Additional Examples and Exercises for Math 401

Contents.

On proving things

Query. When doing an exercise, what can we assume?

Answer NOTHING

For example. *Never assume* that a matrix is invertible or that a matrix is a 2×2-matrix.

One knows that a matrix is invertible, only when

1. It is given (stated in the hypotheses.)

2. It is a consequence of a theorem (or proposition).

Example. If given (or already proven) that A and B are invertible matrices and $C = AB$, then we know that C is also invertible by the proposition on the product rule for inverse matrices.

3. You have calculated the inverse matrix.

Otherwise, one does not know that a matrix is invertible. It would be an unwarranted assumption, to assume that any other matrix is invertible.

Remark. "WOLOG" (WithOut Loss Of Generality). When the method of proof for a 2×2-matrices *is also valid* for all matrices, then one may invoke "WOLOG" (WithOut Loss Of Generality). That is, write "WOLOG, we assume that M is a 2×2-matrix". Do *not* do this if your method of proof is not valid for 7×7-matrices. If the proof uses the formula for the inverse matrix for a 2×2-matrix, then it will be WRONG to use WOLOG.

Warning Do not start a proof with the "To Prove" part.

Otherwise, all sorts of false formulas can be proven.

Example. To prove $x = -x, \forall_x$

Step 1. $x = -x$.

Square both sides:

Step 2. $x^2 = (-x)^2 = x^2$. $\surd$

NOT

Either

1. Start proofs with a known fact, either a given equation or a equation, known from a theorem.

or

2. To prove Formula #1 = Formula #2, you may start with Formula #1 and then simplify or convert it, via a string of valid equations, to Formula #2.

For example:

Formula #1 = Formula #3 = Formula #4 = Formula #2

Always end with the "To Prove" part.

All the proofs in the text use these methods.

Additional Example and Exercise for Ch. III Sec. 1A.

Example III.1.12. *Suppose that there are a batch of objects divided into two groups or states. Suppose that each day half the objects in State 1 go to State 2 and half remain, also each day 0.7 of the objects in State 2 go to State 1 and the rest remain in State 2. Find the transition matrix and its steady-state vector.*

Since there are two states, the transition matrix will be a 2×2-matrix, $M = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$. Using the definition and description of the $p'_{ij}s$, we see that $p_{11} = .5, p_{12} = .7, p_{21} = .5$ and $p_{22} = .3$. Hence $M = \begin{pmatrix} .5 & .7 \\ .5 & .3 \end{pmatrix}$.

Let $v = \binom{x}{y}$ be a candidate for a steady-state vector of this matrix, M. Then $Mv = v$, that is

$$\begin{pmatrix} .5 & .7 \\ .5 & .3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

$$.5x + .7y = x \Rightarrow 7y = 5x$$

$$.5x + .3y = y \Rightarrow 5x = 7y$$

The two equations are the same, and $y = \frac{5}{7}x$. The solutions to $Mv = v$ are $(x, y) = (x, \frac{5}{7}x)$.

But a steady-state vector is a probability vector, hence its coordinates must sum to one. Here $x + y = x + \frac{5}{7}x = \frac{12}{7}x$. We make $x + y = 1$ by choosing $x = \frac{7}{12}$. Thus the steady-state vector is $v = (x, y) = (\frac{7}{12}, \frac{5}{12})$.

Exercise III.1.5. *Given* $M = \begin{pmatrix} .9 & .1 \\ .1 & .9 \end{pmatrix}$. *Find the steady state vector of* M*, call it* p_∞.

Check that your solution is a steady state vector.

Let $p_1 = \binom{1}{0}$ *and let* $p_1 = Mp_0,\ p_2 = Mp_1,$ *and* $p_3 = Mp_2,$. *Find* $p_1, p_2,$ *and* p_3.

Then calculate $||p_\infty - p_0||,\ ||p_\infty - p_1||$ *and* $||p_\infty - p_2||$.

Present your solutions in a table.

Comment on the results.

Additional Example for Ch. III Sec. 1B.

Matrix check

Now we will use this Proposition III. 1B.5 to explain the check for the answers to Exercise 1.55 of Ch. 1. Following Slogan III.1.8, we write Exercise 1.55 of Ch. 1 in matrix vector form and then label the matrices and vectors:

$$\begin{pmatrix} 1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

The solution was

$(u, v, w, x, y, z) = (-4, -2, -1, 0, 0, 0) + x(4, 2, 1, 1, 0, 0) + y(4, 2, 1, 0, 1, 0) + z(4, 2, 1, 0, 0, 1),\ \forall\ x, y$ and z.

Here

$v_0 = (-4, -2, -1, 0, 0, 0),\ v_1 = (4, 2, 1, 1, 0, 0),\ v_2 = (4, 2, 1, 0, 1, 0), v_3 = (4, 2, 1, 0, 0, 1)$ and $w_0 = (-1, -1, -1)$

$$\text{and } M = \begin{pmatrix} 1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 & -1 \end{pmatrix}.$$

Matrix check. Check that

$$Mv_0 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \quad \text{and} \quad Mv_1 = \mathbf{0} = Mv_2 = Mv_3.$$

Then Proposition III. 1B.5 will confirm that *all* the solutions

$$(u, v, w, x, y, z) = (-4, -2, -1, 0, 0, 0) + x(4, 2, 1, 1, 0, 0) + y(4, 2, 1, 0, 1, 0) + z(4, 2, 1, 0, 0, 1),\ \forall_{x,y \text{ and } z}$$

$$= v_0 + Sp\{v_1, v_2, v_3\} = v_0 + Av_1 + Bv_2 + Cv_3, \forall_{A,B \text{ and } C},$$

are correct.

That is, after solving a matrix-vector equation, $Mv = w_0$, and writing the solution in vector form:

$$v = v_0 + Sp\{v_1, v_2, v_3\} = v_0 + Av_1 + Bv_2 + Cv_3, \forall_{A,B \text{ and } C}$$

the check for this infinite set of solutions, is to simply just check that:

$$Mv_0 = w_0 \quad \text{and} \quad Mv_1 = \mathbf{0} = Mv_2 = Mv_3.$$

Then Proposition III. 1B.5 will confirm that *all* the solutions are correct.

Additional Exercises for Ch. III Sec. 2.

Exercise III.2.36. *Given* $M_3 = \begin{pmatrix} .9 & .1 & .1 \\ .1 & .9 & 0 \\ 0 & 0 & .9 \end{pmatrix}$. *Check that* M_3 *is a stochactic matrix. Find the steady state vector for* M_3.

Check that it is a steady state vector.

Exercise III.2.37. *The question is: Which matrices commute with a diagonal matrix?*

Given

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.$$

(i) *Calculate* DM *and* MD. *When is* $DM = MD$? *That is, list or describe the matrices,* M, *which satisfy the equation:* $DM = MD$?

(ii) *What is special about the numbers* $1, 2$ *and* 3 *in the matrix* D? *State a general rule.*

(iii) *Calculate* D^2, D^3 *and* D^4. *Predict a general rule.*

Exercise III.2.38. *Which matrices commute with this diagonal matrix:* $D_2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$? *That is, list or describe all the matrices,* M, *which satisfy the equation:* $D_2M = MD_2$? *Comment!*

Exercise III.2.39. (Complexity) *You need to calculate* $w = MNv$, *when* M *and* N *are* 10×10 *matrices and* v *is a vector in* R^{10}. *The two ways to do this are:*

Method 1. First calculate $A = MN$, *and then calculate* $w = Av$.

Method 2. First calculate $u = Nv$, *and then calculate* $w = Mu$.

For each of Methods 1 and 2, calculate how many multiplications (or dot products) are required to obtain w. *Which is much faster? Which method will you be using?*

Some Quickie questions for Ch. 3

One and two minute problems.

Ch. III Sec. 1

Can write a system of linear equations in matrix-vector form.

Can do matrix-vector multiplication.

What does multiplication by a diagonal matrix do to a vector?

What does multiplication by the vector, $\binom{1}{0}$ do to a 2×2-matrix?

What does multiplication by the vector, $\binom{0}{1}$ do to a $n \times 2$-matrix?

What does multiplication by the one vector do to a matrix?

State the 2×2 identity matrix, the projection-of xy-plane-onto x-axis matrix, the 2×2 switch matrix. What does each do?

What does the rotation matrix do?

Ch. III Sec. 1A

Can state definitions of probability vector and of stochastic matrix.

What is the connection between the census vector and the fraction vector?

Stochastic matrix times probability vector yields another probability vector.

Ch. III Sec. 1B

Write this linear combination of vectors, $a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$ as a matrix-vector product?

What does it mean that matrix transformations are linear?

State the column-vector form of the matrix vector product.

Can use linearity of matrix transformations, to prove Prop. III.1B. 3,4,5

Exer. III.1.14-19

Ch. III Sec. 2

State the reason/rational for the definition of matrix mult. (One equation will suffice.)

Prove that $R_\theta \times R_\phi = R_{\theta+\phi}$. Do not mult. out

What does multiplication by a diagonal matrix do to another matrix?

State two specific 2x2 matrices which do not commute.

Exer. III.2.11, 30,

Prove the Associate Law for matrix. mult. Do not mult. out

Ch. III Sec. 3

What is $(R_\theta)^{-1}$? Why?

Know statements of Examples III.3.12, 13,14

State and prove the Product Rule for inverse matrices.

Exer. III.3.4, the Product Rule for inverse of four matrices.

Exer. III.3.7 each part.

Can prove statements of Examples III.3.12 and Exercise II.3.8.

Exer. III.3.10,

Ch. III Sec. 4B

Can prove superposition for specific DC resistance circuit like exercises on handout.

Ch. III Sec. 5

Can state the Distributive rules for matrices.

III.5.4(a) and 5

Quickie questions

Note "$**$" denotes two-minute items; the others are one -minute items

$**$ Ch. III Sec. 4A. Can do a Superposition problem

Ch. III Sec. 5. Can state Distributive Rules.

Can use Distributive Rules for 3 matrices to prove one for four matrices (Rule III.5.3).

Exer. III.5.4(a), 5

Ch. III Sec. 5B. $**$ Can find a single solution " formally"/"sybolically", for a matrix-vector differential equation. as in Exer. III.5B.7

Ch. III Sec. 6. State the Product Rule for transposes.

$**$ Assuming Product Rule for transpose of two matrices, can prove the Product Rule for transpose of three matrices.

$**$ State and prove the Rule $(A^T)^{-1} = (A^{-1})^T$.

$**$ Exer. III.6.19, 20(a),(b) (See Prop. III.6A.2)

Ch. III Sec. 6A.

State the defining equations for real orthogonal and symmetric matrices.

$**$ Example III.6A.4, Exercises III.6A.6,7,8,10,11

Additional Exercises for Ch. IV Sec. 3.

Exercise IV.3.11. *Let* $A = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$ *and* $\mathbf{1}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. *State/find a specific coordinate vector,* $\mathbf{v}$, *such that* $|\mathbf{v}| < \mathbf{1}_2$, *but* $|A \times \mathbf{1}_2| < |A \times \mathbf{v}|$.

Exercise IV.3.12. *State an example of two specific coordinate vectors* $\mathbf{v}$ *and* $\mathbf{w} \in R^2$, *such that* $\mathbf{v} \neq \mathbf{w}$ *and* $\mathbf{v} \not< \mathbf{w}$ *and* $\mathbf{v} \not> \mathbf{w}$,

Exercise IV.3.13. *State an example of two specific coordinate vectors* $\mathbf{v}$ *and* $\mathbf{w} \in R^2$, *such that* $|\mathbf{v} + \mathbf{w}| \neq |\mathbf{v}| + |\mathbf{w}|$ *and* $|\mathbf{v} + \mathbf{w}| \not< |\mathbf{v}| + |\mathbf{w}|$, *but* $|\mathbf{v} + \mathbf{w}| \leq |\mathbf{v}| + |\mathbf{w}|$.

Exercise IV.3.14. *Use the Triangle Inequality for the absolute value of the sum of two and/or three coordinate vectors to prove the Triangle Inequality for four coordinate vectors, that is prove that*

$$|p + q + r + s| \leq |p| + |q| + |r| + |s|, \ \forall_{p,q,r,s \in R^n}.$$

Exercise IV.3.15. *State and prove the Triangle Inequality for the absolute value of the sum of five coordinate vectors.*

Additional Exercises and Warning for Ch. IV Sec. 3A.

Exercise IV.3.11. *Use the Triangle Inequality for the infinity norm of the sum of two coordinate vectors, to prove the Triangle Inequality for this linear combination of coordinate vectors:*

$$||Ap + Bq||_\infty \le |A|\ ||p||_\infty + |B|\ ||q||_\infty, \ \forall_{p,q \in R^n} \quad \textit{and} \quad \forall_{A,B \in R}.$$

Exercise IV.3.12. *Use the Triangle Inequality for the infinity norm of the sum of two coordinate vectors, to state and prove the Triangle Inequality for the infinity norm of the sum of four coordinate vectors.*

Notation. $J_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, and in general, the matrix, J_n is the $n \times n$-matrix, in which each entrie is one. These "J" matrices are called *One* matrices.

Exercise IV.3.13. (Reversal of the inequality) *Let* $A = \begin{pmatrix} -2 & 1 \\ -1 & 2 \end{pmatrix}$ *and* $J_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. *State/find a specific* 2×2*-matrix,* E*, such that* $|E| < J_2$*, but* $|A \times E| > |A \times J_2|$.

Check that $|A| \times |E| \le |A| \times |J_2|$.

Remark. The next exercise extends the result of Exercise IV.3A.3.

Exercise IV.3.14. ($(M + E)$ not invertible.) *Given* $M = \begin{pmatrix} 4 & 5 \\ 5 & 6 \end{pmatrix}$.

(a) *Check that* $|M^{-1}\frac{1}{10}J_2\ \underline{\mathbf{1}}_2| < \underline{\mathbf{1}}_2$.

(b) *Find a specific matrix,* E*, such that* $|E| \le \frac{1}{10}J_2$ *and* $(M + E)$ *is not an invertible matrix.*

(c) *Check that* $|M^{-1}|\frac{1}{10}J_2\ \underline{\mathbf{1}}_2$ *is not* $< \underline{\mathbf{1}}_2$.

(d) *Note that these results are consistent with Lemma IV.3A.4.*

Warning. Usually the error matrix, E is *unknown.* Hence

$$M^{-1}E, \quad M^{-1}E\underline{\mathbf{1}}_2, \quad |M^{-1}E|, \quad |M^{-1}E\mathbf{v}| \ \text{ and } \ ||M^{-1}E\mathbf{v}||_\infty$$

are all *unknown* and *cannot be calculated.* Calculating them will produce *wrong* amswers. Perhaps concluding that a perturbed matrix is invertible, when it is not; or calculating error bounds that are considerably smaller than the actual error.

What is usually known are just *bounds* on the matrix, E, usually *bounds* on $|E|$. For example, when the entries in matrix, M, have tolerences of $.01$, then $|E| \le .01J_2$. This, can be used with Rules IV.3.4, to calculate error bounds, as follows:

$$|M^{-1}E\mathbf{v}| \ \le \ |M^{-1}| \times |E| \times |\mathbf{v}| \ \le \ |M^{-1}| \times .01J_2 \times |\mathbf{v}|;$$

$$||M^{-1}E\mathbf{v}||_\infty \ \le \ ||\ |M^{-1}| \times |E| \times |\mathbf{v}|\ ||_\infty \ \le \ ||\ |M^{-1}| \times .01J_2 \times |\mathbf{v}|\ ||_\infty.$$

This is what is happening in Equation (IV.3B.4). (Sorry, all the extra absolute value signs are necessary in order to avoid wrong answers.) As the two exercises above demonstrate, it is *crucial* to take the absolute values first. Calculating $|M^{-1} \times J_2|$ or $|M^{-1} \times .01J_2|$ will often produce *wrong* answers.

Additional Exercises for Ch. II Sec. 3.

Exercise II.3.26. *(a) Prove that* $L(x(t)) = t^2\ddot{x} - 2t\dot{x} + 2x$ *is a linear transformation. Use the two defining equations or the single equation of Proposition II.2.1*

(b) Find many solutions to $t^2\ddot{x} - 2t\dot{x} + 2x = 0$*; the educated guess is* $x = x(t) = t^r$*, where* r *is an unknown, to-be-found constant. Indicate which and where the theorems on linear equations are used.*

Check the "basic" solutions.

Exercise II.3.27. (Change of variables) *Find many solutions to each of these equations (short-cuts encouraged):*

(a) $\ddot{y} - 4\dot{y} - 5y = 0$*, the educated guess is* $y = y(t) = e^{rt}$.

(b) $y'' - 4y' - 5y = 0$*, the educated guess is* $y = y(x) = e^{rx}$.

(c) $X'' - 4X' - 5X(x) = 0$*, the educated guess is* $X = X(x) = e^{rx}$.

(d) Comments.

Exercise II.3.28. (Change of variables) *Find many solutions to each of these equations (short-cuts encouraged):*

Write your answer in terms of hyperbolic functions, like $\sinh u = \frac{1}{2}(e^u - e^{-u})$.

Indicate which and where the theorems on linear equations are used.

(a) $\ddot{x} - x = 0$*, with single condition,* $x(0) = 0$*. The educated guess is* $x = x(t) = e^{rt}$.

(b) $y'' - 4y = 0$*, with single condition,* $y(0) = 0$*. The educated guess is* $y = y(x) = e^{rx}$.

(c) $X'' - 4X(x) = 0$*, with single condition,* $X(0) = 0$*. The educated guess is* $X = X(x) = e^{rx}$.

(d) Comments.

Exercise II.3.29. (Change of variables) *Find all solutions to each of these equations (short-cuts encouraged):*

(a) $\ddot{x} + 1x(t) = 0$*, with boundary conditions,* $x(0) = 0$ *and* $x(\pi) = 0$.

(b) $y'' + y(x) = 0$*, with boundary conditions,* $y(0) = 0$ *and* $y(\pi) = 0$.

(c) $X'' + X(x) = 0$*, with boundary conditions,* $X(0) = 0$ *and* $X(\pi) = 0$.

(d) Comments.

Exercise II.3.30. (Change of variables) *Find all solutions to each of these equations:*

(a) $X'' + X(x) = 0$*, with boundary conditions,* $X(0) = 0$ *and* $X(1) = 0$.

(b) $X'' + 4X(x) = 0$*, with boundary conditions,* $X(0) = 0$ *and* $X(4\pi) = 0$.

(c) $X'' + 4X(x) = 0$*, with boundary conditions,* $X(0) = 0$ *and* $X(4) = 0$.

Exercise II.3.31. (Change of variables) *Find all solutions to each of these equations:*

(a) $X'' + X(x) = 0$*, with boundary conditions,* $X'(0) = 0$ *and* $X'(\pi) = 0$.

(b) $X'' + 4X(x) = 0$*, with boundary conditions,* $X'(0) = 0$ *and* $X'(4\pi) = 0$.

(c) $X'' + 4X(x) = 0$*, with boundary conditions,* $X'(0) = 0$ *and* $X(4) = 0$.

Exercise II.3.32. *Quickly, find a pair of functions* $X(x)$ *and* $T(t)$ *such that*

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} = -25, \forall_x \ \ and \ \ t.$$

Questions and Answers on linear equations

(a) *What is a main use of the proposition on basic linear transformations?*

Answer. *It makes it easy to spot and verify many linear transformations.*

(b) *Why is it useful to know that a transformation is a linear transformation?*

Answer. *It is used for quickly identifying linear equations.*

(c) *Why is it useful to know that an equation is a homogeneous linear equation?*

Answer. *Then the theorem on homogeneous linear equations is applicable.*

(d) *Why is the theorem on homogeneous linear equations useful?*

Answer. *It provides an algorithm for solving all homogeneous linear equations.*

(e) *State the algorithm implied by the theorem on homogeneous linear equations.*

(f) *Why is it useful to know that an equation is a non-homogeneous linear equation?*

Answer. *Then the theorem on non-homogeneous linear equations is applicable.*

(g) *Why is the theorem on non-homogeneous linear equations useful?*

Answer. *It provides a formula and an algorithm for solving all non-homogeneous linear equations.*

(h) *State the formula provided by the theorem on non-homogeneous linear equations.*

(i) *State the algorithm implied by the theorem on non-homogeneous linear equations.*

Exercise II.6.11. *Prove that an eigenspace is always a subspace.*

Writing equations in linear equation form

Example II.7.8. (A Heat Equation with a heat source) *Consider a (heat) function;* $u(x,t)$*, on the domain* $0 \le x \le \pi$*, and* $t \ge 0$*, which satisfies this system of equations:*

$$\boxed{u_{xx} = u_t + \sin x \quad \text{and} \quad u(0,t) = 5 \quad \text{and} \quad u(\pi,t) = 7.}$$

Write this system in linear-equation form; then write the system of associated homogeneous equations.

These equations have derivatives and restrictions of the function, $u(x,t)$*. This suggests linear-equations, courtesy of the Proposition II.2.4 on basic linear transformations. Linear-equations have the form,* $L(v) = w_0$*. So let's reorganize these equations in this form.*

$$L_1(u) = u_{xx} - u_t = \sin x,$$

$$L_2(u) = u(0,t) = 5,$$

$$L_3(u) = u(\pi,t) = 7.$$

You may use the Proposition II.2.4 on basic linear transformations to quickly check that these three transformations are indeed linear transformations. Since, there are no u*'s on the right side, this is a system of non-homogeneous linear equations.*

Therefore the Theorem on non-homogeneous linear equations is applicable. One need only find the simplest solution to the non-homogeneous linear equations and the many solutions to the associated homogeneous equations.

In solving non-homogeneous linear equations, Step 1 is to guess the form of a simple solution. For this type of linear partial differential equation, when the functions on the right side are all functions of x *alone, (no* t*) the standard guess is:* $u(x,t) = X(x)$*, where* $X(x)$ *is an unknown, to-be-found function of* x *alone, (no* t*). (Different guess than for ordinary [not partial] differential equations.)*

The associated homogeneous equations are:

$$L_1(v) = v_{xx} - v_t = 0,$$

$$L_2(v) = v(0,t) = 0,$$

$$L_3(v) = v(\pi,t) = 0.$$

In this example, you may recognize this system of equations as the ones from Example II.7.1 and copy the solutions from that example. Usually, you will need to solve them.

Additional Exercises for Ch. II Sec. 7.

Exercise II.7.16. (A Heat Equation with a heat source) *Find an infinite series solution* $u(x,t)$, *to the system of equations:*

$$\boxed{u_{xx} = 9u_t + \sin 3x \quad \textit{and} \quad u(0,t) = 7 \quad \textit{and} \quad u(1,t) = 5.}$$

on the domain $0 \leq x \leq 1$, *and* $t \geq 0$.

Set-up. *Write this system of equations in linear equation form.*

Use Proposition II.2.4 on basic linear transformations to quickly check that the transformations are indeed linear transformations, and hence that the equations are non-homogeneous linear equations.

Find the simplest solution to this system of non-homogeneous linear equations.

Write down the system of associated homogeneous equations.

Solve the system of associated homogeneous equations. Enclose the eigenvalue problem in a box. Check that each "eigenvalue-eigenfunction" solution [$v_n(x,t) = X_n(x)T_n(t)$*] satisfies the equation* $L_1(v) = v_{xx} - 9v_t = 0$. *Indicate where you use the Theorem on homogeneous linear equations.*

Indicate where you use the Theorem on non-homogeneous linear equations.

Exercise II.7.17. (Adding an initial condition) *Find the solution* $u(x,t)$, *(on the domain* $0 \leq x \leq 1$, *and* $t \geq 0$*) to the system of equations:*

$$\boxed{u_{xx} = 9u_t + \sin 3x \quad \textit{and} \quad u(0,t) = 7 \quad \textit{and} \quad u(1,t) = 5 \quad \textit{and} \quad u(x,0) = 4 + 2x + 5\sin \pi x.}$$

Set-up. *Pick out the one solution from the previous exercise, which satisfies the additional equation. Match coefficients to do this. You will need to use the Fourier series for one.*

Additional Exercises for Ch. II Sec. 8.

Exercise II.8.10. *Given that* $u(x,t) = 7e^{-49t}\sin 7x$ *is a solution to the system of equations:*

$$u_{xx} = u_t \quad \textit{and} \quad u(0,t) = 0 = u(\pi,t) \quad \textit{and} \quad u(x,0) = 7\sin 7x,$$

on the domain $0 \le x \le \pi$, *and* $t \ge 0$. *Prove that it is the only solution. Use the corollary of the Maximal Principle. Do not quote the theorem about uniqueness of solutions.*

Exercise II.8.11. (Uniqueness) *Prove that the solution that you found to Exercise II.7.8. is unique, that is, there is no other solution to those four equations.*

Exercise II.8.12. *Given that* $u(x,t) = 7e^{-81t}\sin 9x + 5e^{-144t}\sin 12x$ *is a solution to the system of equations:*

$$u_{xx} = u_t \quad \textit{and} \quad u(0,t) = 0 = u(\pi,t), \quad \textit{and} \quad u(x,0) = 7\sin 9x + 5\sin 12x$$

on the domain $0 \le x \le \pi$, *and* $t \ge 0$. *Show that this is the only solution.*

Exercise II.8.13. *Given functions* $u(x,t)$ *and* $v(x,t)$, *on the domain* $0 \le x \le \pi$, *and* $t \ge 0$. *Also given that:*

$$\boxed{u_{xx} = u_t + \sin x \quad \textit{and} \quad u(0,t) = 5 \quad \textit{and} \quad u(\pi,t) = 7 \quad \textit{and} \quad u(x,0) = f(x).}$$

$$\boxed{v_{xx} = v_t + \sin x \quad \textit{and} \quad |v(0,t) - u(0,t)| \le \tfrac{1}{20} \quad \textit{and} \quad |v(\pi,t) - u(\pi,t)| \le \tfrac{1}{10} \quad \textit{and} \quad |v(x,0) - u(x,0)| \le \tfrac{1}{100}.}$$

Find a good bound on $|v(x,t) - u(x,t)|$ *on the domain* $0 \le x \le \pi$, *and* $t \ge 0$.

Set-up. *Observe that it is not possible to calculate* $v(x,t)$, *and therefore it is not possible to calculate* $|v(x,t)-u(x,t)|$. *A bound on* $v(x,t)-u(x,t)$ *is needed; so set* $w(x,t) = v(x,t)-u(x,t)$. *Then find equations and inequalities for* $w(x,t)$.

Exercise II.8.14. (Uniqueness) *Prove that the solution that you found to Exercise II.7.17 is unique, that is, there is no other solutions to those four equations.*

Set-up. *Write down the system of four associated homogeneous equations. Use the Maximal Principle or its corollary and use the Theorem on non-homogeneous linear equations.*

Exercise II.8.15. *In Exercise II.7.17, suppose that the equipment, for maintaining the temperatures on the sides, was accurate to within* $\frac{1}{100}$, *and the measurement of the initial temperatures were accurate to within* $\frac{1}{10}$. *Let* $v(x,t)$ *be the actual temperature function. Then, the equations for* $v(x,t)$ *are:*

$$v_{xx} = 9v_t + \sin 3x \quad \textit{and} \quad |v(0,t) - u(0,t)| \le \frac{1}{100}$$

$$\textit{and} \quad |v(1,t) - u(1,t)| \le \frac{1}{100} \quad \textit{and} \quad |v(x,0) - u(x,0)| \le \frac{1}{10}.$$

Find a good bound on $|v(x,t) - u(x,t)|$ *on the domain* $0 \le x \le 1$, *and* $t \ge 0$.

Set-up. *Observe that there is not enough information to calculate* $v(x,t)$. *But, only a bound on* $v(x,t) - u(x,t)$ *is needed; so set* $w(x,t) = v(x,t) - u(x,t)$. *Then find equations and inequalities for* $w(x,t)$. *Use Example II.8.4 as a model. Check that* $L_1(w) = w_{xx} - 9w_t = 0$. *Use the Maximal Principle or its corollary on* w.

Formula V.5.6. *(ii)* **(Determinants of triangular matrixs)**

$$\det\begin{pmatrix} d_1 & & & \\ & d_2 & * & \\ & & \ddots & \\ & O & & d_r \end{pmatrix} = d_1 d_2 \cdots d_r = \det\begin{pmatrix} d_1 & & & \\ & d_2 & 0 & \\ & & \ddots & \\ & * & & d_r \end{pmatrix}$$

This formula is often invalid for triangular matrices, not based on the main diagonal, but on the other diagonal:

Example V.5.7.

$$\det\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} = -ab$$

Formula V.5.13. (Product Rule for Determinants)

$$\det M \times N \;=\; \det(M) \times \det(N),\ \forall\ _{n\times n-matrices,\ M \quad and \quad N}.$$

Remark. *The determinant operation commutes with the inverse matrix operation:*

Formula V.5.14. $\det M^{-1} \;=\; (\det M)^{-1},\ \forall_{invertible\ matrices,\ M}.$

Remark. *The similarity operation preserves the determinant, that is:*

Corollary V.5.15. $M = P^{-1}NP \Longrightarrow \det M \;=\; \det N.$

Remark. *The Determinant operation commutes with the transpose matrix operation:*

Formula V.5.16. $\det M^T \;=\; \det M,\ \forall_{square\ matrices,\ M}.$

Exercises

Exercise V.5.11. *Use the Product rule for Determinants to prove that* $\det M^{-1} \;=\; (\det M)^{-1},\ \forall_{invertible\ matri}$

Exercise V.5.12. *Use useful rules for determinants to prove the corollary,*
$M = P^{-1}NP \Longrightarrow \det M \;=\; \det N.$

Exercise V.5.13. *Given that* M *is an* $n \times n$*-matrix, prove that*

$$\det(M^T - 7I_n) = \det(M - 7I_n).$$