Algebraic Word problems

by Jerome Dancis [1]

In this article, we will demonstrate how to solve two Algebraic word problems, one a complicated work problem (Example 1) and the other a complicated mixture problem (Example 2). Students could (and in my opinion should) be taught how to solve these problems in high school, but currently they are excluded from both the high school and college math curriculum.

In addition, an easy and rememberable way to balance the complicated chemical Reduction-Oxidation equations, (Examples 3 and 4) will be presented.

A Complicated Work Problem

We will start with Andrei Toom's "Tom, Dick and Harry work Problem":

Example 1 (Work) *Suppose that it takes Tom and Dick 2 hours to do a certain job, it takes Tom and Harry 3 hours to do the same job and it takes Dick and Harry 4 hours to do the same job. How long would it take Tom, Dick and Harry to do the same job if all 3 men worked together?*

The close-to-unanimous way that students in my senior-level *college* math course solve this problem is to immediately write down the three equations:

$$\begin{aligned} T + D &= 2 \\ T + H &= 3 \\ D + H &= 4; \end{aligned}$$

then they solve these equations for T, D and H and add-up the numbers

$$T + D + H = 4.5 \ \ hours.$$

Wrong answer: 4.5 hours

I ask my students: what do the variables T, D and H stand for? The common response is: Obviously T is for Tom, D is for Dick and H is for Harry.

So I confront them: This makes the equation, "$D + H = 4$" mean that "Dick + Harry = 4". The response is: Of course, T, D and H stand for the *amount* of work that Tom, Dick and Harry, [respectively] did.

Again, I confront them: But then the equation $D + H = 4$ means:

$$\left\{\begin{matrix}\text{The amount of work}\\ \text{that Dick did}\end{matrix}\right\} + \left\{\begin{matrix}\text{The amount of work}\\ \text{that Harry did}\end{matrix}\right\} = 4 \ \text{hours.}$$

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Now the problem is that the *units* do *not* match up. We have amount of work = 4 hours. But work can*not* equal hours! The bulk of my students are majoring in engineering; for sure in their engineering classes they ensure that the units match.

Also, does H represent the amount of work that Harry did when he was working for 4 hours with Dick *or* the amount of work that he did when he was working for 3 hours with Tom? The single letter H cannot represent *both*.

Finally, the answer is wrong, and as we will see, it is not even reasonable.

Remark. My impression is that it is common for high school texts to train students to solve word problems by quickly writing down equations and then solving the equations. Little time is spent or allocated to deciding which equations are to be written down. This works some of the time; it always works for textbooks exercises, since they were chosen so that it will work. It often produces *wrong* equations and *wrong* answers when applied to many of the various problems that arise elsewhere (other than in high school textbooks).

Many college math courses also allocate little time to the translating of word problems into mathematical equations. So it is *predictable* that college seniors will immediately write down the three incorrect equations, listed above.

The work Problems 21 and 22 in my article "Supposedly Difficult Arithmetic Word Problems", are *necessary* background for students to understand this example.

We now present a correct way to approach word problems such as this one:

Step 1. Using common sense and simple observations, estimate or bound the answer.

A simple observation: Tom, Dick and Harry working together can do the job in less time than Tom and Dick can do the job (without Harry's help). Since Tom and Dick can do the job in 2 hours (and since we assumed that Harry is not a saboteur), the three of them together should be able to do the job in less than 2 hours. This is a *tip off* that 4.5 hours must be a wrong answer.

A simple observation: Harry is the slowest worker. The work Problem 21, in "Supposedly Difficult Arithmetic Word Problems", shows that if Harry sped up to the average pace of the other two, then it would take the three of them $4/3$ hours to do the job. But since Harry is remains slower, it will take longer, that is, it should take Tom, Dick and Harry more than $4/3$ hours to do the job.

Step 2. Write down some verbal equations which describe what is happening. Label the quantities in the verbal equations. This will provide both definitions of useful variables and correct algebraic equations.

The information stated in the problem is:

$$\left\{\begin{matrix}\text{The amount of work that}\\ \text{Tom does in 2 hours}\end{matrix}\right\} + \left\{\begin{matrix}\text{The amount of work that}\\ \text{Dick does in 2 hours}\end{matrix}\right\} = 1 \text{ job}$$

$$\left\{\begin{matrix}\text{The amount of work that}\\ \text{Tom does in 3 hours}\end{matrix}\right\} + \left\{\begin{matrix}\text{The amount of work that}\\ \text{Harry does in 3 hours}\end{matrix}\right\} = 1 \text{ job}$$

$$\left\{\begin{matrix}\text{The amount of work that}\\ \text{Dick does in 4 hours}\end{matrix}\right\} + \left\{\begin{matrix}\text{The amount of work that}\\ \text{Harry does in 4 hours}\end{matrix}\right\} = 1 \text{ job}$$

If x is the number of hours it takes the three of them to do the job, then:

$$\left\{\begin{matrix}\text{The amount of work that}\\ \text{Tom does in x hours}\end{matrix}\right\} + \left\{\begin{matrix}\text{The amount of work that}\\ \text{Dick does in x hours}\end{matrix}\right\} + \left\{\begin{matrix}\text{The amount of work that}\\ \text{Harry does in x hours}\end{matrix}\right\} = 1 \text{ job}$$

A useful and relevant simple observation is:

$$\left\{\begin{matrix}\text{The amount of work that}\\ \text{Tom does in 2 hours}\end{matrix}\right\} = 2\left\{\begin{matrix}\text{The amount of work that}\\ \text{Tom does in 1 hour}\end{matrix}\right\}$$

and in general

$$\left\{\begin{matrix}\text{The amount of work that}\\ \text{person A does in N hours}\end{matrix}\right\} = N\left\{\begin{matrix}\text{The amount of work that}\\ \text{person A does in 1 hour}\end{matrix}\right\}$$

Remark. The amount of something that occurs in a unit of time is called a "rate". In general, rates make for very good variables.

Hence

$$\left\{\begin{matrix}\text{The amount of work that}\\ \text{person A does in 1 hour}\end{matrix}\right\} = \left\{\begin{matrix}\text{The rate at which}\\ \text{person A works}\end{matrix}\right\}.$$

This guides us to define the variables as follows:

T = Tom's rate of work = {The amount of work that Tom does in 1 hour}.
D = Dick's rate of work = {The amount of work that Dick does in 1 hour }.
H = Harry's rate of work.

Step 3. With these variables, translate the verbal equations into algebraic equations:

$$\begin{aligned} 2T + 2D \quad\;\; &= 1\\ 3T + \quad\;\; 3H &= 1\\ 4D + 4H &= 1\\ xT + xD + xH &= 1 \end{aligned}$$

One now solves the first three linear equations for T, D and H. Then one plugs the answer into the last equation and solves for $x = 24/13$.

Step 4. One remembers the observations made in Step 1 and then one checks that the answer $24/13$ is indeed less than 2 and greater than $4/3$ as predicted.

A Complicated Mixture Problem

Now lets estimate the gold content of a gold and silver crown.

Example 2 (Mixture) *Prince Braggart of Spoof will be crowned tomorrow as the new king. His 200 year old crown was made from unmixed pure gold and pure silver only. Unfortunately a fire destroyed the record of how much of the crown is gold and how much is silver. Of course, the main topic of royal conversation, among the many kings at the coronation party, is how much gold and how much silver their crowns contain. Prince Braggart wishes to brag about this crown. To do this in a royal manner (no bluffing) he needs to know just how much gold and how much silver his crown contains. As the court mathematician, Prince Braggart has ordered you to find this out before the coronation. Melting down the crown or cutting off a sample for a chemical assay will be harmful to your longevity. (The density of gold is 20 gram/cm^3 and the density of silver is 10 gram/cm^3.)*

Following the lead of the great scientist, Archimedes[2] *(287-212 B.C.) you weigh the crown (3000 grams) and you dip it in water and measure the volume of the overflow (200 cm^3). Now what do you do?*

You use the definition of density; write down verbal equations; label the unknowns; this provides Algebraic equations:

$$\begin{array}{ccccc}
\{\text{weight of gold}\} & = & \{\text{density of gold}\} & \times & \{\text{volume of gold}\} \\
W_g & = & 20 & \times & V_g \\
\{\text{weight of silver}\} & = & \{\text{density of silver}\} & \times & \{\text{volume of silver}\} \\
W_s & = & 10 & \times & V_s
\end{array}$$

You use the fact that weights and volumes are "The whole = $\sum$ parts" types of quantities:

$$\begin{array}{ccccc}
\{\text{Total weight}\} & = & \{\text{weight of gold}\} & + & \{\text{weight of silver}\} \\
3000 & = & W_g & + & W_s \\
\{\text{Total volume}\} & = & \{\text{volume of gold}\} & + & \{\text{volume of silver}\} \\
200 & = & V_g & + & V_s
\end{array}$$

These four simple linear equations may be immediately reduced to two equations:

$$3000 = 20V_g + 10V_s$$

$$200 = V_g + V_s$$

which may be quickly solved.

[2] Read Vitruvius account in <u>World of Mathematics</u>, by James R. Newman, pages 185-186.

Balancing Redox equations

In chemistry classes, students are taught how to balance "Reduction-Oxidation" equations ("Redox" equations), that is, chemical equations which describe the simultaneous "reduction" and "oxidation" of "ions" as well as molecules. In high school, my child was given an involved full-page list of instructions to follow; the list was sophisticated enough, that most students are likely to forget it over winter break.

Here, we will present a high-school-level mathematical approach which can be remembered forever. It does involves solving several, especially simple, linear equations with one more unknown than equations. Two reasons this is avoided in Chem class are * some students have math phobias and * many do not know how to solve 2 linear algebraic equations in 3 unknowns).

Not necessary to know an "ion" is, in order to read the examples.

Definition An *ion* is a "charged" molecule, that is a molecule whose total charge is not zero; this occurs when the total number of protons is different from the total number of electrons. [3]

Definition. A chemical equation, involving ions, is *balanced* when there are the same number of each atom and the same ionic charge on both sides of the arrow.

Remark. Notice that the plus and minus signs need to be "balanced" as well as the number of each atom.

Remark. Aqueous acids contain ions, formed from H_2O and the H^+ part of the acid; basic solutions contain OH^- ions detached from the bases (by definition, a base is a molecule which contains an OH^- ion).

We will now show how to balance the chemical equation which describes the dissolution of copper sulfide CuS, in aqueous nitric acid HNO_3. This reaction produces the highly toxic gas, nitrous oxide, NO. The purpose of the catalitic converter in automobiles is to prevent the nitrous oxide from escaping into the air.

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[3] When, for example, nitric acid HNO_3 is formed, the electron from the hydrogen is attracted to and attaches itself to the NO_3 part, so it is considered as $H^+(NO_3)^-$. Note that the minus sign indicates that the $(NO_3)^-$ ion has an extra electron, that is it has one more electron than it has protons, and that the plus sign indicates that the H^+ ion is short one electron, that is it has one more proton than it has electrons.

When nitric acid is mixed with water, the $(NO_3)^-$ part and the H^+ part separate, with the H^+ part attaching itself to a few water molecules, H_2O. The result is the ionization of nitric acid:

$$HNO_3 + 2H_2O \to NO_3^- + H_5O_2^+.$$

Also

$$HNO_3 + 3H_2O \to NO_3^- + H_7O_2^+.$$

Aqueous nitric acid is a soup of $NO_3^-, H_2O, H_5O_2^+$ and $H_7O_2^+$. It is called "aqueous" nitric acid since "aqueous" is Latin for watery.

Example 3 (Dissolution of copper sulfide in aqueous nitric acid.) *Some copper sulfide, CuS, is poured (carefully and gently) into aqueous nitric acid. The highly toxic gas, nitrous oxide, NO, is formed and leaves (hopefully via a hood), the ions SO_4^{2-} and Cu^{2+} are formed. Describe the reaction by balancing the redox equation.*

Note that the $2-$ indicates that the SO_4^{2-} ion has 2 extra electrons, that is, it has 2 more electrons than it has protons, and that the $2+$ indicates that the Cu^{2+} ion is short 2 electrons, that is it has 2 more protons than it has electrons.

Calculations. The starting equation is

$$CuS + NO_3^- \longrightarrow Cu^{2+} + SO_4^{2-} + NO.$$

Since this is an acid solution, it contains H^+-type ions like $H_5O_2^+$ and $H_7O_3^+$. The "pretend" ion H^+ is a convient placeholder and representative for these ions.

Also water is often an unstated byproduct. Therefore, we add both H^+ and H_2O to the equation. The effective starting equation is:

$$CuS + NO_3^- + H^+ \longrightarrow Cu^{2+} + SO_4^{2-} + NO + H_2O.$$

We need to balance this reaction; so we multiply the molecules by unknown coefficients a, b, c, d, e, f and g :

$$aCuS + b(NO_3)^- + cH^+ \longrightarrow dCu^{2+} + eSO_4^{2-} + fNO + gH_2O.$$

We balance the atoms:

$$\begin{aligned} Cu: &\quad a = d \\ S: &\quad a = e \\ N: &\quad b = f \\ O: &\quad 3b = 4e + f + g \\ H: &\quad c = 2g \end{aligned}$$

We balance the ionic charges:

$$-b + c = 2d - 2e.$$

Thus we have 6 equations in 7 unknowns. Since $a = d = e$, $b = f$ and $c = 2g$, these equations are quickly truncated, (using substitution) to just 2 equations:

$$3f = 4e + f + g \text{ and } -f + 2g = 2e - 2e = 0.$$

The second equation shows that $f = 2g$ and then the first equation simplifies to $g = \frac{4}{3}e$, or without fractions: $3g = 4e$. Hence: $f = 2g = 2(\frac{4}{3}e)$, or without fractions: $3f = 8e$.

Having found all the variables in terms of e, We need only a single solution with just integers. So we choose $e = 3$ (in order to avoid fractions); this yields:

$$a = 3,\ b = 8,\ c = 8,\ d = 3,\ e = 3,\ f = 8,\ g = 4$$

Hence the balanced equation [4] is:

$$3CuS + 8NO_3^- + 8H^+ \longrightarrow 3Cu^{2+} + 3SO_4^{2-} + 8NO + 4H_2O.$$

Remark. We have shown that this is the *only* mathematically possible way to balance this chemical equation.

Remark. In balancing this equation, we have side stepped much of the chemistry. We avoided analyzing the atomic charges and a description of the oxidation of CuS and the reduction of NO_3^-. We were indifferent to the large movement of electons. We made no attempt to explain why the products of the reaction are NO, SO_4^{2-} and Cu^{2+}; we leave that to a chemistry class.

This is the most straight forward way to balance chemical equations. As such it is the one method that is easily remembered even years later. It is much easier to learn than the complicated half-reaction method taught in many chemistry textbooks. It is not always as fast as the atomic charge method.

When simultaneously, both oxidation and reduction of the *same* molecule is occurring, chemists call the process *disproportionation*. In the next example, we will balance a disproportionation equation.

Example 4 (Disproportionation of Nitrous acid). *Nitrous acid, HNO_2 in an acid solution, disproportionates to nitrate ion, NO_3^- and nitrogen oxide NO, a gas, which leaves the solution. Describe the reaction by balancing the redox equation.*

Calculations. The starting equation is

$$HNO_2 \longrightarrow NO_3^- + NO.$$

Since this is an acid solution, it contains H^+; also water is often an unstated byproduct, so we add both H^+ and H_2O to the equation. The effective starting equation is

$$HNO_2 + H^+ \longrightarrow NO_3^- + NO + H_2O.$$

In order to balance this reaction, we add coefficients a, b, c, d and e:

$$aHNO_2 + bH^+ \longrightarrow cNO_3^- + dNO + eH_2O.$$

[4] Remember that H^+ is just a placeholder; one obtains the actual reactions by adding water molecules to both sides. The actual reactions are rarely written; they are:

$$3CuS + 8NO_3^- + 8H_5O_2^+ \longrightarrow 3Cu^{2+} + 3SO_4^{2-} + 8NO + 20H_2O.$$

$$3CuS + 8NO_3^- + 8H_7O_2^+ \longrightarrow 3Cu^{2+} + 3SO_4^{2-} + 8NO + 28H_2O.$$

We balance the atoms:

$$\begin{aligned} N: &\quad a = c + d \\ O: &\quad 2a = 3c + d + e \\ H: &\quad a + b = 2e \end{aligned}$$

We balance the ionic charges:

$$b = -c$$

Thus there are 4 simple linear equations in 5 unknowns. Solving them is simple. The set of (smallest) integer solutions is $a = 3, b = -1, c = 1, d = 2, e = 1$. Hence the almost balanced equation is:

$$3HNO_2 - H^+ \longrightarrow NO_3^- + 2NO + H_2O.$$

Remark. But chemical equations cannot have a *negative* amount of a molecule. So add H^+ to both sides. [5]

Hence the balanced equation is:

$$3HNO_2 \longrightarrow NO_3^- + 2NO + H^+ + H_2O.$$

In this case, both a chemist and a mathematician would have known, at the start, to place the H^+ on the right side where it is needed to balance the negatively charged NO_3^- ion.

As in the preceding example, we avoided the chemistry as we balanced this equation.

[5] This is analogous to how we manipulate mathematical equations.