\pi(X))$, though the latter is probably slightly more natural. (This is because its K-theory is likely to be closer to that of $B\pi(X)$.) For this reason, we shall generally write only C^*_r, with the understanding that C^* is an acceptable alternative.

Since our objective will be to use the C^*-algebra of the fundamental groupoid of a manifold M as an equivariant substitute for the C^*-algebra of the fundamental *group* in Kasparov's work [19], we want to show that when M has a G-fixed basepoint, the theory we get is identical to that which we discussed in [37, Theorem 3.8 and Remark 3.9]. This motivates the following discussion.

2.5. THEOREM. *Let X be an admissible G-space with a G-fixed point x_0. Then $C^*_r(\pi(X))$ and $C^*_r(\pi_1(X, x_0))$ (similarly for the full C^*-algebras) are G-equivariantly strongly Morita equivalent, and there is a natural isomorphism (of $R(G)$-modules) $K^G_*(C^*_r(\pi(X))) \cong K^G_*(C^*_r(\pi_1(X, x_0)))$.*

Proof. An imprimitivity bimodule E between these two algebras was constructed in [27, Theorem 2.8] by completing $C_c(\tilde{X})$ in a suitable norm, where $\tilde{X} = \pi(X)^{x_0}$. It is obvious that if the measure μ is chosen G-invariant as in 2.3, then G acts naturally on E. Now we need a fact which is familiar to anyone who has thought about the equivariant Kasparov functor KK^G. For convenience, we record the easy proof, though this lemma also appears in [19, Theorem 2.18] and in [8].

2.6. LEMMA. *Let G be a compact Lie group and let A and B be separable C^*-algebras equipped with actions of G. Suppose there exists an imprimitivity A-B*

bimodule E (in the sense of [34]) *equipped with an action of G compatible with the actions of G on A and B. Then E defines a KK^G-equivalence between A and B and, in particular, induces an isomorphism of equivariant K-groups $K_*^G(A) \to K_*^G(B)$.*

Proof. Since $\mathscr{K}(E) = A$, the graded Hilbert B-module $E \oplus 0$, together with $0 \in \mathscr{L}(E)$, defines an equivariant Kasparov A-B-bimodule, and thus an element $[E] \in KK^G(A, B)$. Similarly, the dual module $\tilde{E}$ (defined in [34, §6]) defines $[\tilde{E}] \in KK^G(B, A)$. Since, by [34, Lemma 6.22], $\tilde{E} \otimes_A E \cong B$ and $E \otimes_B \tilde{E} \cong A$, we have $[\tilde{E}] \otimes_A [E] = 1_B \in KK^G(B, B)$, $[E] \otimes_B [\tilde{E}] = 1_A \in KK^G(A, A)$. □

Using Theorem 2.5, we can now prove a seemingly much more powerful result (Theorem 2.8) below. First we need a lemma.

2.7. LEMMA. *Let Y be a manifold, $X \subset Y$ a 1-connected compact subcomplex of positive codimension. Then the map $Y \to Y/X$ induces an isomorphism $C_r^*(\pi(Y/X)) \to C_r^*(\pi(Y))$. If a compact Lie group G acts on Y and leaves X invariant, then this isomorphism is G-equivariant.*

Proof. The projection $Y \to Y/X$ obviously induces a continuous function $Y^I \to (Y/X)^I$ (here Y^I denotes the space of continuous maps $I = [0, 1] \to Y$). Since X is 1-connected, there results (say, by Van Kampen) a continuous surjection $\pi(Y) \to \pi(Y/X)$ and, thus, an injection $C_c(\pi(Y/X)) \to C_c(\pi(Y))$. Since X is of positive codimension in Y, $Y \to Y/X$ and $\pi(Y) \to \pi(Y/X)$ are measure-theoretic isomorphisms and, thus, this map preserves convolution multiplication and, thus, induces a map of C^*-algebras. That this is an isomorphism for the reduced C^*-algebras is easy to see, since $Y \to Y/X$ gives an isometry of the Hilbert spaces on which the regular representations act. To prove the corresponding result in the case of the full C^*-algebras, it is easiest to use [27, Theorem 3.1 and Remark on p. 19]. Fix $y_0 \in X$, let $\bar{y}_0$ be its image in Y/X, and choose a Borel cross-section $b\colon Y/X \to \pi(Y/X)^{\bar{y}_0}$. By lifting paths, we get a corresponding Borel map $\tilde{b}\colon Y/X \to \pi(Y)^{y_0}$. Since X has measure 0 in Y, [27] gives compatible isomorphisms

$$C^*(\pi(Y/X)) \cong C^*(\pi_1(Y/X, \bar{y}_0)) \otimes \mathscr{K}(L^2(Y/X))$$

and

$$C^*(\pi(Y)) \cong C^*(\pi_1(Y, y_0)) \otimes \mathscr{K}(L^2(Y/X)),$$

from which we see that our map of C^*-algebras is actually an isomorphism.

The equivariance of $C_r^*(\pi(Y/X)) \to C_r^*(\pi_1(Y))$ (or of the corresponding map of full C^*-algebras) is evident from the equivariance of $Y \to Y/X$ and of $\pi(Y) \to \pi(Y/X)$. □

Remark. The above result can easily be generalized to admissible spaces in the sense of Definition 2.1, assuming a condition as in 2.2(d).

2.8. THEOREM. *Let Y and Y' be connected G-manifolds (or more generally, admissible G-spaces), and suppose $f\colon Y \to Y'$ is a G-map which (nonequivariantly) induces an isomorphism of fundamental groups. (In other words, f is a G-pseudo-1-*

equivalence, but not necessarily an equivariant 1*-equivalence.) Then* f *induces an isomorphism* $K_*^G(C_r^*(\pi(Y))) \to K_*^G(C_r^*(\pi(Y')))$.

Proof. If Y has a fixed point y_0, this follows from Theorem 2.5, since f induces a G-equivariant group isomorphism $\pi_1(Y, y_0) \to \pi_1(Y', f(y_0))$.

Next observe that the result holds if $Y' = Y \times Z$, where Z is a 1-connected G-manifold with a G-fixed point z_0, and $f(y) = (y, z_0)$. The reason is that $\pi(Y') = \pi(Y) \times Z \times Z$, and thus that $C_r^*(\pi(Y')) \cong C_r^*(\pi(Y)) \otimes \mathscr{K}(L^2(Z))$ (canonically and G-equivariantly); in particular, Lemma 2.6 applies. Similarly, the result holds if $Y = Y' \times Z$, where Z is a 1-connected G-manifold and f is the projection onto the first factor, since in this case $C_r^*(\pi(Y)) \cong C_r^*(\pi(Y')) \otimes \mathscr{K}(L^2(Z))$ G-equivariantly.

Now we get the result in general by combining several special cases. First of all, by the result about products we may replace f by

$$f \times \mathrm{id}\colon Y \times \mathbb{R}^3 \to Y' \times \mathbb{R}^3,$$

where $\mathbb{R}^3$ has trivial action. Choose any $y \in Y$; then the suspension $\Sigma(G \cdot y)$ is 0-connected and the double suspension $\Sigma^2(G \cdot y)$ is 1-connected and embeds in $Y \times \mathbb{R}^3$. Crush $\Sigma^2(G \cdot y)$ and $\Sigma^2(G \cdot f(y))$ to points in $Y \times \mathbb{R}^3$ and in $Y' \times \mathbb{R}^3$, respectively. Now by applying Lemma 2.7, we reduce to the case where there are G-fixed basepoints, which we have already handled. □

For technical convenience later (since C^*-algebras of discrete groups have units, but C^*-algebras of fundamental groupoids do not), it will be useful to relate $K_*^G(C_r^*(\pi(X)))$, for X an admissible G-space, to the K-theory of the (reduced) group C^*-algebra of another transformation group.

Therefore fix a universal cover $\tilde{X} \xrightarrow{p} X$ of X and let π_1 be the group of covering transformations of $\tilde{X} \xrightarrow{p} X$. Recall that if we fix a basepoint $x_0 \in X$, then we may choose for $\tilde{X}$ the space of homotopy classes of paths in X beginning at x_0, i.e., $s^{-1}(x_0) \subset \pi(X)$, and then π_1 may be identified with $\pi_1(X, x_0)$. We assume that a compact Lie group G acts on X. Since G may not fix any basepoint, we try to avoid the latter point of view as much as possible. Instead, we may identify $\pi(X)$ (once $\tilde{X} \xrightarrow{p} X$ is given) with the quotient space $(\tilde{X} \times \tilde{X})/\pi_1$, where π_1 acts by the product action. The maps r, $s\colon (\tilde{X} \times \tilde{X})/\pi_1 \to X$ are given by projection onto the first and second factors, respectively.

Now let $\alpha\colon G \to \mathrm{Homeo}\,(X)$ be the action of G. For each $g \in G$, $\alpha(g)\colon X \to X$ can be lifted to a homeomorphism of $\tilde{X}$, which is not unique but is determined up to an element of π_1. Thus we obtain a locally compact transformation group Γ on $\tilde{X}$ which is a group extension

$$1 \to \pi_1 \to \Gamma \to G \to 1. \tag{2.9}$$

(Note Γ is closed in $\mathrm{Homeo}(\tilde{X})$ since $\alpha(G)$ is closed in $\mathrm{Homeo}(X)$, G being compact. Thus, Γ is a Hausdorff topological group and (2.9) is a topological group extension; then Γ is locally compact, being an extension of a compact group by a discrete group.)

The extension (2.9) will split as $\Gamma \cong \pi_1(X, x_0) \rtimes G$ if X has a G-fixed basepoint x_0, but not in general.

2.10. PROPOSITION. *$C_r^*(\pi(X)) \rtimes G$ is strongly Morita equivalent to $C_r^*(\Gamma)$. Thus $K_*^G(C_r^*(\pi(X))) \cong K_*(C_r^*(\Gamma))$.*

Proof. Form a new topological groupoid E, with object space X, and where the morphisms $x \to y$ are given by pairs (g, γ) with $g \in G$, $\gamma \in \pi(X)$, and $r(\gamma) = y$, $s(\gamma) = \alpha(g)(x)$. We write such a morphism for short as

$$x \xrightarrow{g} s(\gamma) \xrightarrow{\gamma} y.$$

Composition of morphisms is then defined by the formula

$$(y \xrightarrow{g_2} s(\gamma_2) \xrightarrow{\gamma_2} z) \circ (x \xrightarrow{g_1} s(\gamma_1) \xrightarrow{\gamma_1} y) = (x \xrightarrow{g_2 g_1} g_2 \cdot s(\gamma_1) \xrightarrow{\gamma_2 \circ g_2 \cdot \gamma_1} z);$$

note that

$$(x \xrightarrow{g} s(\gamma) \xrightarrow{\gamma} y)^{-1} = (y \xrightarrow{g^{-1}} g^{-1} \cdot y \xrightarrow{g^{-1} \cdot \gamma^{-1}} x).$$

It is easily checked that the axioms for a locally compact groupoid are satisfied and that E deserves to be called the semidirect product $\pi(X) \rtimes G$. Furthermore, the same proof that for groups shows that $C_r^*(H) \rtimes_r G \cong C_r^*(H \rtimes G)$ (when a group G acts on a group H) shows that $C_r^*(\pi(X)) \rtimes_\alpha G \cong C_r^*(E)$. (It is not necessary to take the reduced crossed product by α since G is amenable.) Hence we need only show that $C_r^*(E)$ and $C_r^*(\Gamma)$ are strongly Morita equivalent. This now follows immediately from Theorem 3.1 of [27], since E is a second-countable, transitive groupoid, with unit space X, and we may identify Γ with the group $E_{x_0}^{x_0}$ for some $x_0 \in X$.

The last statement about equivariant K-theory follows from a theorem of Green and Julg (see [17] or [2, Theorem 11.7.1]). □

3. The Equivariant Kaminker–Miller and Kasparov Theorems

We come now to the main technical tools of this paper, which are the generalizations to the equivariant setting of the results of [18] and [19]. We begin by defining an equivariant version of the symmetric signature of Ranicki [31] and Miščenko [25], and by noting its invariance properties.

3.1. DEFINITION. Let M^{2n} be a closed, connected, oriented, smooth manifold of even dimension, and suppose a *finite* group G acts smoothly on M. Let $\tilde{M}$ be a universal cover of M and let Γ be the group acting on M generated as in (2.9) by lifts of the action of G and by the group π_1 of covering transformations. Choose a finite cell decomposition of M, with respect to which G acts cellularly. This gives a cell decomposition of $\tilde{M}$, usually infinite, with respect to which Γ acts cellularly, and the associated cellular chain complex may be viewed as a finite complex $C_.(M; \mathbb{C}[\Gamma])$ of $\mathbb{C}[\Gamma]$-modules.

3.2. LEMMA. *The* $\mathbb{C}[\Gamma]$*-modules* $C_j(M; \mathbb{C}[\Gamma])$ *are finitely generated and projective.*

Proof. Note Γ acts properly discontinuously on $\tilde{M}$, so all its isotropy groups are finite. Thus $C_j(M; \mathbb{C}[\Gamma])$ is a finite direct sum of modules of the form $\mathbb{C}[F\backslash\Gamma]$ with $F \subseteq \Gamma$ a finite subgroup. But $\mathbb{C}[F\backslash\Gamma] = p\mathbb{C}[\Gamma]$, where $p \in \mathbb{C}[F] \hookrightarrow \mathbb{C}[\Gamma]$ is the idempotent corresponding to the trivial representation of F, and so is projective, in fact a direct summand of a rank-one free module. (This works even with $\mathbb{Q}$ in place of $\mathbb{C}$, but not with $\mathbb{Z}$, since we need to invert $|F|$. Cf. Proof of Theorem 3.1 in [37].) □

Now the Poincaré duality pairing produces as usual a symmetric signature $\sigma(M, G) \in L^{2n}(\mathbb{C}[\Gamma])$ $(= L_0(\mathbb{C}[\Gamma])$, since we are working over $\mathbb{C}$, which contains $\frac{1}{2}$ and $i = \sqrt{-1})$. A basic observation is Proposition 3.3.

3.3. PROPOSITION. *The symmetric signature* $\sigma(M, G)$ *is invariant under orientation-preserving pseudoequivalences.*

Proof. Let $M \xrightarrow{h} M'$ (M and M' as in Definition 3.1 above) be a G-map and (forgetting the G) an orientation-preserving homotopy equivalence. Then M and M' have the same fundamental group, and lifting h to the universal covers, we see the corresponding Γ's are the same and we have a Γ-map $\tilde{M} \to \tilde{M}'$. This gives a chain map

$$C_{\cdot}(M; \mathbb{C}[\Gamma]) \xrightarrow{h_{\cdot}} C_{\cdot}(M'; \mathbb{C}[\Gamma]).$$

Let $C(h_{\cdot})$ be the algebraic mapping cone. Since $M \xrightarrow{h} M'$ is a homotopy equivalence (forgetting the G), $C(h_{\cdot})$ is acyclic as a complex of $\mathbb{C}[\pi_1]$-modules, hence acyclic as a complex of $\mathbb{C}[\Gamma]$-modules (a $\mathbb{C}[\Gamma]$-module which is zero as a $\mathbb{C}[\pi_1]$-module is zero, period!). Thus $h_{\cdot}$ is a chain equivalence and $\sigma(M, G) = \sigma(M', G)$. □

Next, we have to relate the invariant $\sigma(M, G)$ to G-equivariant analysis on M. This is where the C^*-algebra of the fundamental groupoid comes in. But first note that the inclusion $\mathbb{C}[\Gamma] \hookrightarrow C_r^*(\Gamma)$ sends $\sigma(M, G)$ to an invariant in

$$L_0(C_r^*(\Gamma)) \cong K_0(C_r^*(\Gamma)) \cong K_0(C_r^*(G \ltimes \pi(M))) \cong K_0^G(C_r^*(\pi(M)))$$

which we call $\bar{\sigma}(M, G)$. The equivariant Kaminker–Miller theorem will identify this with a suitable index of an equivariant signature operator on M. First we need a suitable flat bundle on M on whose sections this operator will act.

Recall that if A is a C^*-algebra, an A-bundle over a topological space X is defined the same way as a vector bundle, except that the fibers of the bundle are right A-modules and the transition functions are A-linear. When A has a unit and X is compact, $K^0(X; A)$ denotes the Grothendieck group of such bundles with fibers that are finitely generated and projective over A. This group is isomorphic to $K_0(C(X) \otimes A)$. When A does not have a unit, the definition is adjusted so that one still has $K^0(X; A) \cong K_0(C(X) \otimes A)$. Everything can be made equivariant for a compact group G in the usual way.

3.4. PROPOSITION. *Let* M *be a compact* G*-manifold, where* G *is a compact Lie* *group, and let $A = C^*_r(\pi(M))$ (which carries an obvious action of G). Define a bundle $\mathscr{Y}$ of Hilbert A-modules over M by*

$$\mathscr{Y}_y = \text{completion of } C_c^\infty(r^{-1}(y), \Omega^{1/2})$$

as in [15]. *Note that G operates on $\mathscr{Y}$ in an obvious fashion compatible with the actions on M and on A. They $\mathscr{Y}$ is a flat A-bundle over M and defines a canonical class in $K^0_G(M; A)$, hence (by Kasparov product) a map $\beta: K^G_*(M) \to K^G_*(A)$.*

Proof. As in [15], we want to prove that $\mathscr{K}(\mathscr{Y}_y) = C^*_r(\pi_1(M, y))$. Since this has an identity and the bundle of such algebras is obviously flat over M, this will prove the result, since $\mathscr{Y}$ is G-equivariant.

First, $C^*_r(\pi_1(M, y))$ acts on $\mathscr{Y}_y$ on the left, since if $\gamma \in \pi_1(M, y)$ and $\xi \in C_c^\infty(r^{-1}(y), \Omega^{1/2})$, so is $\gamma \cdot \xi$, defined by $\gamma \cdot \xi(\eta) = \xi(\gamma^{-1}\eta)$, and this action obviously commutes with the A-action on the right and preserves the A-valued inner product. The fact that $C^*_r(\pi_1(M, y))$ thereby injects in $\mathscr{L}(\mathscr{Y}_y)$ is as in [15]. If ξ, $\xi_1, \xi_2 \in C_c^\infty(r^{-1}(y), \Omega^{1/2})$, then

$$\begin{aligned}(\xi_1 \cdot \langle \xi_2, \xi \rangle)(\gamma) &= \int_{r(\gamma_1) = y} \xi_1(\gamma_1) \langle \xi_2, \xi \rangle (\gamma_1^{-1}\gamma) \\ &= \int_{r(\gamma_1) = y} \xi_1(\gamma_1) \sum_{\substack{s(\gamma_2) = r(\gamma_1^{-1}) \\ r(\gamma_2) = y}} \overline{\xi_2(\gamma_2)} \xi(\gamma_2 \gamma_1^{-1}\gamma) \\ &= \sum_{\eta \in \pi_1(M, y)} f(\eta)\xi(\eta\gamma)\end{aligned}$$

for suitable f, where

$$f(\eta) = \int_{\gamma_2\gamma_1^{-1} = \eta} \xi_1(\gamma_1)\overline{\xi_2(\gamma_2)}.$$

This shows $\mathscr{K}(\mathscr{Y}_y) \subseteq C^*_r(\pi_1(M, y))$ and in particular that $\mathscr{K}(\mathscr{Y}_y)$ has a unit. The equality is verified as in [15]. □

Next, we want to view the invariant $\tilde{\sigma}(M, G)$ as coming from the bundle $\mathscr{Y}$ and the ring $C^*_r(\pi(M))$. To this end, observe that since the group G acts on $C^*_r(\pi(M)))$, it makes sense to talk about an equivariant L-group $L^G_0(C^*_r(\pi(M)))$ constructed from Hermitian forms over $C^*_r(\pi(M))$ which are invariant under the G-action. (If lack of a unit in the ring makes the reader nervous, note that we can adjoin an identity and work with the ring $A^+ = C^*_r(\pi(M))^+$, then take the reduced group

$$\ker(L^G_0(A^+) \to L^G_0(\mathbb{C})) \text{ .)}$$

Note further that by Proposition 3.4, the fibers of $\mathscr{Y}$ are Hilbert A-modules for which $\mathscr{K}(\mathscr{Y}_y)$ is unital, and are thus finitely generated projective A^+-modules.

3.5. PROPOSITION. *The invariant $\tilde{\sigma}(M, G) \in K^G_0(C^*_r(\pi(M)))$ agress with the equivariant symmetric signature of the chain complex of twisted cellular chains of M with coefficients in the A-bundle $\mathscr{Y}$.*

Proof. If we forget the G for the moment, note that $\bar{\sigma}$ is defined using the canonical $C_r^*(\pi_1)$-bundle $\mathcal{V} = \tilde{M} \times_{\pi_1} C_r^*(\pi_1)$ over M. The sections of this bundle are obtained by completing $C_c(\tilde{M})$ for a suitable $C(M) \otimes C_r^*(\pi_1)$-valued inner product defined by

$$E(\langle \xi, \eta \rangle_{C(M) \otimes C_r^*(\pi_1)} \gamma)(m) = \sum_{p(\tilde{m}) = m} \bar{\xi}(\tilde{m}) \eta(\gamma \cdot \tilde{m}),$$

$$\xi, \eta \in C_c(\tilde{M}), \quad m \in M.$$

Here

$$\gamma \in \pi_1 \hookrightarrow \mathbb{C} \otimes C_r^*(\pi_1) \hookrightarrow C(M) \otimes C_r^*(\pi_1)$$

and $E : C(M) \otimes C_r^*(\pi_1) \to C(M)$ is the conditional expectation coming from the canonical trace on $C_r^*(\pi_1)$.

On the other hand, the sections $\Gamma(M, \mathcal{Y})$ are obtained by completing $C_c(\pi(M))$ for a suitable $C(M) \otimes C_r^*(\pi(M))$-valued inner product. We claim that

$$\Gamma(M, \mathcal{V}) \otimes_{C_r^*(\pi_1)} X \cong \Gamma(M, \mathcal{Y}),$$

where X is the $(C_r^*(\pi_1), C_r^*(\pi(M)))$-equivalence bimodule $\mathcal{Y}_{x_0}$ defined by a choice of basepoint x_0 in M, as constructed in the proof of Proposition 3.4, together with the corresponding identification of π_1 with $\pi_1(M, x_0)$. This can be seen by looking at the map sending $f \otimes \xi \mapsto \Xi$, where

$$f \in C_c(\tilde{M}) \cong C_c(s^{-1}(x_0)), \qquad \xi \in C_c(r^{-1}(x_0)),$$

and

$$\Xi(\gamma) = \sum_{\substack{\gamma = \gamma_1 \gamma_2 \\ r(\gamma_2) = s(\gamma_1) = x_0}} f(\gamma_1) \xi(\gamma_2).$$

Note that the map is $\mathbb{C}[\pi_1(M, x_0)]$-balanced, so does indeed give a map

$$C_c(\tilde{M}) \otimes_{\mathbb{C}[\pi_1]} C_c(r^{-1}(x_0)) \to C_c(\pi(M))$$

with dense image, which on completion yields the desired isomorphism of $C(M) \otimes C_r^*(\pi(M))$-modules.

The upshot of all this is that the map $\beta : K_*(M) \to K_*(A)$ defined by $\mathcal{Y}$ agrees (modulo the Morita equivalence of Theorem 2.5) with the map $\beta : K_*(M) \to K_*(C_r^*(\pi_1))$ of [19], and similarly the cellular symmetric signature invariant defined using $\mathcal{V}$ and $C_r^*(\pi_1)$ agrees with that defined using $\mathcal{Y}$ and A. So we have only to match up the G-actions. That this can be done will be another consequence of Proposition 2.10, for the same considerations as above will show that $\bar{\sigma}(M, G) \in K_0(C_r^*(\Gamma))$ corresponds under the Morita equivalence to a symmetric signature for the ring $C_r^*(G \ltimes \pi(M))$. This in turn evidently corresponds to the G-equivariant symmetric signature defined using $\mathcal{Y}$ and A. □

3.6. THEOREM (Equivariant Kaminker–Miller). *Let the manifold M^{2n} and the finite group G be as in Definition* 3.1. *Choose a G-invariant Riemannian structure on*

M and a G-invariant flat connection on the A-vector bundle $\mathcal{Y}$, and let D be the Atiyah–Singer signature operator, $D_{\mathcal{Y}}$ the operator with coefficients in $\mathcal{Y}$, as defined by the connection. Then

$$G\text{-}Ind(D_{\mathcal{Y}}) = \bar{\sigma}(M, G) \quad \text{in } K_0^G(C_r^*(\pi(M))).$$

Proof. We use the interpretation of $\bar{\sigma}(M, G)$ given by Proposition 3.5, and then observe that the argument of Kaminker and Miller in [18] can be made equivariant for a finite group of isometries. The only difficulty comes from the fact that the ring A does not have an identity, so it is not immediately clear that the equivariant A-index of $D_{\mathcal{Y}}$ is well-defined. Therefore we appeal to §6 of [18], which shows that everything will work provided the complex $\Omega^{\cdot}(M, \mathcal{Y})$, completed in suitable Sobolev norms, gives a *quasi-regular* Hermitian Fredholm A-complex or, in other words, that we can go up to the unitalization A^+. This again follows from the proof of Proposition 3.4, which showed that the fibers of $\mathcal{Y}$ may be regarded as finitely generated projective A^+-modules. □

3.7 THEOREM (Equivariant Kasparov). *Let W be a complete Riemannian manifold of non-positive curvature on which a compact Lie group G acts by isometries, and let $A = C_r^*(\pi(W))$. Then the map $\beta : K_*^G(W) \to K_*^G(A)$ of Proposition* 3.4 *is split injective.*

Proof. Let $\tilde{W}$ be the universal cover of W and let Γ as in (2.9) be the group of isometries of $\tilde{W}$ generated by the group π_1 of covering transformations and by lifts of the elements of G. As we saw in Proposition 2.10, there is a natural isomorphism $K_*^G(A) \cong K_*(C_r^*(\Gamma))$. As Kasparov points out in [19], β really comes from an element

$$[d_{\tilde{W}}] \in KK_{\mathrm{Isom}(\tilde{W})}(C_\tau(\tilde{W}), \mathbb{C}),$$

where $C_\tau(\tilde{W})$ is the algebra of sections vanishing at infinity of the Clifford algebra bundle of the cotangent bundle of $\tilde{W}$. The main results of [19], Theorems 5.3 and 6.7, say there is also an element

$$\eta_{\tilde{W}} \in KK_{\mathrm{Isom}(\tilde{W})}(\mathbb{C}, C_\tau(\tilde{W})),$$

with

$$[d_{\tilde{W}}] \otimes_{\mathbb{C}} \eta_{\tilde{W}} = 1_{C_\tau(\tilde{W})}.$$

Clearly, we may restrict attention from $\mathrm{Isom}(\tilde{W})$ down to Γ. Then we obtain classes

$$[d_{\tilde{W}}] \in KK_\Gamma(C_\tau(\tilde{W}), \mathbb{C}), \qquad \eta_{\tilde{W}} \in KK_\Gamma(\mathbb{C}, C_\tau(\tilde{W})),$$

and applying the induction functor j of [19] gives

$$j([d_{\tilde{W}}]) \in KK(C_\tau(\tilde{W}) \rtimes \Gamma, C_r^*(\Gamma)), \qquad j(\eta_{\tilde{W}}) \in KK(C_r^*(\Gamma), C_\tau(\tilde{W}) \rtimes \Gamma),$$

with

$$j([d_{\tilde{W}}]) \otimes_{C_r^*(\Gamma)} j(\eta_{\tilde{W}}) = 1_{C_\tau(\tilde{W}) \rtimes \Gamma}.$$

Now we are done, since as Kasparov shows, β is the composite

$$K_*^G(W) \xrightarrow[\cong]{\text{Poincaré duality}} K_*^G(C_\tau(W))$$
$$= K_*^G(C_\tau(\tilde{W}/\pi_1)) \xrightarrow[\cong]{\text{Green-Julg}} K_*(C_\tau(\tilde{W}/\pi_1) \rtimes G)$$
$$\xrightarrow[\cong]{\text{Morita equivlence}} K_*(C_\tau(\tilde{W}) \rtimes \Gamma) \xrightarrow{j([d_{\tilde{W}}])} K_*(C_r^*(\Gamma)),$$

and Kasparov product with $j(\eta_{\tilde{W}})$ followed by Poincaré duality will provide a splitting map. □

4. Construction of Equivariant Maps

In this section, we shall construct equivariant maps from a given manifold with a group action to certain equivariant aspherical manifolds. Our method is similar to that in [40] and is based on the theory of harmonic maps. We begin with the most elementary case, a theorem for which we know no short comprehensible purely topological proof, and which is, in fact, a little trickier than the more exotic cases. A *standard cubical n-torus* will just mean T^n realized as $\mathbb{R}^n/\mathbb{Z}^n$.

4.1. THEOREM. *Let M be a smooth manifold and suppose a compact Lie group G acts smoothly on M. There is then an affine G-action on a (standard cubical) torus T, and an equivariant map $\phi: M \to T$ inducing an isomorphism on $H^1(; \mathbb{Z})$.*

Remark. Of course, the theorem implies the PL version, or even the version where the manifold is homotopy equivalent to a G-CW complex. Nonetheless, the proof is decidedly smooth.

Proof. Give M a G-invariant Riemannian metric. Let $b \in M$ be a basepoint picked arbitrarily from M. We define $T = \mathrm{Hom}(H^1(M; \mathbb{Z}), \mathbb{R}/\mathbb{Z})$. We define an affine action on T as follows. If $\alpha \in T$ and c is a cohomology class with harmonic representative ω, let

$$(g \cdot \alpha)(c) = \alpha(g^*c) + \int_b^{g \cdot b} \omega \bmod \mathbb{Z}.$$

One readily checks that this defines a group action.

The map $\phi: M \to T$ is defined by analogy to the Jacobi map in the classical theory of algebraic curves and is given in terms of harmonic forms. For a cohomology class c with harmonic representative ω, one defines

$$\phi(m)(c) = \int_b^m \omega \in \mathbb{R}/\mathbb{Z}.$$

The integral is taken over any path, and is well defined because the ambiguity is the integral of ω over a closed loop, which is in integer, as c is an integral class. (Note this map is canonically given by a Riemannian manifold with a choice of base point.) One readily checks that ϕ is equivariant, and the theorems of Hodge and DeRham easily combine to yield the isomorphism on $H^1(\,;\mathbb{Z})$. □

4.2. (Well Known) COROLLARY. *Every flat manifold has a flat metric whose toral cover is cubical.*

Proof. From Bieberbach's theorem, the flat manifold can be described as the quotient of a torus under some affine group of transformations. The associated action constructed above has cubical cover.

4.3. COROLLARY. *Suppose $\pi_1(M) = \pi_1(F)$, where F is a closed flat manifold. If G acts smoothly on M, then there is another flat Riemannian manifold F', diffeomorphic to F, on which G acts affinely, and for which one can construct an equivariant map $\phi: M \to F'$ inducing an isomorphism on π_1. F' can be taken with cubical cover.*

Proof. One considers the cover of M with a characteristic free Abelian fundamental group, and considers the group of all lifts of all group elements, and applies the previous theorem to this larger group action. Then one takes the quotient under the group of covering translates.

We conjecture (but can scarcely believe) that a similar result might be true for closed manifolds of nonpositive curvature. In general, the above shows that it is necessary to deform metrics. In cases where the moduli space is a point, no deformations should be necessary. In general, it seems more delicate to construct the group action on the 'model manifold' than to construct the equivariant map. Where we succeed, the problem is either trivial or a consequence of a rigidity theorem.* In the 'rigid' cases below, the group G is to be finite.

4.4. THEOREM. *Let W^n be a locally symmetric space of finite volume, of noncompact type and with no local one- or two-dimensional factors. Suppose G acts on M and one has homorphisms $\pi_1 M \to \pi_1 W$, with noncentral image, and $G \to \mathrm{Out}\ \pi_1 W$ that are compatible (e.g., one has an isomorphism $\pi_1 M \to \pi_1 W$). Then there is an action of G on W by isometries, and an equivariant map $\phi: M \to W$ realizing all the above data.*

Proof. The action is constructed by a direct application of Mostow rigidity [26], since each self-homotopy-equivalence of W is realized by an isometry. Because W has nonpositive curvature, according to [9] there is a harmonic map in any free homotopy class of maps $M \to W$. According to [40], the harmonic map is unique in our case. Since the compositions of harmonic maps with isometries are harmonic, uniqueness implies equivariance. □

In the hyperbolic case, one can also handle dimension 2.

4.5. THEOREM. *Let W^2 be a surface of genus $\geqslant 2$. Suppose G acts on M and one has homomorphisms $\pi_1 M \to \pi_1 W$, with noncentral image, and $G \to \mathrm{Out}\ \pi_1 W$ that are compatible (e.g., one has an* isomorphism *$\pi_1 M \to \pi_1 W$). Then there is an action of G*

* To get a feeling for what is involved in an incredibly special case, we propose the following question: Suppose that a torsion-free group contains as a normal subgroup of finite index the fundamental group of a nonpositively curved closed manifold. Is the group such itself?

on W by isometries for some hyperbolic metric, and an equivariant map $\phi: M \to W$, realizing all the above data.

Proof. The construction of the metric and action on W is produced by Kerckhoff's proof of the Nielsen realization conjecture [20]. The remainder of the argument is identical. □

Finally, we point out that if the map $G \to \mathrm{Out}\, \pi_1 W$ is trivial, then the action on W can be trivially found. In this case, the existence of a harmonic map implies that $\phi: M \to W$ factors through M/G. This condition is very useful, and the equivariant Novikov conjecture in this case was already studied in [37].

4.6. THEOREM. *Let W^n be a closed manifold of nonpositive curvature or a complete locally symmetric space of noncompact type, perhaps with infinite volume. Suppose G acts on M and one has a homomorphism $\pi_1 M \to \pi_1 W$, with noncentral image, or in the second case, Zariski dense image. Assume that the map $G \to \mathrm{Out}\, \pi_1 M$ induced by the action is trivial. Then there is a map $\phi: M/G \to W$ whose composition with the projection induces the given homomorphism on π_1.*

Proof. The harmonic maps are constructed by [9] and [7] in the two cases respectively. □

4.7. *Remark.* This statement should be true in much greater generality. For instance, it should apply to any $K(\pi, 1)$ for π a hyperbolic group in the sense of [13]. By consideration of free actions, this would have interesting group-theoretic implications.

5. Applications

This section combines the results of the previous four sections to yield some topological and differential geometric conclusions. For more information on the geometrical topological aspects of the problem, we refer the reader to the second author's survey of the Novikov conjecture [44].

5.1. THEOREM. *Suppose W is a complete manifold of nonpositive curvature, and a compact Lie group G acts by isometries on W. If G is not finite, also assume that W has the proper G-homotopy type of a finite G-CW complex, or more generally, that $K_G^*(W)$ is finitely generated over $R(G)$. Let M and M′ be any closed oriented n-manifolds with G-actions, and suppose $M' \xrightarrow{h} M \xrightarrow{f} W$ are G-equivariant maps for which the first map h is orientation-preserving and a(n unequivariant) homotopy equivalence. Then $f_*(\Delta(M)) = (f \circ h)_*(\Delta(M'))$ in $K_n^G(W)$. Equivariant maps f to W exist in the situations described in Section 4.*

Proof. Because of technical difficulties in proving the analogue of the Kaminker–Miller theorem for general compact Lie groups, we reduce part of the proof to the case where G is finite. For this we need a slight variant of a theorem of J. McClure.

5.2. PROPOSITION. *Let X be a (possibly noncompact) smooth G-manifold, and let $\delta \in K_*^G(X)$ or $KO_*^G(X)$. (Recall this is $\varinjlim K_*^G(X_n)$ or $\varinjlim KO_*^G(X_n)$, where $\{X_n\}$ is a* *sequence of finite G-CW-complexes exhausting X.) Assume X has the proper G-homotopy type of a finite G-CW-complex, or more generally, that* $K_G^*(X)$ *is finitely generated over R(G) (or in the real case, that* $KO_G^*(X)$ *is finitely generated over* $KO_G^*(pt)$*). Then* $\delta = 0$ *if and only if the restriction of* δ *to* $K_*^H(X)$ *or* $KO_*^H(X)$ *vanishes for each finite subgroup H of G.*

Proof. Recall [22] that McClure proved the dual statement for *K-co*homology of *finite* G-CW-complexes. We may choose the X_n's to be compact submanifolds of X (with boundary). Assuming for simplicity that X is spin^c in the complex case, spin in the real case (otherwise one merely needs to twist appropriately), we can choose the X_n's to be spin^c or spin as well. Then by Poincaré duality (see [Kasparov, Theorem 4.10]), $K_*^G(X_n) \cong K_G^*(X_n, \partial X_n)$, and these converge to $K_G^*(X)$ (K-theory *with compact support*). Since the latter is assumed finitely generated and $R(G)$ is Noetherian, the sequence stabilizes and we may replace X by X_n with n sufficiently large. Now $\delta = 0$ if and only if the Poincaré dual $D\delta$ of δ is stably trivial on X_n. Since X_n is a finite G-CW-complex, we can apply McClure's Theorem. □

Remark. One can easily extend this from smooth G-manifolds to arbitrary G-CW-complexes satisfying the finite generation condition.

Proof of 5.1. (contd): Let δ be the difference of the images of $\Delta(M)$ and of $\Delta(M')$ in $K_n^G(W)$. We must show $\delta = 0$, so by Proposition 5.2, it is enough to assume G is finite. By the equivariant Kasparov theorem (Theorem 3.7 above), $\beta: K_*^G(W) \to K_*^G(C_r^*(\pi(W)))$ is injective, so we need only show that $\beta(\delta) = 0$. Now this follows from Proposition 3.3, from the equivariant Kaminker–Miller theorem (Theorem 3.6 above), and from the KK-index theorem that says that G-$\text{Ind}(D_{f^*(\mathscr{V})}) = \beta(\Delta)$ (cf. [19] or [2, §24]). □

Following [29] we call an equivariant map which is an unequivariant homotopy equivalence a pseudoequivalence. Unhappily, pseudoequivalence is not an equivalence relation, and the classification of manifolds pseudo-equivalent to a given one (up to concordance) is in terrible shape. (Petrie gives some sufficient conditions for constructing pseudoequivalences under very restrictive fundamental group hypotheses, and brilliantly applies this machinery to construct many rather exotic actions on familiar manifolds like the sphere.) It is a challenging problem to understand even all 'signature' type invariants. We find the following examples interesting:

5.3. EXAMPLE. Let M be a simply connected manifold. Consider a semifree group action on a sphere with a circle as fixed set. Manifolds pseudo-equivalent to the product of M and this sphere have as fixed set manifolds $\mathbb{Z}_{(p)}$-homology equivalent to $M \times S^1$. (This is because of P. A. Smith theory; see [3].) In addition to just the ordinary signature of the fixed set, many of the invariants of knot theory are defined in this generality. (See [5].) (If M is a sphere, one performs surgery on the generating circle, and obtains a knot in a rational homology sphere.) It is well known that there are many signature-related invariants of knots. (See, for a textbook reference, [35].)

Another peculiar feature of the equivariant setting is that unlike the manifold case, we do not know whether the smooth and topological versions of the Novikov conjecture are equivalent. Unequivariantly, one knows that Top/O has finite homotopy groups and the Novikov conjecture is a rational statement. Equivariantly, however, the classifying spaces for bundle theories are radically different (see [6, 23, 32]) and one cannot argue that way. However, our theorem is true topologically (for locally linear G-actions), i.e. we have the following theorem.

5.4. THEOREM. *The result of Theorem* 5.1 *still holds if M and M' are only topological manifolds with locally linear G-actions. (W is still as before.)*

Proof. We can sketch two rather different arguments. The first makes use of the theorem of [39] that produces equivariant Lipschitz structures for topological locally linear G-manifolds. At that point, the analysis from [38] (based on earlier work of [41] and [14]) takes over and allows one to redo all the above arguments in the Lipschitz category.

The second argument is more indirect and makes use of the results of [11] and [12] on equivariant topological rigidity. We shall sketch it below.

For manifolds, it is well known that the Novikov conjecture is equivalent to the injectivity of a certain 'assembly map' from group (K-)homology to (localized) L-theory. We have the following proposition.

5.5. PROPOSITION. *For a finite group G and a G-aspherical G-space W, the following are equivalent:*

(a) *the equivariant Novikov conjecture for pseudoequivalences between topological locally linear G-manifolds**, *i.e. the conclusion of Theorem* 5.1, *but with M and M' only topological manifolds with locally linear G-actions;*
(b) *the equivariant Novikov conjecture for equivariant homotopy equivalences between topological locally linear G-manifolds**, *i.e. the same statement, but with h required to be an equivariant homotopy equivalence;*
(c) *the rational injectivity of an assembly map*

$$K_*^G(W) \to L^*(\Gamma) \otimes \mathbb{Q},$$

where Γ is the 'orbifold' fundamental group of W/G, which equals the fundamental group of the quotient of the principal orbits (assuming each fixed set has codimension at least three).

Clearly (a) implies (b). The method of proof of (a) in this paper is basically that (c) implies (a). The fact that (b) implies (c) follows from equivariant surgery due to Madsen-Rothenberg [23] for the odd-order case (assuming gap hypotheses) and the second author in general. In short, if the assembly map were not injective and one had an element in the kernel, then surgery would enable one to construct an equivariant homotopy equivalence, which is a homeomorphism on the lower strata,

* Or more generally, 'weakly stratified actions' in the sense of [30].

for which the difference between equivariant signature classes (the normal invariant) is, when pushed into $K_*^G(W)$, the given element. □

Proof of 5.4. (contd). The second proof of pseudoequivalence invariance uses the implication (b) ⇒ (a) in Proposition 5.5. It stems from the result of [12] (see also [11]) that any manifold equivariantly homotopy equivalent rel ∞ to an equivariant nonpositively curved manifold is homeomorphic to it after crossing with Euclidean space. (As with ordinary surgery, a sufficiently good understanding of any particular G-manifold implies an understanding of all manifolds of the same stratified dimension of the same isovariant 2-type. In particular, the equivariant Borel conjecture (which is false, as noted in [11]), which asserts the topological rigidity of equivariant aspherical manifolds, would imply all the necessary facts for classification of G-manifolds of the given 2-types.) □

5.6. *Remark.* Another application of the current results concerns maps that are even less than pseudoequivalences. The idea is this. Any equivariant map between G-spaces induces an $R\mathbb{Q}(G)$-module map between their homology groups. If the quotient map splits on π_1, the induced map on twisted homology also is a map of $R\mathbb{Q}(G)$-modules. (See Section 4 and [37] for cases where this is automatic.) For pseudoequivalence, one asks that the map be an isomorphism, but it is possible to ask what happens with the assumption that the map on homology be an isomorphism only at certain irreducible representations. For instance, if one asks for a map to a point which is an equivalence with respect to the augmentation ideal, then one is studying the type of homologically trivial action studied in [42, 43]. One can break up $K_*^G(B\pi)$ and $K_*(C_r^*(\Gamma))$ similarly, and a little thought shows that localized pesudoequivalences preserve the pieces corresponding to the relevant representations. Finally, it is possible to apply the K-theory localization formula to the situation to get information about fixed sets and the like. We shall not formulate the most general theorem of this sort, but just point out:

5.7. COROLLARY (of the discussion). *If $\mathbb{Z}_n$ acts homologically trivially on a manifold M with fundamental group π which is the fundamental group of a manifold of nonpositive curvature, then there is a characteristic class $D(\xi)$ of the equivariant normal bundle of the fixed set F, such that*

$$f_*(\mathbb{L}(M) \cap [M]) = (fi)_*(D(\xi) \cdot \mathbb{L}(F) \cap [F]) \in H_*(B\pi; \mathbb{Q}),$$

where $f\colon M \to B\pi$ and $i\colon F \hookrightarrow M$.

If the action were semifree, this would be true under the weaker condition that the unequivariant Novikov conjecture is true for π. For this, and a discussion of D, see [43, II].

We close the paper with the observation that as in [37] we can deduce the following from our method:

5.8. THEOREM. *If G acts smoothly by spin-preserving isometries on a closed spin manifold M^n of positive scalar curvature, and $f\colon M \to W$ is an equivariant map to a* *complete G-manifold of nonpositive sectional curvature (satisfying the finiteness condition of Theorem 5.1 if G is not finite), then* $f_*[\not{D}] = 0 \in KO_n^G(W)$, *where* $\not{D}$ *denotes the Dirac operator on M.*

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Appendix: G-Spaces and Fundamental Groupoids by J. P. May*

For purposes of equivariant generalization of the Novikov conjecture, Rosenberg and Weinberger want a functorial map of G-spaces $f: X \to B\pi(X)$, where $B\pi(X)$ is a G-space which encodes the structure of the fundamental groupoids of all the fixed point spaces X^H while discarding all higher homotopy groups and where f preserves components of fixed point spaces and their fundamental groups. We use work of Fiedorowicz [2] and Elmendorf [1] to produce such gadgets.

As usual in this kind of work, we allow functoriality to include natural arrows which point the wrong way but are homotopy equivalences. This is weaker than space level functoriality but much stronger than mere functoriality up to homotopy. When we work on the fixed-point-data level, this kind of functoriality is good enough to allow us to apply Elmendorf's coalescence functor to get the same kind of functoriality on the equivariant, G-space, level.

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We shall need to define the fundamental groupoid of a space in terms of Moore loops. For a space X, let ΠX denote the set of paths (β, s), $\beta : [0, s] \to X$. We can set $\beta(t) = \beta(s)$ for $t \geqslant s$ and so regard (β, s) as a point of $\mathrm{Map}([0, \infty], X) \times [0, \infty]$; we give ΠX the resulting subspace topology. We regard ΠX as a topological category with object space X and morphism space ΠX. The structural maps source, target, identity, and composition are specified by $S(\beta, s) = \beta(0)$, $T(\beta, s) = \beta(s)$, $I(x) = (c(x), 0)$, where $c(?)$ denotes the constant function at ?, and

$$C((\alpha, r), (\beta, s)) = (\alpha \cdot \beta, r + s), \quad \text{if } \alpha(0) = \beta(s),$$

where

$$\begin{cases} (\alpha \cdot \beta)(t) = \beta(t), & \text{if } 0 \leqslant t \leqslant s \\ \quad \text{and} & \\ (\alpha \cdot \beta)(t) = \alpha(t - s), & \text{if } s \leqslant t \leqslant r + s. \end{cases}$$

We abbreviate $(\beta, s) = \beta$ henceforward.

Regard X itself as a topological category with object and morphism space X and S, T, I and C all the identity map of X. We have an obvious continuous functor $\iota : X \to \Pi X$ given by the identity maps of ΠX. The resulting map from X to the space of q-tuples of composable arrows of ΠX is clearly a homotopy equivalence for each q, and it follows that ι induces a homotopy equivalence $X = BX \to B\Pi X$ on passage to classifying spaces.

Define an equivalence relation on ΠX as follows: two paths α and β from x to y are equivalent if there is a path $h : I \to \Pi X$ such that $h(0) = \alpha$, $h(1) = \beta$, and each $h(t)$ is a path from x to y. Let πX (small π) denote the quotient *set* $\Pi X/\approx$, and regard πX as a discrete category with object *set* X in the evident way. This is the usual fundamental groupoid, defined in terms of Moore paths. Let $\pi^\tau X$ denote $\Pi X/\approx$ with the quotient space topology, and regard $\pi^\tau X$ as a quotient topological category of ΠX, with object space X. (Aside to point-set topology worriers: the possibly lousy nature of this topology should not cause problems.) We then have a continuous quotient functor $\phi : \Pi X \to \pi^\tau X$. We also have a continuous functor $v : \pi X \to \pi^\tau X$ given by the set-theoretic identity functions on objects and morphisms.

PROPOSITION. *The functor v induces a homotopy equivalence on passage to classifying spaces.*

Proof. For a topological category $\mathscr{C}$, let $\mathscr{C}^I$ denote the topological category obtained by applying $\mathrm{Map}(I, ?)$ to the object and morphism spaces of $\mathscr{C}$ and to the structural maps, S, T, I and C. Let $J : \mathscr{C} \to \mathscr{C}^I$ be the functor which sends objects and morphisms to the corresponding constant functions and let $K : \mathscr{C}^I \to \mathscr{C}$ be the functor obtained by evaluation at zero. According to [2, 1.4], it suffices to construct a continuous natural transformation $\eta : JK \to \mathrm{Id}$ of functors $(\pi^\tau X)^I \to (\pi^\tau X)^I$. For an object $f : IK \to X$ of $(\pi^\tau X)^I$, define a morphism $\eta(f) : I \to \Pi X$ of $(\pi^\tau X)^I$ by letting

$\eta(f)(t) = (f \mid [0, t], t)$. The source of $\eta(f)$ is $c(f(0)) = JK(f)$; the target is f. To see the naturality of η, consider a morphism $\omega: f \to g$ in $(\pi^\tau X)^I$. Then

$$[\omega \cdot \eta(f)](t) = \omega(t) \cdot (f \mid [0, t]) \quad \text{and} \quad [\eta(g) \cdot JK(\omega)](t) = (g \mid [0, t]) \cdot \omega(0).$$

These are not equal, but they are equivalent; this is one reason why we had to introduce the topologized version $\pi^\tau X$ of the fundamental groupoid. To see the equivalence, consider the path $h(t): [0, t] \to \Pi X$ specified by

$$h(t)(s) = (g \mid [s, t]) \cdot \omega(s) \cdot (f \mid [0, s]), \quad \text{for } 0 \leqslant s \leqslant t.$$

Summarizing, for a space X we have the chain of functors

$$X \xrightarrow{\iota} \Pi X \xrightarrow{\phi} \pi^\tau X \xleftarrow{\nu} \pi X$$

in which ι and ν induce homotopy equivalences on passage to classify spaces. The discrete category ΠX has as skeleton the groupoid $\coprod \pi_1(X, x)$, where one point x is chosen from each path component of X, hence $B\Pi X \simeq \coprod K(\pi_1(X, x), 1)$. The following is obvious from the constructions.

LEMMA. *The map $B\phi: B\Pi X \to B\pi^\tau X$ induces a bijection on components and induces isomorphisms of their fundamental groups.*

We now turn to the equivariant world. Let G be any topological group; subgroups are to be closed. Let $\mathscr{G}$ be the category of orbit spaces G/H and G-maps between them. A $\mathscr{G}$-space is a continuous contravariant functor $\mathscr{G} \to$ spaces. Passage to fixed point subspaces gives a forgetful functor Φ from G-spaces to $\mathscr{G}$-spaces. The work of [1] gives a functor Ψ from $\mathscr{G}$-spaces to G-spaces together with a natural spacewise equivalence $\varepsilon: \Phi\Psi F \to F$ for $\mathscr{G}$-spaces F. In particular, ε restricts on the orbit G/e to give a G-equivalence $\zeta: \Psi\Phi X \to X$ for G-spaces X. When G is discrete, the following chain gives the natural 'map' f sought by Rosenberg and Weinberger, where continuous functors and natural maps on spaces are extended spacewise to functors and natural maps on $\mathscr{G}$-spaces:

$$X \xleftarrow{\zeta} \Psi\Phi X = \Psi B\Phi X \xrightarrow{\Psi B\iota} \Psi B\Pi\Phi X \xrightarrow{\Psi B\phi} \Psi B\pi^\tau\Phi X \xleftarrow{\Psi B\nu} \Psi B\pi\Phi X \equiv B\pi(X).$$

For general topological groups G, there is a catch. The functor $B\pi\Phi X: \mathscr{G} \to$ Spaces is continuous trivially if G is discrete, but it fails to be continuous in general. To see this, note that *any* subset of X^K is a closed subspace of the subspace of vertices of $B\pi(X^K)$ and consider a typical evaluation function

$$(G/K)^H \times B\pi(X^H) \to B\pi(X^K)$$

determined by the functor $B\pi\Phi X$: clearly this function need not be continuous. The remedy is built into our definitions. The functor $B\pi^\tau\Phi X: \mathscr{G} \to$ Spaces is continuous, and we define $B\pi(X) = \Psi B\pi^\tau\Phi X$. The desired natural map $f: X \to B\pi(X)$ is obtained by deleting the undefined last arrow from the chain above. We have that

$B\pi(X)^H$ is equivalent to $B\pi^\tau(X^H)$ and thus to $B\pi(X^H)$. Thus, for general topological groups G, it is only the less intuitive, and nonstandard, topologized fundamental groupoid that leads to a correct construction of the fundamental groupoid G-space desired by Rosenberg and Weinberger.

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