Non-locally modular regular types in classifiable theories

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Abstract

We introduce the notion of strong p-semi-regularity and show that if p is a regular type which is not locally modular then any p-semi-regular type is strongly p-semi-regular. Moreover, for any such p-semi-regular type, "domination implies isolation" which allows us to prove the following: Suppose that T is countable, classifiable and M is any model. If $p \in S(M)$ is regular but not locally modular and b is any realization of p then every model N containing M that is dominated by b over M is both constructible and minimal over Mb.

1 Introduction

In obtaining the uncountable spectrum of any classifiable theory T in [2], localizations of ω-stability near certain regular types were considered. A regular type $p \in S(M)$ over a countable M is *locally totally transcendental* (locally t.t.) if it is not orthogonal to a $q \in S(M)$ that is strongly regular and for which there is a constructible (and hence prime) model over M and any realization of q. There are examples of depth zero non-trivial regular types in classifiable theories which are not locally t.t (see for instance Example 2.3). We intend to consider the manner in which models dominated by such types are constructed in future papers. In this paper, we concentrate on *non-locally modular* regular types p and prove that they are all locally t.t. in a very strong way. The two main results build on the dichotomy theorem of Hrushovski and Shelah in [4]. Here, we prove that if a stationary $q \in S(A)$ is p-semiregular then

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- q is strongly p-semiregular (see Definition 3.1); and
- q is depth-zero like and "domination implies isolation" (DI), (see Definitions 5.12 and 6.2) hence if q is based on a model M then there is a constructible model $N \supseteq Mb$ for any realization b of $q|M$, and moreover, any $N \supseteq Mb$ that is dominated by b over M is constructible.

It is this last result which produces in particular the following easy to state theorem:

Theorem 5.17 Suppose that T is classifiable and M is any model. If $p \in S(M)$ is regular but not locally modular and b is any realization of p then every model N containing M that is dominated by b over M is both constructible and minimal over Mb.

Section 2 contains some background information and as well there is an appendix (section 7) with a review of basic geometric stability theory. Section 3 contains the introduction of the notion of strongly semi-regular types and before proceeding to the main theorem, we provide some applications in section 4. The main theorem is found in section 5 and the paper proper concludes with an examination of other circumstances under which domination implies isolation in section 6.

2 Preliminaries and historical background

One of the major accomplishments of stability theory was Shelah's proof of the Main Gap in [10] where the notion of classifiable was introduced.

Definition 2.1 A complete theory T in a countable language is *classifiable* if T is superstable, has prime models over pairs (PMOP) and does not have the dimension order property (NDOP).

We assume that the reader is familiar with superstability and adopt the usual convention when working with stable theories that we are working in a large, saturated model $\mathfrak{C}$ of the theory and all models mentioned are small elementary submodels of $\mathfrak{C}$, sets are small subsets of $\mathfrak{C}$ and tuples are from $\mathfrak{C}$. We will also assume that T eliminates imaginaries i.e. $T = T^{eq}$.

We include for the reader's convenience an appendix at the end of this paper where we recall many of the basic definitions and notions from classification theory which we will refer to throughout.

The two properties, NDOP and PMOP, refer to independent triples $\mathcal{M} = (M_0, M_1, M_2)$ of models of T, where $M_0 \subseteq M_1$, $M_0 \subseteq M_2$, and $M_1 \underset{M_0}{\downarrow} M_2$. A superstable theory *does not have the dimension order property* (NDOP) if, for every triple $\mathcal{M}$ of a-saturated models, the a-prime model M^* over M_1M_2 (which exists in any superstable theory) is minimal among all a-saturated models containing M_1M_2. Shelah proves that NDOP is equivalent to the statement that any regular type q non-orthogonal to M^* is either non-orthogonal to M_1 or M_2.

A superstable theory T has *prime models over pairs* (PMOP) if, for any independent triple (M_0, M_1, M_2) of models, there is a constructible[1] model over M_1M_2.

After Shelah defined PMOP, Harrington gave another treatment, which was developed in [1]. Given two independent triples $\mathcal{M} = (M_0, M_1, M_2)$, $\mathcal{N} = (N_0, N_1, N_2)$ of models, say that $\mathcal{N}$ *extends* $\mathcal{M}$ if $M_0 \subseteq N_0$, $N_0 \underset{M_0}{\downarrow} M_1M_2$, and N_i is dominated by M_i over N_0 for $i = 1, 2$. Call a strong type $\mathrm{stp}(b/M_1M_2)$ V*-isolated* if $b \underset{M_1M_2}{\downarrow} N_1N_2$ for every extension $\mathcal{N}$ of $\mathcal{M}$. Among countable superstable theories with NDOP, PMOP is equivalent to "every V-isolated strong type is isolated."

This notion of V-isolation will not be used directly in the paper, but it is an example of how, in classifiable theories, certain forms of domination or "weak" isolation, imply actual isolation. This is a theme that occurs throughout the paper and is highlighted in section 6.

In [9] but building extensively on [10], Shelah and Buechler prove that among complete countable theories T, T is classifiable if and only if every model is prime and minimal over an independent tree of countable, elementary substructures. We shall call such a tree a classifying tree.

Since then, there has been a considerable amount of work analyzing the 'fine structure' of classifiable theories. The fine structure to a large extent revolves around understanding the leaves of classifying trees. The leaves are controlled by depth zero types and so we remind the reader of the definition (for more definitions and classical results see the appendix, section 7).

Definition 2.2 A regular type p in a superstable theory is said to have *depth zero* if for any a-model M on which p is based and any realization b of $p|M$, any type q over $M[b]$ (the a-saturated prime

[1]Recall that a model N is constructible over a set B if its universe can be enumerated as $\{c_i : i < \alpha\}$ with $\mathrm{tp}(c_i/B \cup \{c_j : j < i\})$ isolated for each i. Any two constructible models over B are isomorphic.

model over Mb), is non-orthogonal to M i.e. p does not support a regular type.

A *leaf* is a triple (M, b, N) where $M \subset N$, $b \in N$, $tp(b/M)$ is regular and depth zero, and N is dominated by b over M. As one can imagine, the computation of the uncountable spectrum of a countable theory depends on, among other things, understanding the isomorphism types of leaves that appear in the classifying trees of models. Through the coarseness of cardinal arithmetic, in [2], it was not necessary to determine all possible isomorphism types of leaves in order to determine the uncountable spectra of countable theories. Still, the techniques available from classification theory are suitable to do this and demonstrate a deeper understanding of the structure of models of countable, classifiable theories. In this paper and subsequent papers, we intend to explore the isomorphism types of leaves (M, b, N) at least when $tp(b/M)$ is non-trivial; in a classifiable theory, non-trivial types necessarily have depth zero. We recall two separate facts about leaves that were known before this investigation.

First of all, if (M, b, N) is a leaf in a classifiable theory, $tp(b/M)$ is non-trivial and $M \subseteq_a \mathfrak{C}$ (all strong types of finite tuples from $\mathfrak{C}$ over finite tuples in M are realized in M) then the isomorphism type of N is determined up to isomorphism over M; see [10]. Of course such an M would typically have size at least the continuum but this still shows that the geometry of $tp(b/M)$ plays a role in the isomorphism types of leaves.

Secondly, recall that if T is countable and stable, then for any set A, there is an ℓ-constructible model N over A (see Definition in Section 7.3). In particular, if $M \subset N$ and N is ℓ-constructible over Mb, then N is dominated by b over M. In fact, if T is countable, superstable, and NDOP, and (M, b, N) is a leaf then N is ℓ-constructible over Mb. This doesn't say that the isomorphism type of N is determined by Mb but it does put constraints on how such N can be built.

Our goal is to measure the extent to which these results can be extended to constructible models assuming classifiability. Within the context of ω-stable theories, this is easy. As ω-stable theories have constructible models over any set, an ω-stable theory is classifiable if and only if it has NDOP. For such theories, if (M, b, N) is a leaf then N is constructible over Mb. Unfortunately, for an arbitrary classifiable theory, there are leaves (M, b, N) for which N is not constructible over Mb as this following example shows.

Example 2.3 The language will consist of countably many sorts U_n and a collection of relations R^n_η for $\eta \in 2^{<\omega}$ for $n \in N$. There will also be a function $+_n$ for each n. Between sorts U_{n+1} and U_n there will be a function f_n. The canonical model of our theory in this language is as follows:

1. U_n will be interpreted as the product of $n+1$ many copies of 2^ω;
2. R^n_η will hold of an $n+1$-tuple $x_0, \ldots, x_n$ iff η is an initial segment of x_n;
3. $+_n$ is interpreted as coordinatewise addition modulo 2, and
4. f_n is the projection onto the first n coordinates.

The theory T of this structure is classifiable; in fact, it is superstable and unidimensional but not ω-stable. Now suppose that M is a model of T and N is an elementary weight one extension of M - the type of thing that would happen with leaves on a classifying tree. The claim is that if $b \in N \setminus M$ is any finite tuple then N is not constructible over M. To see this, suppose we have such a b. By presence of addition in all the sorts and the model M, we can assume that b is a singleton in some sort U_n. But then if one considers the preimage of b under f_n, one sees that this formula in the sort U_{n+1} does not contain an isolated type - the predicates R^n_η preclude this.

This example is suggestive of the result we will prove in subsequent papers: if we don't restrict ourselves to finite tuples then of course N is determined by making a coordinated choice of elements from each sort.

In the example, all the regular types are locally modular (non trivial). It is not clear in advance that this is important but in this paper we will show that if (M, b, N) is a leaf in a countable, classifiable theory and $tp(b/M)$ is *not* locally modular then N is constructible over Mb.

3 Strongly p-semiregular types

Definitions of semi-regular types, p-simplicity and other related notions can be found in the appendix.

Definition 3.1 A stationary type $q \in S(B)$ is *strongly p-semiregular* of weight $k > 0$ if q is p-semiregular of weight k and there is a p-simple formula $\theta(x) \in q$ of weight k such that if d realizes θ and $C \supseteq B$ with $d \underset{B}{\perp} C$ and $w_p(d/C) = w_p(q)$, then $\mathrm{tp}(d/C) = q|C$, the non-forking extension of q to $S(C)$.

The goal of this whole section is to prove the following Theorem. Its proof is patterned after the argument in [4], where Hrushovski and

Shelah prove that in a classifiable theory, every non-locally modular stationary, regular type is strongly regular (i.e., strongly p-semiregular of p-weight one). We will refer to precise Lemmas in their paper quite a few times in this section. When there is a risk of ambiguity, these Lemmas will be denoted as "Lemma x.y[4]".

Theorem 3.2 *If T is classifiable and p is a non-locally modular regular type, then every stationary p-semiregular type is strongly p-semiregular.*

Remark 3.3 It appears that all is needed is T superstable, PMOP, and p is non-locally modular of depth zero.

Remark 3.4 By the open mapping theorem, this notion is parallelism invariant. In particular, if $d \perp E$ with $\mathrm{tp}(d/\emptyset)$ stationary, then if $\mathrm{tp}(d/E)$ is strongly p-semiregular via $\theta(x, \bar{e})$, then $\mathrm{tp}(d/\emptyset)$ will be strongly p-semiregular via $d_r y\theta(x, y)$, where $r = \mathrm{stp}(e/\emptyset)$.

We will use this Remark as justification for freely adding independent parameters in many places.

3.1 Triples

In this subsection, T is superstable and p is a stationary regular type over $\emptyset$. We adopt the data structure of a *triple* and then show that a given triple can be massaged to get matching triples with more and more desirable properties. The key will be to obtain a *minimal triple* as a normal cover of a given one.

Definition 3.5

- A *triple* is a sequence (a, b, C) such that $a, b \in D(p, C)$, i.e., $\mathrm{stp}(ab/C)$ is p-simple.
- A triple (a, b, C) is *normal* if the three strong types $\mathrm{stp}(ab/C)$, $\mathrm{stp}(a/Cb)$, and $\mathrm{stp}(b/Ca)$ are all p-semiregular.
- A triple (a, b, C) is *p-disjoint* if $\mathrm{cl}_p(Ca) \cap \mathrm{cl}_p(Cb) = \mathrm{cl}_p(C)$.
- Two triples (a, b, C), (a', b', C') are *matching* if $w_p(ab/C) = w_p(a'b'/C')$, $w_p(a/Cb) = w_p(a'/C'b')$, and $w_p(b/Ca) = w_p(b'/C'a')$.

There is a canonical way of extending a p-disjoint triple (a, b, C) to a matching, normal p-disjoint triple (a', b', C').

Definition 3.6 A *soft extension* of a triple (a, b, C) is a triple (a', b', C') such that

1. $a'b'C' \subseteq \mathrm{dcl}(abC)$;

2. $C \subseteq C' \subseteq \mathrm{cl}_p(C)$ with $C' \setminus C$ finite and $C' \mathop{\downarrow}_C ab$
3. $a \subseteq a' \subseteq \mathrm{cl}_p(C'a)$ and $b \subseteq b' \subseteq \mathrm{cl}_p(C'b)$.

Note that $\mathrm{cl}_p(C') = \mathrm{cl}_p(C)$, $\mathrm{cl}_p(C'a') = \mathrm{cl}_p(Ca)$, and $\mathrm{cl}_p(C'b') = \mathrm{cl}_p(Cb)$ in any soft extension.

Lemma 3.7 (Normalization) *Given any (a, b, C), there is a soft extension to a matching, normal triple (a', b', C'), called the normalization of (a, b, C). If (a, b, C) is p-disjoint, then (a', b', C') will be p-disjoint as well.*

Proof. First, apply Lemma 7.14 to ab/C to get $C' \subseteq \mathrm{dcl}(Cab) \cap \mathrm{cl}_p(C)$ such that $\mathrm{stp}(ab/C')$ is p-semi-regular. Note that $\mathrm{cl}_p(C') = \mathrm{cl}_p(C)$, $\mathrm{cl}_p(C'a) = \mathrm{cl}_p(Ca)$, and $\mathrm{cl}_p(C'b) = \mathrm{cl}_p(Cb)$. Next, apply the Lemma to b/aC' to get a' and $\mathrm{stp}(b/C'a')$ p-semi-regular, and finally apply the Lemma to $a'/C'b$ to get b'. It is easily checked that (a', b', C') is normal and matches (a, b, C). The preservation of p-disjointness is clear because of the equality of the p-closed sets mentioned above. $\square$

Next we describe three ways of extending a given triple (a, b, C) to a larger, matching (a', b', C') that preserves p-disjointness.

Definition 3.8 A *simple extension* of a triple (a, b, C) is any of:

1. (a^*, b, C), where $a \subseteq a^* \subseteq \mathrm{cl}_p(Ca)$;
2. (a, b^*, C), where $b \subseteq b^* \subseteq \mathrm{cl}_p(Cb)$;
3. (a, b, C^*), where $C^* \supseteq C$ and $C^* \mathop{\downarrow}_C ab$.

Lemma 3.9 *Given any p-disjoint, normal (a, b, C), the normalization (a', b', C') of any simple extension is a p-disjoint, matching extension of (a, b, C).*

Proof. That all three species of simple extensions are matching is clear. Next, we show that each of the simple extensions preserves p-disjointness. This is clear for the first two, as $\mathrm{cl}_p(Ca^*) = \mathrm{cl}_p(Ca)$ in the first case and $\mathrm{cl}_p(Cb^*) = \mathrm{cl}_p(Cb)$ in the second. As (a, b, C) normal implies $\mathrm{stp}(ab/C)$ is p-semiregular, p-disjointness of the third species is preserved by Lemma 7.20.

The Lemma now follows from Lemma 3.7. $\square$

Definition 3.10 Suppose that (a, b, C) is a p-disjoint normal triple.

- A *normal cover* is any p-disjoint normal (a', b', C') obtained as a sequence of extensions as in Lemma 3.9.
- The *strength* of (a, b, C) which we denote by $\alpha(a, b, C)$, is equal to $R^\infty(a/bb'C)$, where b' is (any) element satisfying $\mathrm{stp}(b'/Ca) = \mathrm{stp}(b/Ca)$ and $b' \underset{Ca}{\perp} b$. (This is well-defined, as $\mathrm{stp}(abb'/C)$ is independent of our choice of b'.)
- A normal triple (a, b, C) is *minimal* if $\alpha(a, b, C) \leq \alpha(a', b', C')$ for all of its normal covers (a', b', C').

Clearly, by superstability and the transitivity of being a normal cover, every normal triple (a, b, C) has a minimal, normal cover (a', b', C'). In fact, one can find one with the additional property that $a' \in \mathrm{dcl}(C'a)$. At present, this improvement does not seem to be necessary.

Lemma 3.11 *Given any p-disjoint, normal triple (a, b, C), there is a matching minimal, normal, p-disjoint triple (a', b', C') that is a normal cover of (a, b, C).*

Proof. Among all normal covers (a', b', C') of (a, b, C), choose the one of smallest strength. □

We close with two lemmas concerning p-disjointness.

Lemma 3.12 *Suppose a/C and b/C are both p-semi-regular and $a \underset{C}{\perp} b$. Then the triple (a, b, C) is p-disjoint.*

Proof. Choose any $e \in \mathrm{cl}_p(Ca) \cap \mathrm{cl}_p(Cb)$. Then $w_p(e/Ca) = w_p(e/Cb) = 0$. As $\mathrm{stp}(b/Ca)$ is p-semi-regular, this implies $b \underset{Ca}{\perp} e$, hence $b \underset{C}{\perp} e$. But this, coupled with $w_p(e/Cb) = 0$ implies $e \in \mathrm{cl}_p(C)$. □

Lemma 3.13 *Suppose (a_1, b_1, C) and (a_2, b_2, C) are both normal and p-disjoint. If, moreover, $a_1b_1 \underset{C}{\perp} a_2b_2$, then the triple (a_1a_2, b_1b_2, C) is also p-disjoint.*

Proof. In light of Lemma 7.20, then p-disjointness of (a_1, b_1, C) implies that $\mathrm{cl}_p(a_1a_2b_2C) \cap \mathrm{cl}_p(b_1a_2b_2C) \subseteq \mathrm{cl}_p(a_2b_2C)$, so

$$\mathrm{cl}_p(a_1a_2C) \cap \mathrm{cl}_p(b_1b_2C) \subseteq \mathrm{cl}_p(a_2b_2C)$$

Arguing in reverse, the p-disjointness of (a_2, b_2, C) yields

$$\mathrm{cl}_p(a_1a_2C) \cap \mathrm{cl}_p(b_1b_2C) \subseteq \mathrm{cl}_p(a_1b_1C)$$

Furthermore, it follows immediately from Lemma 3.12 that

$$\mathrm{cl}_p(a_1b_1C) \cap \mathrm{cl}_p(a_2b_2C) \subseteq \mathrm{cl}_p(C)$$

and the result follows. □

3.2 Triples over semi-regular types

Suppose that we have a fixed p-semi-regular type $\mathrm{stp}(e/\emptyset)$. To find a formula as in Theorem 3.2 one first encapsulates e into the first component of a triple.

Definition 3.14 A triple (a, b, C) is *over* e if $e \in \mathrm{dcl}(a)$ and $e \perp Cb$.

Lemma 3.15 *Suppose that (a, b, C) is over e. Then every soft extension (a', b', C') is also over e. In particular, the normalization (a', b', C') given by Lemma 3.7 is also over e.*

Proof. Since $a \subseteq a'$, $e \subseteq \mathrm{dcl}(a')$. Also, since $C' \underset{C}{\perp} ab$, we get $C' \underset{Cb}{\perp} e$, implying that $\mathrm{stp}(e/C'b)$ does not fork over $\emptyset$. Again, by p-semiregularity, we have $e \underset{C'b}{\perp} \mathrm{cl}_p(C'b)$, so $\mathrm{stp}(e/C'b')$ does not fork over $\emptyset$. □

3.3 Using triples

We recall in the appendix basic definitions and facts about regular locally modular types (Section 7.1)

The following Proposition is the content of Lemma 3.2 of [4].

Proposition 3.16 *Suppose $p \in S(\emptyset)$ is a stationary, non-locally modular regular type. Then, for every realization d of p, there is a minimal, normal, p-disjoint triple (a, b, C) over d. Moreover, $w_p(a/C) = w_p(b/C) = 2$, $w_p(ab/C) = 3$, and $w_p(a/bC) = w_p(b/aC) = 1$.*

Proof. By the first paragraph of the proof of Lemma 3.2 in [4], there is a triple (a, b, M), where M is an a-model, a and b each consist of two M-independent realizations of $p|M$, with $w_p(a/Mb) = w_p(b/Ma) = 1$ and the triple (a, b, M) being p-disjoint.

By the p-weight computations, at least one of $\{a_1, a_2\}$ must be independent from b over M, so by passing to an automorphism of $\mathfrak{C}$, we may assume that $d \subseteq a$ and $d \perp Mb$. Thus, (a, b, M) is over d. Now, apply the Normalization Lemma 3.7, and then choose a minimal normal cover via Lemma 3.11. □

The following generalization is the point of all these definitions.

Proposition 3.17 *Suppose $p \in S(\emptyset)$ is stationary, regular, but non-locally modular. For any $k \geq 1$ and any e such that $\mathrm{stp}(e/\emptyset)$ is p-semi-regular with $w_p(e) = k$, there is a minimal, normal, p-disjoint (a, b, C) over e such that $w_p(ab/C) = 3k$, $w_p(a/C) = w_p(b/C) = 2k$, and $w_p(a/bC) = w_p(b/aC) = k$.*

Proof. First, by p-semi-regularity, choose E and an E-independent $(d_i : i < k)$ realizations of $p|E$ such that $e \perp E$ and e and $(d_i : i < k)$ are domination equivalent over E. Without loss, we may assume $E = \emptyset$.

Fix (a_0, b_0, C_0) as in Proposition 3.16. Without loss, we may assume that it is over d_0. Next, choose $\{a_i b_i : i < k\}$ to be C_0-independent with $\text{stp}(a_i b_i / C_0) = \text{stp}(a_0 b_0 / C_0)$ for each i. Again, we may assume that each d_i is over (a_i, b_i, C_0). As notation, let $\bar{a}$ denote $\{a_i : i < k\}$ and $\bar{b}$ denote $\{b_i : i < k\}$.

By iterating Lemma 3.13, we have that the triple $(\bar{a}, \bar{b}, C_0)$ is p-disjoint. As well, it follows from the independence that $w_p(\bar{a}/C_0) = w_p(\bar{b}/C_0) = 2k$, $w_p(\bar{a}\bar{b}/C_0) = 3k$, and $w_p(\bar{a}/C_0\bar{b}) = w_p(\bar{b}/C_0\bar{a}) = k$. It also follows that $(\bar{a}, \bar{b}, C_0)$ is normal.

As $w_p(e/d_0, \ldots, d_{k-1}) = 0$, it follows that $w_p(e/\bar{a}C_0) = 0$. Thus, the triple $(e\bar{a}, \bar{b}, C_0)$ is a simple extension of $(\bar{a}, \bar{b}, C_0)$. By employing Lemmas 3.9, 3.11, and 3.15, there is a matching, minimal, normal cover (a, b, C) of $(e\bar{a}, \bar{b}, C_0)$ that is p-disjoint and over e.

□

We continue literally along the lines of Section 3 of [4], with our Proposition 3.17 taking the place of their Lemma 3.2. A key point is that we have the following Lemma, which takes the place of their Lemma 3.3.

From now on, fix a triple (a, b, C) as in Proposition 3.17.

Lemma 3.18 *Choose any b' realizing $\text{stp}(b/Ca)$ with $b' \underset{Ca}{\perp} b$. Then:*

1. $w_p(a/bb'C) = 0$*;*
2. $b' \underset{C}{\perp} b$*;*
3. $\text{stp}(a/Cbb')$ *is isolated. (This is slightly stronger than what is stated in the text. The 'improvement' is not needed.)*

Proof. (1) Here is where p-disjointness plays a leading role. Recall that (a, b, C) covers $(\bar{a}, \bar{b}, C_0)$, where for each $i < k$, $w_p(a_i/b_i C_0) = 1$. As $\text{stp}(a_i/b_i C_0)$ is p-semi-regular, we additionally have $w_p(a_i/bC) = 1$ as well. So, in order to establish (1), it suffices to prove that $w_p(a_i/bb'C) = 0$ for each $i < k$.

Fix an $i < k$. We will actually prove that $a_i \underset{C_0 b_i}{\not\perp} b'$, which suffices. To see this, by p-disjointness we have

$$\text{cl}_p(Ca) \cap \text{cl}_p(Cb) \subseteq \text{cl}_p(C)$$

Thus, $\text{acl}(C_0 a_i) \cap \text{acl}(C_0 b_i) \subseteq \text{cl}_p(C)$. Coupling this with the fact that $\text{stp}(a_i b_i / C_0)$ is p-semi-regular implies $a_i b_i \underset{C_0}{\perp} \text{cl}_p(C)$, hence

$$\text{acl}(a_i C_0) \cap \text{acl}(b_i C_0) \subseteq \text{acl}(C_0)$$

Now, on one hand, $b' \underset{Ca}{\perp} b$, so in particular $b_i \underset{Ca}{\perp} b'$. By p-semi-regularity, $a_i b_i \underset{C_0}{\perp} Cab$, so $b_i \underset{C_0 a_i}{\perp} Ca$ hence, by transitivity

$$b_i \underset{C_0 a_i}{\perp} b'$$

On the other hand, since $\mathrm{stp}(b'/Ca) = \mathrm{stp}(b/Ca)$, there is $b'_i \in b'$ (corresponding to b_i) such that $b'_i \underset{C_0}{\not\perp} a_i$, hence

$$b' \underset{C_0}{\not\perp} a_i b_i$$

Combining the last three displayed expressions with Lemma 7.22 (where b' takes the role of X) we obtain $a_i \underset{C_0 b_i}{\not\perp} b'$.

(2) Given (1), this is identical to the p-weight computation given in Lemma 3.3[4] just multiplied by k.

(3) This is just like 3.3(c)[4], with the minimality playing the same role here as it did there. More precisely, in order to establish $a \underset{Cbb'}{\perp} BB'$, one splits BB' into its the p-weight zero part and its p-semi-regular part. The independence of the first half is due to (a, b, C) having minimal strength, and the independence of the second half is due to the fact that $w_p(a/Cbb') = 0$ from (1). □

Continuing, Lemma 3.4[4] goes through, with the following changes:

- In (1), multiply all inequalities by k;
- Replace (2)ii by '$w_p(b'/Cab) < k$';
- (2)iii remains as stated, '$\rho(a, b, b')$ holds and $w_p(a/Cbb') = 0$.'

With these changes, Lemma 3.5[4] goes through. Finally, the Proof of 3.1[4] goes through verbatim, noting our definition of (a, b, C) being 'over e' implies $e \perp Cb$. In particular, $\mathrm{stp}(e/Cb)$ is strongly p-semiregular. In light of Remark 3.4, so is $\mathrm{stp}(e/\emptyset)$. □

4 Applications of Theorem 3.2

In this brief section, we give two applications of Theorem 3.2, although they will not be used in the proof of our main results.

Lemma 4.1 *If T is superstable, if H is an infinitely definable connected group (over A), with generic strongly p-semiregular, then H is definable over A.*

Proof. Let r be the generic of H and $\varphi(x) \in L(A)$ be the formula such that r is the unique type of p-weight k in φ. By superstability, there is a definable group H' with connected component H. The formula $\varphi(x)$ contains the principal generic, hence H' is a finite union of translates of φ. Each translate of φ contains a unique type of p-weight equal to k. Every other generic type of H' must also be of p-weight k, so there are only finitely many generic types in H'. This means that H, its connected component, has finite index in H', hence that H itself is in fact definable (H is a closed subgroup in H', if it has finite index it must be also open). $\square$

Together with Theorem 3.2, this immediately yields the following corollary:

Corollary 4.2 *Let T be classifiable and p a non-locally modular regular type. If H is an infinitely definable connected group (over A), with generic p-semiregular, then H is definable over A.*

The second application connects binding group constructions with isolation of p-semiregular types, in the spirit of [5] or more recently [6].

First, an easy remark :

Remark 4.3 *In general, if there is a B-*definable *group G acting transitively on a complete type q over B, then q is isolated over B: Let e realize q, consider the formula $\varphi(x) \in L(Be) : \exists g \in G\ x = g.e$. Then $\varphi(x)$ holds if and only if $x \models q$. So q is isolated by a formula over e. On the other hand , q is B-invariant, so in fact, q is isolated by a formula over B.*

Let us now recall some definitions and notation.

Definition 4.4 Let $B = acl(B)$ and q be a (partial) type over B and r a complete type over B. We say that r is *q-internal* over B if there is $D \supset B$ such that for every $e \models r|D$, $e \in dcl(q())$.

Proposition 4.5 ([7] or [8]) *Let $B = acl(B) \supset M$, $q \in S(B)$, $q \not\perp M$, but $q \perp^a M$. Let $p \in S(M)$ such that $q \not\perp p$. Then for all $a \models q$, there is $a' \in dcl(Ba) \setminus dcl(B)$ such that $tp(a'/B)$ is p-internal.*

Lemma 4.6 *Let $B = acl(B) \supseteq M$, $q \in S(B)$, $q \not\perp M$, but $q \perp^a M$. Let $p \in S(M)$ be regular such that $q \not\perp p$ and suppose that q is p-semiregular. Then $a \underset{B}{\perp} p(\mathfrak{C})$ for any a realizing q.*

Proof. Let a realize q and let $E \subseteq p(\mathfrak{C})$ be finite. Let $E_1 \subseteq E$ be a maximal M-independent subset of E, and let E_2 be a maximal subset of E_1 satisfying $E_2 \underset{M}{\perp} B$. Note that E_2 realizes a Morley sequence $(p|B)^{(k)}$ of the non-forking extension of p to B.

Claim. $w_p(E/E_2B) = 0$.

Proof. First, choose any $e \in E_1 \setminus E_2$. Because maximality implies that $eE_2 \underset{M}{\not\perp} B$, $\mathrm{tp}(e/BE_2)$ is a forking extension of a regular type, hence $w_p(e/E_2B) = 0$. It follows that $w_p(E_1/E_2B) = 0$ as well. Now choose any $e \in E \setminus E_1$. By the maximality of E_1, $e \underset{M}{\not\perp} E_1$ so $w_p(e/E_1) = 0$. Thus, $w_p(e/E_1B) = 0$. As this holds for every $e \in E \setminus E_1$, $w_p(E/E_1B) = 0$. Combining the two arguments yields $w_p(E/E_2B) = 0$. $\square$

Because $\mathrm{tp}(E_2/B)$ does not fork over M and $q \perp^a M$, we have that $a \underset{B}{\perp} E_2$. As q is p-semiregular, that $a \underset{B}{\perp} E$ follows immediately from the Claim. $\square$

Theorem 4.7 (Binding group) ([7] 7.4.8 or [8] 2.2.20) *Let $B = acl(B)$ $q \in S(B)$, $p \in S(B)$ such that q is $p(\mathfrak{C})$-internal.Let $G := Aut(q(\mathfrak{C})/B \cup p(\mathfrak{C}))$. Then G and its action are infinitely definable in the following sense: there is G_1 an infinitely definable group over B and a B-definable action of G_1 on $q(\mathfrak{C})$ such that, as permutation groups of $q(\mathfrak{C})$ over C, G and G_1 are isomorphic. If $a \underset{B}{\perp} p(\mathfrak{C})$ for all a realizing q, then G and hence also G_1 act transitively on $q(\mathfrak{C})$.*

Note that by stability, the group G is also the group of restrictions to $q(\mathfrak{C})$ of the automorphisms of $\mathfrak{C}$ fixing $p(\mathfrak{C}) \cup B$ pointwise. Recall that G is infinite if and only if q is not algebraic over $p(\mathfrak{C})) \cup B$.

We can always suppose that the action is faithful (that is, that the subgroup of G_1 which fixes all of $q(\mathfrak{C})$ pointwise is trivial).

The next proposition tells us that we can always suppose that G is connected.

Proposition 4.8 *If the infinitely definable group G and its connected component G^0 are defined over $B = acl(B)$ and G acts definably transitively on the type q over B, then so does its connected component G^0.*

Proof. Let Q denote the set of realizations of the type q in $\mathfrak{C}$ and let $B = \emptyset$.
As G^0 is a subgroup of G, everything is clear except that G^0 acts transitively on Q. We obtain this via two claims and a brief argument.

Claim 1. There is some pair (e, e^*) realizing $q \otimes q$ and some $h \in G^0$ such that $h.e = e^*$.

Proof. Choose a set of representatives $R \subseteq G(\mathfrak{C})$ such that every $g \in G$ can be written as ch for some $c \in R$ and $h \in G^0$, i.e., R contains an element of every G^0-coset of G. As the index $[G : G^0]$ is bounded, we may choose R of bounded size ($2^{\aleph_0}$ if the language is countable). Choose any e realizing $q|R$ and any e' realizing $q|Re$.

By the transitivity of the action, as both e, e' realize q, choose $g \in G$ such that $g.e = e'$. By choice of R, choose $c \in R$ and $h \in G^0$ such that $g = ch$.

Now, as $c \in R$, both g and h are equi-definable over R. Thus, $g.e$ and $h.e$ are equi-definable over Re. As $g.e = e'$ and $e' \underset{\emptyset}{\perp} Re$, we conclude that $h.e \underset{\emptyset}{\perp} Re$. Thus, $e^* := h.e$ satisfies the Claim. $\square$

Claim 2. For every (e, e') realizing $q \otimes q$ there is $h \in G^0$ such that $h.e = e'$.

Proof. Homogeneity of $\mathfrak{C}/B$. Fix (e, e^*) and h as in Claim 1, and let (e_1, e_2) be any other realization of $q \otimes q$. Choose an automorphism σ of $\mathfrak{C}$, fixing B pointwise with $\sigma(e) = e_1$ and $\sigma(e^*) = e_2$. Then, as G and the action are B-definable, $\sigma(h) \in G^0$ and $\sigma(h).e_1 = e_2$. $\square$

To complete the proof of the Lemma, choose any $e, f \in Q$. Choose e^* realizing $q|\{e, f\}$. By Claim 2, choose $h_1 \in G^0$ such that $h_1.e = e^*$ and choose $h_2 \in G^0$ such that $h_2.e^* = f$. Then $h_2h_1 \in G^0$ and $h_2h_1.e = f$, so G^0 acts transitively on Q.

$\square$

The following results give a sufficient condition for a non locally modular type to be isolated. These results will be used as part of the forthcoming work of the authors on the analysis of weight one models in classifiable theories.

The following definition appears already in [5].

Definition 4.9 We say that $q \in S(B)$ is c-isolated (following Hrushovski-Shelah in [5]) if there is a formula $\theta(x) \in L(B)$, $q \vdash \theta(x)$ such that $R^\infty(\theta(x) = R^\infty(q) = \alpha$ and furthermore, for any $r \in S(B)$, such that $r \vdash \theta(x)$, $R^\infty(r) = \alpha$.

Proposition 4.10 *Let $q \in S(B)$ be p-strongly semiregular and c-isolated via the formula $\varphi(x)$ and let G_1 be an infinitely definable group over B, with a B-definable faithful transitive action of G_1 on $q(\mathfrak{C})$. Then q is isolated.*

Proof. By the usual construction, find a B-definable overgroup of G_1, H, a B-definable set X containing q, and a B-definable (faithful) transitive action of H on X which extends the action of G_1 on $q(\mathfrak{C})$, and such that $X \subset \varphi(\mathfrak{C})$.

So X has the property that every type over B in X has same R^∞ rank as q, say α, that every type in X is p-simple, of p-weight at most equal to k, the p-weight of q, and that q is the unique type in X of p-weight exactly k. We show that X isolates q.

Claim 1. Let e realize q, and let $h \in H$ be independent from e over B, then $h.e$ also realizes q.

Proof. Then e and h, h^{-1} are also independent over B. If $d = h.e$, then d and e are interdefinable over B, h, h^{-1}. As $X \subset \varphi(\mathfrak{C})$, $t(d/B)$ is p-simple of p-weight at most k. As e is independent from h, h^{-1} over B, it remains of p-weight k. By interdefinablity, d and e must have same p-weight over B, h, h^{-1}, that is k. It follows that $t(d/B)$ must already have p-weight k over B, and hence must be equal to q. $\square$

Claim 2. Let h be any element of H, not necessarily independent from e, and let $d = h.e$. Then d realizes q.

Proof. Let g be a generic in H, independent from h, e (and hence from h, e, d) over B. It follows that $g^{-1}h$ and e are independent over B : $g^{-1} \underset{Bh}{\perp} e$ by choice of g, hence $(g^{-1}h) \underset{Bh}{\perp} e$. As g^{-1} is generic and independent from h, $g^{-1}h$ is also generic independent from h, so it follows that $g^{-1}h$ and e are independent over B. Hence by Claim 1, $f := (g^{-1}h).e$ realizes q.

But, by c isolation of q, g and f are also independent over B: Note that d and f are interdefinable over Bg, as $f = g^{-1}.d$, hence they must have same ∞-rank over Bg. By our choice of g, $g \underset{B}{\perp} d$, hence $R^\infty(d/Bg) = R^\infty(d/B) = \alpha = R^\infty(f/Bg) = R^\infty(f/B)$. It follows that f and g are independent over B. Thus, by Claim 1, $g.f$ must realize q, but $g.f = d$. $\square$

As the action of H on X is transitive, we have shown that any element in the formula X must realize q, that is, that the type q is isolated. $\square$

Corollary 4.11 *Let T be classifiable and p regular non-locally modular. If $q \in S(B)$ $(B = acl(B))$ is p-semiregular, c-isolated and $q \perp^a M$, then q is isolated.*

Proof. First by Theorem 3.2, q is p-strongly semiregular. By classifiability, $q \not\perp M$ (Fact labelNDOP); by Proposition 4.5 and the binding group Theorem (Proposition 4.7), there is an infinitely definable group G, defined over B which acts transitively on $q(\mathfrak{C})$. It now follows by Proposition 4.10 that q is isolated. $\square$

Note that we have used two conditions on the type q to prove isolation: the strong p-semiregularity condition goes up to non forking extensions, but the c-isolation does not necessarily.

Let us finish this section by mentioning another way to prove isolation, without group actions, in the case of a strongly regular type. It is not clear if this could be generalized to the case of strong p-semiregularity.

Proposition 4.12 *T superstable $M \subseteq_{na} \mathfrak{C}$. Suppose B is algebraically closed with $M \subseteq B$, $tp(c/B)$ is strongly regular depth zero, and that $tp(c/B) \perp^a M$. Then either $tp(c/B) \perp M$ or $tp(c/B)$ is isolated.*

Proof. If $tp(c/B)$ is trivial, $tp(c/B) \perp^a M$ implies that $tp(c/B) \perp M$. So we can suppose that $tp(c/B)$ is non trivial.

Suppose that $tp(c/B) := q$ is non-orthogonal to M. Let p be a regular type based over M. As $q \perp^a M$, in particular, over B, q is almost orthogonal to $p^{(\omega)}$. By Fact 7.16 choose $\varphi \in q$ that is p-simple, such that p-weight is defined and continuous in φ. As q is strongly regular, by strengthening φ we may additionally assume q is the only type over B containing φ of positive p-weight.

Now choose n least such that there are $\bar{a} = (a_1, \ldots, a_n)$ with each a_i realizing φ and $tp(\bar{a}/B)$ is not almost orthogonal to $p^{(\omega)}$. We know that such a finite n exists, since q non-orthogonal to p implies that some $q^{(\ell)}$ is not almost orthogonal to $p^{(\omega)}$. Remark: Here, however, we are minimizing n without assuming $\bar{a}$ is B-independent.

Note that $n \geq 2$. Indeed, if a_1 realizes q, then a_1/B is almost orthogonal to $p^{(\omega)}$ by assumption. On the other hand, if a_1 realizes φ but not q, then $w_p(a_1/B) = 0$, so a_1 cannot fork over B with any any independent set of realizations of p.

Once n is fixed, choose k such that $\bar{a}/B$ is not almost orthogonal to $p^{(k)}$. To save writing, let $n = m + 1$ and $r(\bar{y}) := (p|B)^{(k)}$. Choose a specific realization $\bar{c}$ of r such that $\bar{a} \not\perp_B \bar{c}$, and choose an $L(B)$-formula $\theta(\overline{x}, \bar{y}) \in \mathrm{tp}(\bar{a}\bar{c}/B)$ witnessing the forking. Let

$$\gamma(x_0) := \varphi(x_0) \wedge d_r\bar{y} \left[\exists x_1 \ldots \exists x_m (\bigwedge \varphi(x_i) \wedge \theta(x_0, x_1, \ldots, x_m, \bar{y}))\right]$$

As B is algebraically closed, γ is over B. We argue that γ isolates q.

To see this, choose any b_0 realizing γ. Choose $\bar{d}$ realizing $r|Bb_0$, and choose witnesses $b_1, \ldots, b_m$. Thus, the n elements $b_0, \ldots, b_m$ each realize φ and $\theta(\bar{b}, \bar{d})$ holds. We argue that in fact, every b_i realizes

q. Let $I = \{i \leq m : b_i$ realizes $q\}$. Let $\bar{b}_I$ be the subsequence of $\bar{b}$ induced by I. By way of contradiction, assume $|I| < m + 1 = n$. By the minimality of n, we must have $\bar{b}_I \underset{B}{\perp} \bar{d}$. But also, by our choice of φ, if $i \notin I$, then $w_p(b_i/B) = 0$. Thus, $w_p(\bar{b}/B\bar{b}_I) = 0$. But $\bar{d}$ is a Morley sequence in p, hence $\bar{d}/B\bar{b}_I$ is p-semiregular. Thus, $\bar{b} \underset{B\bar{b}_I}{\perp} \bar{d}$. It follows by transitivity that $\bar{b} \underset{B}{\perp} \bar{d}$, which is contradicted by $\theta(\bar{b}, \bar{d})$. □

Corollary 4.13 *Let T be classifiable, $M \subseteq_{\mathrm{na}} \mathfrak{C}$, $M \subset B$ and $tp(c/B)$ be regular non locally modular. Suppose that $tp(c/B) \perp^a M$. Then either $tp(c/B) \perp M$ or $tp(c/B)$ is isolated.*

Proof. By classifiability of T, $tp(c/B)$ is then also strongly regular and depth zero. So the above Proposition applies. □

5 Constructible, minimal models over realizations of non-locally modular types

The basic definitions and facts about locally modular regular types can be found in the appendix (Section 7.1).

5.1 Witnesses to non-modularity

Definition 5.1 Let M be any model (not necessarily an a-model) and let $p \in S(M)$ be regular. A quadruple (a, b, c, d) is *4-dependent* if

1. Any three elements realize $p^{(3)}$, but
2. $w_p(abcd/M) = 3$.

A *witness to non-modularity for p over M* is a set of parallel lines, i.e., some $4 - \textit{dependent}$ quadruple (a, b, c, d) such that $\mathrm{cl}_p(Mab) \cap \mathrm{cl}_p(Mcd) = \mathrm{cl}_p(M)$.

The following proposition follows from Section 7.1 of the appendix:,

Proposition 5.2 *Over any a-model M, if $p \in S(M)$ is a regular type, then p is non-locally modular if and only if there is a witness to non-modularity for p over M.*

Lemma 5.3 *Suppose M is any countable model, and $p \in S(M)$ is regular, but not locally modular. There is an ϵ-finite set E (E is contained in the algebraic closure of some finite set) such that for any countable model M' containing $M \cup E$, there is a witness (a,b,c,d) to non-modularity over M'.*

Proof. Let $M^* \supseteq M$ be any a-model, and let q denote the non-forking extension of p to M^*. As q is not locally modular, it follows from Proposition 5.2 that there is a witness (a,b,c,d) to non-modularity over M^*. Choose a countable, ϵ-finite E over which $\mathrm{tp}(abcd/M^*)$ is based and stationary. To see that E suffices, choose any countable M' containing $M \cup E$. As M^* is sufficiently saturated, we may assume that $M' \preceq M^*$. We argue that (a,b,c,d) is a witness to non-modularity over M'. To see this, note that $abcd \mathop{\perp}_{M'} M^*$. Thus, (a,b,c,d) is 4-dependent with respect to the non-forking extension of p to M'. For the final clause, let $C := \mathrm{dcl}(M'abcd) \cap \mathrm{cl}_p(M')$. Note that $abcd \mathop{\perp}_{C} M^*$ and $\mathrm{stp}(abcd/C)$ is p-semi-regular by Criterion 7.12. Thus, $\mathrm{cl}_p(M^*ab) \cap \mathrm{cl}_p(M^*cd) = \mathrm{cl}_p(M^*)$ implies that $\mathrm{cl}_p(Cab) \cap \mathrm{cl}_p(Ccd) = \mathrm{cl}_p(C)$ by Proposition 7.20. However, $\mathrm{cl}_p(C) = \mathrm{cl}_p(M')$, $\mathrm{cl}_p(Cab) = \mathrm{cl}_p(M'ab)$, and $\mathrm{cl}_p(Ccd) = \mathrm{cl}_p(M'cd)$, so we finish. $\square$

5.2 A definable witness to non-modularity

In this section we assume throughout that T is classifiable.

By Lemma 5.3, if $p \in S(B)$ is any non-locally modular stationary, regular type over a countable set B, then there is a countable M containing B with a witness to non-modularity (a,b,c,d) over M (relative to the non-forking extension of p to M). We first explore how definable such a witness is. For this, we recall some theorems of Hrushovski and Shelah in [4].

Theorem 5.4 *If T is classifiable, B is algebraically closed, and $p \in S(B)$ is a regular, non-locally modular type, then there is a formula $\theta \in p$ such that (recall Definition 7.7):*

1. *θ is p-simple of p-weight one*
2. *p-weight is defined and continuous inside θ*
3. *p is strongly regular via the formula θ, i.e., for every $C \supseteq B$ and every $e \in \theta(\mathfrak{C})$, if $w_p(e/C) > 0$, then $\mathrm{tp}(e/C)$ is the non-forking extension of p to C.*

Lemma 5.5 *Suppose that $p \in S(M)$ is regular and (a,b,c,d) is a 4-dependent sequence of realizations of p. Then there is a symmetric* *formula $S(x_1,\dots,x_4) \in \mathrm{tp}(abcd/M)$ such that for any (b',c',d') realizing $p^{(3)}$ and for any element a' realizing $S(x_1,b',c',d')$, we have (a',b',c',d') is 4-dependent.*

Proof. For a given ordering of the variables, let S_0 be the conjunction of $\bigwedge_{i=1}^4 \theta(x_i)$ with formulas in $\mathrm{tp}(a,b,c,d)$ demonstrating that each variable has p-weight zero over the other three. Then, for any three coefficients, which we write as (x_2,x_3,x_4) for definiteness, if (b',c',d') realizes $p^{(3)}$ and $S_0(a',b',c',d')$ holds, then $\mathrm{tp}(d'/Mb'c'a')$ forks over $Mb'c'$. As $\mathrm{tp}(d'/Mb'c')$ is a non-forking extension of p, this implies $w_p(a'/M) > 0$. Combined with $\theta(a')$, this implies that $\mathrm{tp}(a'/M) = p$. Easy p-weight computations show that (a',b',c',d') is 4-dependent. To find a symmetric S, take the disjunction of the 24 S_0's, with respect to each ordering of $(x_1,\dots,x_4)$. $\square$

Next, among all 4-dependent quadruples (a,b,c,d), we want to distinguish those that are witnesses to non-modularity over M. This will require some Lemmas.

Lemma 5.6 *Suppose $p \in S(M)$ is a regular, non-locally modular type and (a,b,c,d) is a witness to non-modularity over M. Then for every e realizing $p|Mabcd$ and for every $f \in \theta(\mathfrak{C})$ that satisfies $b \in \mathrm{cl}_p(Maef)$, we have (c,d,e,f) realizes $p^{(4)}$ and $w_p(ab/Mcdef) = 0$.*

Proof. First, since $\theta(f)$ holds, $\mathrm{tp}(f/M)$ is p-simple. As b realizes p, $b \underset{M}{\perp} ae$, but $b \in \mathrm{cl}_p(Maef)$, we must have $w_p(f/M) > 0$, hence $\mathrm{tp}(f/M) = p$. Thus, all six elements a,b,c,d,e,f realize p. Next, note that $w_p(abcdef/M) = 4$, since $w_p(abcd/M) = 3$ (by 4-dependence), $e \underset{M}{\perp} abcd$, and $w_p(f/Mabcde) = 0$ by exchange. Next, we show that $w_p(cdef/M) = 4$, i.e., that (c,d,e,f) realizes $p^{(4)}$. By way of contradiction, assume that this were not the case, i.e, that $w_p(cdef/M) \le 3$. We compute the following p-weights:

- $w_p(ef/Mabcd) = 1$ [it is ≥ 1 because of e, but < 2 since $f \in \mathrm{cl}_p(Mabe)$].
- $w_p(ef/Mcd) = 1$ [it is ≥ 1 from the former line, but if it was $= 2$, then we would have $w_p(cdef/M) = 4$].
- $w_p(ef/Mab) = 1$ [it is > 0 because of e, but < 2 because $f \in \mathrm{cl}_p(Mabe)$].

Thus, as both $\mathrm{stp}(ef/\mathrm{cl}_p(Mab))$ and $\mathrm{stp}(ef/\mathrm{cl}_p(Mcd))$ are p-semiregular, we have

$$ef \underset{\mathrm{cl}_p(Mab)}{\perp} abcd \qquad \text{and} \qquad ef \underset{\mathrm{cl}_p(Mcd)}{\perp} abcd$$

Hence,

$$Cb(ef/Mabcd) \subseteq \mathrm{cl}_p(Mab) \cap \mathrm{cl}_p(Mcd) = \mathrm{cl}_p(M)$$

with the last equality holding since (a, b, c, d) is a witness to non-modularity. This would imply $ef \underset{\mathrm{cl}_p(M)}{\perp} abcd$, which contradicts $f \in \mathrm{cl}_p(Mabe)$.

Finally, since $w_p(abcdef/M) = w_p(cdef/M) = 4$, it follows immediately that $w_p(ab/Mcdef) = 0$. □

For the Corollary that follows, note that if (a, b, c, d) is 4-dependent, then there is an M-definable formula $\beta(x_2, a, c, d) \in \mathrm{tp}(b/Macd)$ that implies both $\theta(x_2)$ and $w_p(x_2/Macd) = 0$.

Corollary 5.7 *Suppose (a, b, c, d) is a witness to non-modularity over M and fix any formula $\beta(x_2, a, c, d) \in \mathrm{tp}(b/Macd)$ as above. Then there is an M-definable $\delta(x_1, x_2, x_3, x_4, y, z)$ such that*

$$R(x_1, \ldots, x_4) := d_p y \forall z\, [(\theta(z) \wedge \beta(x_2, x_1, y, z)) \rightarrow \delta(x_1, \ldots, x_4, y, z)]$$

is in $\mathrm{tp}(abcd/M)$.

Proof. Immediate, by Lemma 5.6 and compactness. □

We now consider the negation of the condition.

Lemma 5.8 *Suppose that $p \in S(M)$ is a regular, non-locally modular type and (a, b, c, d) is 4-dependent, but is not a witness to non-modularity over M. Then there are realizations e of $p|abcd$ and f of p such that $\mathrm{stp}(ef/Mab) = \mathrm{stp}(cd/Mab)$, yet $w_p(ab/Mabcd) \neq 0$.*

Proof. Choose e realizing $p|Mabcd$ and choose any f such that ef and cd have the same strong type over $\mathrm{cl}_p(Mab)$ and $ef \underset{\mathrm{cl}_p(Mab)}{\perp} cd$. By superstability, there is a finite g such that $\mathrm{cl}_p(Mg) = \mathrm{cl}_p(Mab) \cap \mathrm{cl}_p(Mcd)$. Now $\mathrm{tp}(g/M)$ is p-simple and, since (a, b, c, d) is 4-dependent but not a witness to non-modularity, $w_p(g/M) = 1$. Next, as $w_p(cd/M) = 2$ and $cd \underset{\mathrm{cl}_p(M)}{\not\perp} g$, we have $w_p(cd/Mg) \leq 1$. As $Mg \subseteq \mathrm{cl}_p(Mab)$ we have $\mathrm{stp}(ef/Mg) = \mathrm{stp}(cd/Mg)$, so $w_p(ef/Mg) \leq 1$ as well. Thus, $w_p(cdefg/M) \leq 3$, hence $w_p(cdef/M) \leq 3$. But, since $w_p(abcd/M) = 3$ and e realizes $p|abcd$, we have $w_p(abcdef) \geq 4$. It follows that $w_p(ab/Mcdef) > 0$. □

Proposition 5.9 *Suppose that $p \in S(M)$ is a regular, non-locally modular type, and that (a, b, c, d) is a witness to non-modularity over M. Then there is a symmetric formula $R^* \in \mathrm{tp}(abcd/M)$ such that for any (b', c', d') realizing $p^{(3)}$ and any a', if $R^*(a', b', c', d')$, then (a', b', c', d') is a witness to non-modularity over M.*

Proof. Fix an enumeration $(x_1, \ldots, x_4)$ of the variables. Choose a formula $S(x_1, \ldots, x_4) \in \mathrm{tp}(abcd/M)$ as in Lemma 5.5 and choose $\beta(x_2, a, c, d) \in \mathrm{tp}(b/Macd)$ as in the note preceding Corollary 5.7. Take $R(x_1, \ldots, x_4)$ from Corollary 5.7 and let $R' := S \wedge R$. Now, if (b', c', d') realizes $p^{(3)}$ and a' satisfies $R'(x_1, b', c', d')$, then (a', b', c', d') is 4-dependent by Lemma 5.5, and hence is a witness to non-modularity by Lemma 5.8. We obtain a symmetric R^* by taking a disjunction over all orderings of $(x_1, \ldots, x_4)$. $\square$

Corollary 5.10 *If T is classifiable, then for any model M and any $p \in S(M)$ that is not locally modular, there is a witness (a, b, c, d) to the non-modularity of p over M.*

Proof. Choose any triple (abc) realizing $p^{(3)}$ and choose any a-model $M^* \succeq M$ independent from abc over M. By Proposition 5.2 there is a witness (a', b', c', d') to the non-modularity of $p|M^*$ over M^*. As $\mathrm{tp}(abc/M^*) = \mathrm{tp}(a'b'c'/M^*)$, there is d such that (a, b, c, d) is a witness to the non-modularity of $p|M^*$ over M^*. Choose $R^* \in \mathrm{tp}(a, b, c, d/M^*)$ as in Proposition 5.9, and let e be a finite subset from M^* over which $\mathrm{tp}(abcd/M^*)$ is based and which contains the parameters over which R^* is defined. Write R^* as $R^*(x_1, \ldots, x_4, e)$. So $\mathrm{tp}(abce/M)$ implies $\exists x_4 R^*(a, b, c, x_4, e)$. As $abc \underset{M}{\perp} e$, it follows from finite satisfiability that there is some $e' \in M$ with $\exists x_4 R^*(a, b, c, x_4, e')$. Choose any d witnessing this and check that (a, b, c, d) is a witness to the non-modularity of p over M. $\square$

5.3 Depth-zero like types and minimality

The following Lemma is routine, but it is interesting that its proof does not require NDOP (although the verification that a non-trivial regular type has depth zero does).

Lemma 5.11 *Suppose T is superstable, $M \subseteq_{\mathrm{na}} N$, and $\{c_i : i < n\} \subseteq N$ are M-independent, with $\mathrm{tp}(c_i/M)$ regular of depth zero for each i. Let $M^* \preceq N$ be any model dominated by $\{c_i : i < n\}$ over M. Then:*

1. *M^* is minimal over $M \cup \{c_i : i < n\}$;*
2. *$M^* \subseteq_{\mathrm{na}} N$;*
3. *If q is regular and $q \not\perp M^*$, then $q \not\perp M$.*

Proof. We argue by induction on n. For $n = 0$, $M^* = M$, so there is nothing to prove. Assume the Lemma holds for sets of size

n, and choose $c_0 \ldots, c_n$ from N satisfying the hypotheses. Let M^* be dominated by $\{c_i : i \leq n\}$ over M. To see that M^* is minimal, choose any $M' \preceq M^*$ containing $M \cup \{c_i : i \leq n\}$. Let $M_0 \preceq M'$ be dominated by $\{c_i : i < n\}$ over M. As $M \subseteq_{\mathrm{na}} M'$, we can apply the Lemma to conclude that both $M_0 \subseteq_{\mathrm{na}} M'$ and that any regular type q non-orthogonal to M_0 is non-orthogonal to M. Now choose $M_1 \preceq M'$ to be dominated by c_n over M_0, by induction $M_1 \subseteq_{\mathrm{na}} M^*$. In order to show that M^* is minimal over $M \cup \{c_i : i \leq n\}$, it certainly suffices to prove that $M_1 = M^*$. If this were not the case, then there would be a regular type $q = \mathrm{tp}(d/M_1)$ realized in the difference. As $\mathrm{tp}(c_n/M_0)$ is a non-forking extension of $\mathrm{tp}(c_n/M)$, it is regular and of depth zero. Thus, by definition of depth zero, $q \not\perp M_0$. Thus, by the inductive hypothesis, q is non-orthogonal to M. So, by the 3-model Lemma (Fact 7.4) applied to the triple (M, M_1, M^*), there would be $e \in M^*$ such that $\mathrm{tp}(e/M)$ is regular and non-orthogonal to q, with $e \underset{M}{\perp} M_1$. As $\{c_i : i \leq n\}$, this implies $e \underset{M}{\perp} c_0, \ldots, c_n$, which contradicts M^* being dominated by $\{c_i : i \leq n\}$ over M. Thus, M^* is minimal over $M_{\{} c_i : i \leq n\}$, proving (1).

Next, by Fact 7.3 there is a model $N' \subseteq_{\mathrm{na}} N$ that contains and is dominated by M^* over M. Such an N is clearly dominated by $\{c_i : i \leq n\}$ over M, so the previous paragraph applies to N'. Thus, $N' = M^*$, so $M^* \subseteq_{\mathrm{na}} N$, which gives (2).

Finally, choose any regular q non-orthogonal to $M^* = M_1$. As $\mathrm{tp}(c_n/M_0)$ has depth zero, q is non-orthogonal to M_0, hence is non-orthogonal to M by our inductive assumption. $\square$

The following definition extends the concept of depth zero to both finite, independent tuples of depth zero types as well as to types dominated by such tuples.

Definition 5.12 A strong type p is *depth-zero like* if every regular type q non-orthogonal to p is of depth zero.

As examples, if a regular type p has depth 0, then any p-semiregular type q is depth zero-like. The following Proposition uses classifiability to obviate the need for $M \subseteq_{\mathrm{na}} \mathfrak{C}$ in Lemma 5.11(1).

Proposition 5.13 *Suppose p is depth zero-like.*

1. *If T is superstable and M is an a-model on which p is based, then for any realization b of $p|M$, every model N that is dominated by b over M is minimal over Mb.*
2. *If T is classifiable, then (1) holds for every model M on which p is based.*

Proof. (1) First, only assume T is superstable. Fix an a-model M on which p is based and a realization b of $p|M$. We first argue that if a regular type q is non-orthogonal to any a-prime model $M[b]$ over Mb, then q is non-orthogonal to M. To see this, choose a maximal M-independent set $\{c_1, \ldots, c_n\} \subseteq M[b]$ with $\operatorname{tp}(c_i/M)$ regular. Then $M[b] = M[c_1, \ldots, c_n]$ and as each $\operatorname{tp}(c_i/M)$ is of depth zero, we finish by Lemma 5.11(3). Next, we argue that the same holds for any model $N_0 \supseteq Mb$ dominated by b over M. Choose any regular q non-orthogonal to N_0. By superstability, choose a finite $d \in N_0$ such that $q \not\perp Mbd$. As bd is dominated by b over M, choose an a-prime $M[b]$ over Mb that contains d. Then q is non-orthogonal to M by the argument above.

Now, choose any model N that is dominated by b over M. To see that N is minimal over Mb, choose any $N' \preceq N$ containing Mb and assume by way of contradiction that $N' \neq N$. By superstability, choose $c \in N \setminus N'$ such that $q = \operatorname{tp}(c/N')$ is regular. As N' is dominated by b over M, it follows from the previous paragraph that $q \not\perp M$. By the 3-model Lemma (Fact 7.4) , there is $c^* \in N - M$ that does not fork with N' over M. As $b \in N'$ we conclude that $c^* \underset{M}{\perp} b$, which contradicts N being dominated by b over M.

(2) Now assume that T is classifiable. Fix any model M on which p is based and fix a realization b of $p|M$. Suppose N is dominated by b over M. To show that N is minimal over Mb, choose any $N_0 \preceq N$ containing Mb. To see that $N_0 = N$, choose any $c \in N$ and we will conclude that $c \in N_0$. To start, choose any a-model $M^* \succeq M$ with $b \underset{M}{\perp} M^*$. Note that N continues to be dominated by b over M^*. By PMOP, let N_2 be constructible over $N \cup M^*$. Thus, N_2 is also dominated by b over M^*. It follows from (1) that N_2 is minimal over M^*b. Also, $c \in N_2$.

As $N_0 \preceq N$, by PMOP again we can find $N_1 \preceq N_2$ that is constructible over $N_0 \cup M^*$. The minimality of N_2 over M^*b implies that $N_2 = N_1$, hence $c \in N_1$. As N_1 is atomic over N_0M^*, we have that $\operatorname{tp}(c/N_0M^*)$ is isolated. However, as cN_0 is dominated by b over M, the fact that $b \underset{M}{\perp} M^*$ implies that $cN_0 \underset{M}{\perp} M^*$. As $M \preceq N_0$, the Open Mapping Theorem implies that $\operatorname{tp}(c/N_0)$ is isolated, hence $c \in N_0$. $\square$

5.4 The main theorem

Proposition 5.13 suggests the following notation. If $M \subseteq_{\text{na}} \mathfrak{C}$ and $\{c_i : i \leq n\}$ are M-independent and realize regular, depth zero types over M, then the notation $M(c_0, \ldots, c_n)$ refers to any model dominated by $c_0 \ldots, c_n$ over M. Thus, even when $\operatorname{tp}(b/M) = \operatorname{tp}(c/M)$ are the same regular type, the notations $M(b)$ and $M(c)$ represent possibly different isomorphism types of models dominated over M by b or c, respectively. If the theory is classifiable, then given any $M(c)$, we let $M(c)^{(4)}$ denote the prime model over four M-independent copies of $M(c)$. (It exists by PMOP and, via the Proposition above, it is both prime and minimal over $\bigcup_{i<4} M(c_i)$. Thus, the isomorphism type of $M(c)^{(4)}$ over M is uniquely determined by $M(c)$.

Proposition 5.14 is the most technical result of this paper.

Proposition 5.14 *Suppose T is classifiable, $M \subseteq_{\mathrm{na}} \mathfrak{C}$, $p \in S(M)$ is regular and there is a witness to the non-modularity of p over M. Then for any realizations b, c of p and any choice of models $M(b)$ and $M(c)$ as above, $M(b)$ embeds into $M(c)^{(4)}$ over M.*

Proof. Without loss of generality, we may assume b and c are independent. Fix choices for $M(b)$ and $M(c)$. Choose d realizing $p|Mbc$ and choose $M(d)$ to be isomorphic to $M(c)$ over M. Let $M(bcd)$ denote the prime model over $M(b)M(c)M(d)$. As there is a witness to the non-modularity of p over M and as (b, c, d) realizes $p^{(3)}$, by Proposition 5.9 there is a "definable" one, and hence there is $a \in M(bcd)$ such that (a, b, c, d) is a witness to non-modularity over M. Since $M \subseteq_{\mathrm{na}} M(bcd)$, we can use Fact 7.3 to find $M(a) \preceq M(bcd)$. Note that $M(a) \subseteq_{\mathrm{na}} M(bcd)$ by Lemma 5.11. As $M(a), M(b), M(c) \preceq M(bcd)$, by PMOP there is a prime model $M^* \preceq M(bcd)$ over $M(a)M(b)M(c)$. Clearly, M^* is dominated by abc over M and (a, b, c) realizes $p^{(3)}$, so we can write it as $M(abc)$.

Claim 1. $M^* = M(bcd)$.

Proof. If not, then choose $g \in M(bcd) \setminus M^*$ to realize a regular type q. By Lemma 5.11 we have $q \not\perp M$, so by the 3-model Lemma (Fact 7.4) there would be $h \in M(bcd)$ such that $\mathrm{tp}(h/M)$ is regular and non-orthogonal to q, with $h \underset{M}{\perp} M^*$. This h is dominated by bcd over M, so $\mathrm{tp}(h/M)$ must be non-orthogonal to p. But then h, a, b, c are independent realizations of regular types non-orthogonal to p, which is impossible since $w_p(M(bcd)/M) = 3$. $\square$

So M^* is equal to both $M(abc)$ and $M(bcd)$. Next, let $M(ab) \preceq M(abc)$ be the unique model that is prime over $M(a) \cup M(b)$. For the moment we work over $M(ab)$. Choose e to realize $p|Mabcd$ fix an isomorphism

$$\Phi : M(abc) \to M(abe)$$

fixing $M(ab)$ pointwise. As both d and $M(d)$ are contained in $M(abc)$, we let $f := \Phi(d)$ and $M(f) := \Phi(M(d))$. Let N^* be prime over $M(abc) \cup M(abe)$ over $M(ab)$. Note that N^* is dominated by ce over $M(ab)$.

We now use the fact that (a, b, c, d) is a witness to non-modularity. By construction, ef and cd have the same type over $M(ab)$, so it follows from Lemma 5.6 that $\{c, d, e, f\}$ are independent over M. It follows that the four models $M(c), M(d), M(e), M(f)$ are M-independent, and the construction shows that they are pairwise isomorphic over M.

As each of $M(c), M(d), M(e), M(f)$ are contained in N^*, let $N \preceq N^*$ be prime over $M(c) \cup M(d) \cup M(e) \cup M(f)$. Note that N is isomorphic to $M(c)^{(4)}$ over M and $M(b) \preceq M(ab) \preceq N^*$. Thus, the Proposition is proved once we establish the following claim.

Claim 2. $N = N^*$.

Proof. If not, then choose $g \in N^* \setminus N$ with $q := \mathrm{tp}(g/N)$ regular. By Lemma 5.11, q is non-orthogonal to M. Thus, by the 3-model Lemma (Fact 7.4), there is $h \in N^* \setminus M$ such that $\mathrm{tp}(h/M)$ is regular and non-orthogonal to q, with $h \underset{M}{\perp} N$. We split into cases depending on the non-orthogonality class of q.

First, assume that q is non-orthogonal to p. On one hand, $\{h, c, d, e, f\} \subseteq N^*$ consist of 5 independent realizations of regular types non-orthogonal to p. On the other hand, $w_p(M(ab)/M) = 2$ and $w_p(N^*/M(ab)) = 2$, so $w_p(N^*/M) = 4$, which is a contradiction.

Finally, assume that q is orthogonal to p. Then clearly $\mathrm{tp}(h/M(ab))$ does not fork over M. But N^* is dominated by ce over $M(ab)$, and by the orthogonality, $h \underset{M(ab)}{\perp} ce$. Thus, by transitivity, $h \underset{M}{\perp} N^*$, which is absurd since $h \in N^* \setminus M$. $\square$

Corollary 5.15 *T classifiable. Suppose $M \subseteq_{\mathrm{na}} \mathfrak{C}$ is countable,* $\mathrm{tp}(c/M) =$ *p, and p has a witness to non-local modularity over M. Then for each n, $Q_n := \{q \in S_n(Mc) : \bar{d}c$ is dominated by c over M and $\bar{d} \models q\}$ is countable.*

Proof. Fix n. For any $q \in Q_n$, choose a realization $\bar{d}_q$ of q and let N_q be any countable, ℓ-constructible model over $Mc\bar{d}_q$. Note that N_q is dominated by $\bar{d}_q c$ (and hence by c) over M. By Proposition 5.14, choose an embedding $f_q : N_q \to M(c)^{(4)}$ fixing M pointwise. If Q_n were uncountable, there would be distinct $q \neq q'$ with $f_q(\bar{d}_q c) = f_{q'}(\bar{d}_{q'} c)$. As both f_q and $f_{q'}$ fix M pointwise, $f_q(c) = f_{q'}(c)$ realizes p, which would imply $q = q'$, a contradiction. $\square$

Theorem 5.16 *T classifiable. Suppose $M \subseteq_{\mathrm{na}} \mathfrak{C}$ is countable,* $\mathrm{tp}(c/M) =$ *p, and p has a witness to non-local modularity over M. Then there is a constructible, minimal model over Mc.*

Proof. Assume by way of contradiction that there is no constructible model over Mc. As Mc is countable, there is a finite $\bar{b}$ such that $\mathrm{tp}(\bar{b}/Mc)$ is isolated, and a consistent formula $\theta(x)$ over $Mc\bar{b}$ that has no complete extension $\varphi(x) \vdash \theta(x)$ over $Mc\bar{b}$. Fix such a $\bar{b}$ and θ. Choose a consistent formula $\psi(x) \vdash \theta(x)$, also over $Mc\bar{b}$ of smallest R^∞-rank. As there is no complete $\varphi \vdash \psi$, there are $2^{\aleph_0}$ types in $S_1(Mc\bar{b})$ containing ψ. Because of the minimality of $R^\infty(\psi)$, if d realizes any one of these types, then, as in the proof of Lemma 5.3 in [9], $d\bar{b}c$ is dominated by $\bar{b}c$ over M. However, $\mathrm{tp}(\bar{b}/Mc)$ isolated implies $\bar{b}c$ is dominated by c over M. Thus, $\mathrm{tp}(d\bar{b}/Mc) \in Q_{n+1}$ (where $n = \mathrm{lg}(\bar{b})$) for every such d, contradicting Corollary 5.15. Thus, there is a constructible model N over Mc. As N is dominated by c over Mc, its minimality over M follows from Lemma 5.11(1). $\square$

The following is the main result of this section.

Theorem 5.17 *Suppose that T is classifiable and M is any model. If $p \in S(M)$ is regular but not locally modular and b is any realization of p then every model N containing M that is dominated by b over M is both constructible and minimal over Mb.*

Proof. As N is dominated by b over M, it is minimal over Mb by Proposition 5.13(2).

Now first assume that N and hence M are countable. Choose, by Lemma 5.3, a countable $M^* \succeq M$ with $N \underset{M}{\perp} M^*$ such that $M^* \subseteq_{\mathrm{na}} \mathfrak{C}$ and for which there is a witness to non-local modularity over M^*. By PMOP, there is a countable N^* that is prime over $N \cup M^*$. By Theorem 5.16, there is S, constructible minimal over M^*b, $S \preceq N^*$. We claim that $N^* = S$. As N and M^* are independent over M, and N is dominated by b over M, N is also dominated by b over M^*. As N^* is prime over $M^* \cup N$ (and M^* is a model) it is dominated by N over M^*, hence it is dominated by b over M^*, and by Lemma 5.11, N^* is minimal over M^*b, hence $N^* = S$. As $N \subset N^*$, N (as a set) must be atomic over M^*b. By the Open mapping theorem, it follows that N is already atomic over Mb. As N is a countable model, it is constructible over Mb.

Now suppose $M \preceq N$ are arbitrary such that $b \in N$, $\mathrm{tp}(b/M)$ is regular and not locally modular, and N is dominated by b over M. Consider the pair (N, M) in a language with a predicate U for M. Take a countable elementary substructure in the pair language, $(N', M') \preceq (N, M)$ with $b \in N'$. It follows that, in the original language, N' and M are independent over M'.

As $N' \subset N$, b dominates N' over M. By the independence of N' and M over M', b also dominates N' over M'. From the paragraph above, N' is constructible over $M'b$ by a construction sequence which, by PMOP, remains a construction sequence over Mb. Now takes N^*, a constructible model over N' and M, inside N. As N is minimal over Mb, $N^* = N$ so N is constructible over Mb. $\square$

Corollary 5.18 *T classifiable. Suppose that $M \preceq N$ and N/M has weight one, non-orthogonal to a non-locally modular type p. Then N is both constructible and minimal over Mb for any element $b \in N \setminus M$.*

Proof. Recall that N/M having weight one means that $wt(b/M) = 1$ and N is dominated by b over M for any $b \in N \setminus M$. Choose any such b. As b/M is not orthogonal to p, there is $a \in \mathrm{dcl}(Mb) \setminus M$ that is p-simple of positive p-weight (Fact 7.8). As $a \in N$ and as N/M has weight one, it follows that $\mathrm{tp}(a/M)$ is regular, non-orthogonal to p. Thus, by Theorem 5.17, N is constructible and minimal over Ma. As $a \in \mathrm{dcl}(Mb)$, it follows that N is constructible and minimal over Mb as well. $\square$

6 When domination implies isolation

We begin this section with a recasting of Theorem 5.17.

Corollary 6.1 *Suppose that A is any set, $p \in S(A)$ is a regular, stationary, non-locally modular type, and b is any realization of p. Then for any e, if be is dominated by b over A, $\mathrm{tp}(e/Ab)$ is isolated.*

Proof. Let $M \subseteq_{\mathrm{na}} \mathfrak{C}$ be free from b over A and choose an ℓ-constructible model $N \supseteq Mbe$ over Mbe. As be is dominated by b over M, hence N is dominated by b over M. By Theorem 5.17, N is constructible, hence atomic over Mb, so $\mathrm{tp}(e/Mb)$ is isolated. Also, $e \underset{Ab}{\perp} Mb$, hence $\mathrm{tp}(e/Ab)$ is isolated by the Open Mapping Theorem. $\square$

This result suggests the following definition.

Definition 6.2 A strong type p satisfies *DI* (read 'domination implies isolation') if, for every set A on which p is based and stationary and for every realization b of $p|A$, for every $c \in \mathfrak{C}$, if bc is dominated by b over A, then $\mathrm{tp}(c/Ab)$ is isolated.

In the remainder of this section, we explore this notion in classifiable theories. We begin with two results that only require stability.

Lemma 6.3 *Suppose T is stable, M is a model, $A \supseteq M$ is any set, and $\varphi(x, a)$ isolates a type $p \in S(A)$ Then:*

1. *For every $B \underset{M}{\perp} A$, p has a unique extension $q \in S(AB)$ that is also isolated by $\varphi(x, a)$ and*
2. *For any c realizing $\varphi(x, a)$, cA is dominated by A over M.*

Proof.
For (1), the unique type $q := \{\psi(x, b, a') : \varphi(x, a) \vdash d_r y \psi(x, y, a')$, where $a' \in A$, $b \in B$, $\psi(x, y, z)$ over M, and $r = \mathrm{tp}(b/M)\}$.

For (2), choose any B satisfying $B \underset{M}{\perp} A$. As p has a non-forking extension to $S(B)$, it must be the q from (1). $\square$

Among depth zero-like types, the notion of DI has many equivalents.

Proposition 6.4 *Suppose T is classifiable. The following are equivalent for a depth zero-like strong type p:*

1. *p is DI;*
2. *For every countable M on which p is based and for every b realizing $p|M$, and for every n, the isolated types in $S^n(Mb)$ are dense;*
3. *For every countable M on which p is based, for every b realizing $p|M$, there is a constructible model N over Mb. Moreover, every model N that is dominated by b over M is constructible over Mb;*
4. *Same as (3), but for every model M on which p is based;*
5. *There is some a-model M on which p is based and some b realizing $p|M$ for which there is a constructible model N over Mb.*

Proof. We prove (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1). Then (4) $\Rightarrow$ (5) is trivial and we will show (5) $\Rightarrow$ (2).

(1) $\Rightarrow$ (2). Assume (1) and choose a countable M, b, and n as in (2). Let $\varphi(x, b, m)$ be any consistent formula with $\lg(x) = n$. Choose any model N that is dominated by b over M (e.g., an ℓ-constructible one). Choose any $c \in N$ realizing $\varphi(x, b, m)$. As cb is dominated by b over M, it follows from (1) that $\mathrm{tp}(c/Mb)$ is isolated.

(2) $\Rightarrow$ (3). Fix M and b as in (3). As M is countable and the isolated types are dense, it follows from Vaught that a constructible model N over Mb exists. For the final sentence, let N^* be any model dominated by b over M. As N is prime over Mb, we may assume $N \preceq N^*$. But, as N^* is minimal over Mb by Proposition 5.13(2), $N = N^*$, so N^* is constructible over Mb.

(3) $\Rightarrow$ (4). Fix M and b as in (4). Choose a countable model $M_0 \preceq M$ such that $b \underset{M_0}{\perp} M$. By (3), let N_0 be constructible over $M_0 b$.

Fix a construction sequence $\langle c_i : i < \omega\rangle$ for N_0 over Mb. By iterating Lemma 6.3, $\langle c_i : i < \omega\rangle$ is a construction sequence over Mb. Now, by PMOP, let N be constructible over N_0M. It follows that N is constructible over Mb. For the final sentence, let N^* be any model dominated by b over M. As N is prime over Mb, we may assume $N \preceq N^*$. But then $N = N^*$ by minimality.

(4) $\Rightarrow$ (1). Choose any set A on which p is based and stationary and let b be any realization of $p|A$. Choose any element $c \in \mathfrak{C}$ such that bc is dominated by b over M. Choose any model $M \supseteq A$ satisfying $M \underset{A}{\perp} bc$. It follows that bc is dominated by b over M. Choose any model N dominated by bc over M. By transitivity we have that N is dominated by b over M. Thus, by (4), N is constructible, and hence atomic over Mb. In particular, $\mathrm{tp}(c/Mb)$ is isolated.

(4) $\Rightarrow$ (5) is immediate.

(5) $\Rightarrow$ (2). Let M^* and b^* be any a-model and witness exemplifying (5). Given any countable M on which p is based and b realizing $p|M$, the saturation of M^* implies there is an $M' \preceq M^*$ such that $\mathrm{tp}(Mb) = \mathrm{tp}(M'b^*)$. Thus, without loss, assume $M \preceq M^*$ and $b = b^*$. As b realizes $p|M^*$, we have that $b \underset{M}{\perp} M^*$. Let $\varphi(x, b, m)$ be any consistent formula with $m \in M$. By (5), choose a constructible (and hence atomic) N over M^*b. Choose any $c \in N$ realizing $\varphi(x, b, m)$. Then $\mathrm{tp}(c/M^*b)$ is isolated, hence $\mathrm{tp}(c/Mb)$ is as well by the Open Mapping Theorem. $\square$

Proposition 6.5 *Let T be classifiable.*

1. *Suppose* $\mathrm{stp}(b/A)$ *is depth zero-like and DI and bc is dominated by b over A. Then* $\mathrm{stp}(bc/A)$ *is both depth zero-like and DI.*
2. *Suppose* $\mathrm{tp}(bc/M)$ *is depth zero-like and DI, and* $\mathrm{tp}(c/Mb)$ *is isolated. Then* $\mathrm{tp}(b/M)$ *is depth zero-like and DI as well.*
3. *If p and q are both depth zero-like and DI, then so is $p \otimes q$.*

Proof. (1) That $\mathrm{stp}(bc/A)$ is depth zero-like is clear. As for DI, choose an a-model M and b as in Proposition 6.4(5). Without loss, we may assume $A \subseteq M$, so $b \underset{A}{\perp} M$. It follows that bc is dominated by b over M, hence $\mathrm{tp}(c/Mb)$ is a-isolated. Thus, there is an a-prime model $M[b]$ over Mb with $c \in M[b]$. [Think of c as being the first step of an a-construction sequence over Mb.] Then $M[b]$ is also a-prime over Mbc. However, $M[b]$ is constructible over Mb, so $\mathrm{tp}(c/Mb)$ is actually isolated. It follows that $M[b]$ is constructible over Mbc, so $\mathrm{stp}(bc/A)$ is DI by Proposition 6.4(5).

(2) That $\mathrm{tp}(b/M)$ is depth zero-like is immediate. For DI, choose an a-model M^* witnessing that $\mathrm{tp}(bc/M)$ is DI. Without loss, we

may assume $M \preceq M^*$, so $bc \underset{M}{\downarrow} M^*$. It follows from Lemma 6.3 that $\mathrm{tp}(c/M^*b)$ is isolated as well. As $\mathrm{tp}(bc/M^*)$ is DI, let N^* be constructible over Mbc. As $\mathrm{tp}(c/M^*b)$ is isolated, N^* is also constructible over M^*b. Thus, $\mathrm{tp}(b/M^*)$ is DI by Proposition 6.4(5).

(3) As both p and q are depth zero-like, it is immediate that $p \otimes q$ is as well. As for DI, let M be any model on which $p \otimes q$ is based and let (c_1, c_2) realize $p \otimes q$. As both p and q are DI, there is a constructible model N_1 over Mc_1 and a constructible model N_2 over Mc_2. As $c_1 \underset{M}{\downarrow} c_2$, by iterating Lemma 6.3 it follows that N_1 is constructible over Mc_1c_2 and N_2 is constructible model over N_1c_2. However, by PMOP there is a model N^* that is constructible over N_1N_2. It follows that N^* is constructible over Mc_1c_2, so $p \otimes q$ is DI by Proposition 6.4(4). $\square$

We can combine several of our results in the following Corollary which generalizes Corollary 5.18.

Corollary 6.6 *T classifiable. Suppose p is a regular depth zero DI type (for example a non-locally modular regular type) and let* $\mathrm{tp}(a/M)$ *be p-semiregular of weight k. Then there is a constructible, minimal model N over Ma.*

Proof. Choose an a-model M^* independent from a over M and choose an M^*-independent tuple $\bar{c} = \langle c_i : i < k \rangle$ of realizations of $p|M^*$ such that a and $\bar{c}$ are domination equivalent over M^*.

As $\mathrm{tp}(a/M^*)$ is p-semiregular, it is depth-zero like. By Proposition 6.5(3) and (1), $\mathrm{tp}(\bar{c}/M^*)$ is depth zero-like and DI. To show that $\mathrm{tp}(a/M^*)$ is DI, by Proposition 6.5(2) it suffices to show $\mathrm{tp}(\bar{c}/M^*a)$ is isolated. First choose $\varphi(y) \in p|M$ witnessing that $p|M$ is strongly regular. Next, for each $i < k$, choose a formula $\theta_i(y, x) \in \mathrm{tp}(c_i/M^*a)$ such that $\theta_i(y, a)$ forks over M^*. Finally, as $w_p(a/M^*\bar{c}) = 0$, it does so provably: tp(a/M*) is p-semiregular so, by Fact 7.16, we can find a formula $\psi(x, \bar{y}) \in \mathrm{tp}(a\bar{c}/M^*)$ such that $w_p(a', M^*\bar{c}') = 0$ whenever $a'\bar{c}'$ realizes $\psi(x, \bar{y})$.

Then $\mathrm{tp}(\bar{c}/M^*a)$ is isolated by

$$\bigwedge_{i<k} \varphi(y_i) \wedge \bigwedge_{i<k} \theta_i(y_i, a) \wedge \psi(\bar{y}, a)$$

As $\mathrm{tp}(a/M^*)$ is depth zero-like and DI, an application of Proposition 6.4(3) completes the proof. $\square$

7 Appendix

In this appendix, we bring together for the reader's convenience, many of the basic definitions and facts from classification theory and geometric stability theory which are used throughout the paper and can be found for example in [7]. We assume that the reader is familiar with stability theory, independence and the basics of superstability. Throughout this appendix, T will be a stable theory.

Definition 7.1
- We say that M is an *a-model* if every strong type over every finite $B \subseteq M$ is realized in M. In the original notation of Shelah ([10]) this corresponds to $\mathbf{F}^{\mathbf{a}}_{\aleph_0}$-saturation.
- Let $M \preceq N$ be models of T, N is an *na-extension* of M, denoted $M \subseteq_{\mathrm{na}} N$, if for every formula $\varphi(x, y)$, for every tuple a from M and every finite subset F of M, if N contains a solution to $\varphi(x, a)$ not in M, then M contains a solution to $\varphi(x, a)$ that is not algebraic over F.

Definition 7.2
- If $B \subseteq A$, types $p \in S(A)$ and $q \in S(B)$ are *almost orthogonal*, denoted $p \perp^a q$, if for all a realizing p and b realizing q, if $b \mathop{\smile\!\!\!\!\!\!|}_B A$, then $a \mathop{\smile\!\!\!\!\!\!|}_B b$. p is almost orthogonal to the set B, $p \perp^a B$ if $p \perp^a q$ for every $q \in S(B)$.
- Two types $p \in S(A)$ and $q \in S(B)$ are *orthogonal*, denoted $p \perp q$, if $p|C \perp^a q$ for all $C \supseteq AB$. $p \in S(A)$ is *orthogonal to a set* B, $p \perp B$, if $p \perp q$ for every $q \in S(B)$.
- A set C is *dominated* by E over D if for all b such that b is independent from E over D, b is independent from C over D.
- A stationary type p is regular if it is orthogonal to all its forking extensions.

Note that if $B \subseteq A$ and $p \in S(A)$, then $p \perp^a B$ if and only if Aa is dominated by A over B for some/every a realizing p.

The following fact is a consequence of Lemma 5.3 in [9]:

Fact 7.3 *Suppose T is superstable, $M \subseteq_{\mathrm{na}} N$ are models of T and $M \subseteq A \subseteq N$. Then theres exists a model N', $M \subseteq A \subseteq N' \subseteq_{\mathrm{na}} N$ such that N' is dominated by A over M.*

Fact 7.4 *[The 3-model lemma]* (Proposition 3.6, Chapter 8 in [7]) *Suppose T superstable. Let $M_0 \preceq M_1 \preceq M_2$ be models of T such that $M_0 \subseteq_{\mathrm{na}} M_2$. Suppose $a \in M_2$ and $tp(a/M_1)$ is regular non-orthogonal to M_0. Then there is $b \in M_2$ such that $tp(b/M_1)$ is regular and does not fork over M_0 and b is not independent from a over M_1.*

7.1 p-simplicity and locally modular regular types

Let p be any stationary regular type, which for convenience we take to be over $\emptyset$. Consider $p(\mathfrak{C})$ the set of realizations of the type p, then $(p(\mathfrak{C}), \mathrm{cl}_{fork})$ forms a homogeneous pre-geometry (see e.g., Chapter 7 of [7]), where $a \in \mathrm{cl}_{fork}(B)$ means that a forks with B over $\emptyset$.

Now suppose that M is an a-model, let $p|M$ denote the non-forking extension of p to $S(M)$. Call a subset $X \subseteq p|M(\mathfrak{C})$ *closed* if whenever $a \in p(\mathfrak{C})$ does not fork over M, but $a \not\perp_M X$, then $a \in X$. Because of the van der Waerden axioms, closed sets X come equipped with a dimension $\dim(X)$, which here is equal to the cardinality of a maximal M-independent subset $I \subseteq M$.

Definition 7.5 The regular type p is *locally modular* if, for any a-saturated M and for any closed subsets $X, Y \subseteq (p|M)(\mathfrak{C})$,

$$\dim(\mathrm{cl}(X \cup Y)) + \dim(\mathrm{cl}(X \cap Y)) = \dim(X) + \dim(Y)$$

Proposition 7.6 *[7] Suppose that M is an a-model. The following are equivalent for a regular type $p \in S(M)$:*

1. *The type p is not locally modular;*
2. *There is a set of four realizations $\{a_1, a_2, b_1, b_2\}$ of p, that are dependent over M, yet the closures of $\{a_1, a_2\}$ and $\{b_1, b_2\}$ in $p|M(\mathfrak{C})$ are disjoint.*

In fact it will be useful to work in a wider space than $(p(\mathfrak{C}), \mathrm{cl}_{\mathrm{fork}})$. We recall the definition of p-simplicity.

Definition 7.7
- A strong type $\mathrm{stp}(a/A)$ is *hereditarily orthogonal to p* if $\mathrm{stp}(a/B)$ is orthogonal to p for every $B \supseteq A$.
- A strong type $\mathrm{stp}(a/A)$ is *p-simple* if for some a-model M independent from a over A, there is an M-independent set $\{b_1, \ldots, b_k\}$ of realizations of $p|M$ such that $\mathrm{stp}(a/Mb_1, \ldots, b_k)$ is hereditarily orthogonal to p. We say that $\mathrm{stp}(a/A)$ is *p-simple of weight k* if k is least such.
- If $\mathrm{stp}(a/A)$ is p-simple of p-weight k we write $w_p(a/A) = k$.
- A formula $\theta(x)$ over A is p-simple of p-weight k if every type extending θ is p-simple and k is the maximum of $\{w_p(a/A) : \theta(x) \in \mathrm{stp}(a/A)\}$.
- For a p-simple $\theta(x)$ over A, we say *p-weight is definable and continuous inside* $\theta(x)$ if, for all $C \supseteq A$ and for all a realizing θ, if $w_p(a/C) = m$, then there is a formula $\varphi(x, c) \in \mathrm{tp}(a/C)$ of p-weight m and a formula $\psi(y) \in \mathrm{tp}(c/A)$ such that $w_p(\varphi(x, c')) \leq m$ for all c' realizing ψ.

Non-orthogonality to p gives the existence of p-simple types:

Fact 7.8 *(Lemma 1.17, Chapter 7 in [7]). Let X be algebraically closed and $tp(a/X)$ be non-orthogonal to p. Then there is $e \in dcl(aX)$ such that $tp(e/X)$ is p-simple of positive p-weight.*

Following [3] and [7], for a set C, let

$$D(p, C) := \{a \in \mathfrak{C} : \operatorname{stp}(a/C) \text{ is } p\text{-simple of finite } p\text{-weight}\}$$

We equip $D(p, C)$ with a closure operator cl_p, namely for a, B from $D(p, C)$, $a \in \mathrm{cl}_p(B)$ if and only if $w_p(a/BC) = 0$. Technically, the closure relation cl_p depends on C, but much of the time we will take $C = \emptyset$, so we do not muddy our notation by referring to it explicitly. Because of the regularity of p, the closure space $(D(p, C), \mathrm{cl}_p)$ is well-behaved, but formally is not a pre-geometry as the Exchange Axiom fails.

Definition 7.9 Fix any set C. $D(p, C)$ is *modular* if $w_p(a/C) + w_p(b/C) = w_p(ab/C) + w_p((\mathrm{cl}_p(a) \cap \mathrm{cl}_p(b))/C)$ for all $a, b \in D(p, C)$.

As shown for example in 7.2.4 of [7]:

Fact 7.10 *The regular type p is locally modular (as defined above) if and only if $D(p, C)$ is modular for all sets C*

7.2 p-semiregular types

Within this space, it will be useful to identify the p-semiregular types.

Definition 7.11 $\operatorname{stp}(a/A)$ is p-semi-regular of weight k if it is p-simple and is (eventually) domination equivalent to $p^{(k)}$ for some finite $k \geq 1$, i.e., for some (equivalently, for all) a-models M independent from a over A, there is an M-independent sequence $\bar{b} = \langle b_1, \ldots, b_k \rangle$ of realizations of the non-forking extension $p|M$ witnessing the p-simplicity of $\operatorname{stp}(a/A)$, with a and $\bar{b}$ domination equivalent over M (for any set X, $X \underset{M}{\perp} a$ if and only if $X \underset{M}{\perp} \bar{b}$.)

There is a natural Criterion for determining whether a p-simple type is p-semiregular. This Criterion appears as either Fact 1.4 of [4] or 7.1.18 of [7]):

Criterion 7.12 *Suppose* $\operatorname{stp}(a/X)$ *is p-simple of positive p-weight, and choose* $Y \subseteq \operatorname{dcl}(aX)$*. Then* $\operatorname{stp}(a/Y)$ *is p-semi-regular of positive p-weight if and only if $a \notin \operatorname{acl}(Y)$, but $w_p(e/Y) > 0$ for every $e \in$* $\operatorname{dcl}(aY) \setminus \operatorname{acl}(Y)$.

To get the existence of a p-semiregular type nearby a given p-simple type, we couple this with the following easy Lemma, whose proof only requires superstability.

Lemma 7.13 *Suppose a and X are given with a finite, and Y is chosen arbitrarily such that $X \subseteq Y \subseteq \mathrm{acl}(Xa)$. Then there is a finite sequence b from Y such that $Y \subseteq \mathrm{acl}(Xb)$.*

Proof. Recursively construct a sequence $\langle b_i : i\rangle$ from Y of maximal length such that $a \not\downarrow_{B_i} b_i$, where $B_i := X \cup \{b_j : j < i\}$. Clearly, $R^\infty(a/B_i)$ is strictly decreasing with i, so any such sequence has finite length. But, for the sequence to terminate, it must be that $Y \subseteq \mathrm{acl}(B_{i^*})$ for the terminal i^*. $\square$

Finally, we get our existence lemma.

Lemma 7.14 *If $\mathrm{stp}(a/X)$ is p-simple of positive p-weight, then there is a finite b from $\mathrm{dcl}(aX) \cap \mathrm{cl}_p(X)$ such that $\mathrm{stp}(a/Xb)$ is p-semiregular and $w_p(a/Xb) = w_p(a/X)$.*

Proof. Let $Y = \mathrm{dcl}(aX) \cap \mathrm{cl}_p(X)$ and choose a finite b from Y such that $Y \subseteq \mathrm{acl}(Xb)$. Now $\mathrm{dcl}(Ya) = \mathrm{dcl}(Xa)$, so if $e \in \mathrm{dcl}(Ya) \setminus \mathrm{acl}(Y)$, we must have $w_p(e/Y) > 0$, lest we would have $e \in Y$. Thus, Criterion 7.12 for p-semi-regularity applies. $\square$

Next we record ways in which an existing p-semi-regular type is persistent.

Lemma 7.15 *Suppose $\mathrm{stp}(e/X)$ is p-semi-regular.*

1. *If $e' \in \mathrm{acl}(eX) \setminus \mathrm{acl}(X)$, then $\mathrm{stp}(e'/X)$ is p-semi-regular;*
2. *If $\mathrm{stp}(e/Y)$ is parallel to $\mathrm{stp}(e/X)$, then $\mathrm{stp}(e/Y)$ is p-semi-regular of the same p-weight;*
3. *If $X \subseteq Y \subseteq \mathrm{cl}_p(X)$, then $\mathrm{stp}(e/Y)$ is p-semi-regular of the same p-weight.*

Proof. (1) As $\mathrm{dcl}(e'X) \subseteq \mathrm{dcl}(eX)$, the result follows by Criterion 7.12.

(2) This is immediate, as 'domination equivalent to $p^{(k)}$' is preserved.

(3) Since $\mathrm{stp}(e/X)$ is p-semi-regular, we automatically have $e \downarrow_X Y$, so (3) follows from (2). $\square$

The following fact is Theorem 2(b) in [4].

Fact 7.16 *Let T be superstable, let p be a non trivial regular type of depth zero and let $stp(a/B)$ be p-semiregular. Then a lies in some $acl(B)$-definable set D such that p-weight is continuous and definable inside D.*

7.3 Classifiable theories and isolation

We now recall definitions and facts about isolation and constructibility.

A type $p \in S(A)$ is *isolated* if there is some formula $\varphi(x,a) \in p$ such that $\varphi(x,a) \vdash p$. A *construction sequence over* A is a sequence $\langle a_\alpha : \alpha < \beta \rangle$ such that $\mathrm{tp}(a_\alpha / A \cup \{a_\gamma : \gamma < \alpha\})$ is isolated for every $\alpha < \beta$. A model N is *constructible over* A if there is a construction sequence over A whose union is N. If N is constructible over A then it is both prime and atomic over A. Any two constructible models over A are isomorphic over A. As T is countable, it follows from results of Vaught that for A countable, there is a constructible model over A.

If T is $\aleph_0$-stable, then constructible models exist over every set A. In a superstable theory it is not always true that there are constructible models over all sets. Indeed, one of the main goals of this paper is to determine when constructible models over particular sets exist.

A weaker notion is ℓ*-isolation*. A type $p \in S(A)$ is ℓ*-isolated* if, for every formula $\varphi(x,y)$ there is a formula $\psi(x,a) \in p$ such that $\psi(x,a) \vdash p{\restriction}_\varphi$, the restriction of p to instances of $\pm\varphi(x,b)$ for $b \in A$. ℓ-construction sequences and N being ℓ-constructible over A are defined analogously. An advantage is that for a superstable theory T, ℓ-constructible models over A exist. However, there can be many non-isomorphic ℓ-constructible models over A.

Recall the definitions of NDOP and PMOP from the introduction:

Definition 7.17 *1. A superstable theory* does not have the dimension order property *(NDOP) if, for every independent triple* $\mathcal{M} = (M_0, M_1, M_2)$ *of a-saturated models, the a-prime model* M^* *over* M_1M_2 *(which exists in any superstable theory) is minimal among all a-saturated models containing* M_1M_2.
2. A superstable theory has prime models over pairs *(PMOP) if, for any independent triple* (M_0, M_1, M_2) *of models, there is a constructible model over* M_1M_2.
3. A complete theory T *in a countable language is* classifiable *if* T *is superstable, has prime models over pairs (PMOP) and does not have the dimension order property (NDOP).*

The following are essential facts:

Fact 7.18
- *T has NDOP if and only if, for $\mathcal{M} = (M_0, M_1, M_2)$ any independent triple of a-saturated models and M^* a-prime model over M_1M_2, any regular type q non-orthogonal to M^* is either non-orthogonal to M_1 or M_2.*

- *If T has NDOP, then any non trivial regular type must have depth zero.*

7.4 p-disjointness

Definition 7.19 Suppose $a, b \in D(p, C)$. We say that *a and b are p-disjoint over C* if $\mathrm{cl}_p(Ca) \cap \mathrm{cl}_p(Cb) = \mathrm{cl}_p(C)$.

The next two Lemmas discuss the relationship between p-disjointness and forking, at least when $\mathrm{stp}(ab/C)$ is p-semiregular.

Lemma 7.20 *Suppose that $\mathrm{stp}(ab/C)$ is p-semiregular, $C \subseteq D$ and $\mathrm{stp}(ab/D)$ does not fork over C. Then $\mathrm{cl}_p(Ca) \cap \mathrm{cl}_p(Cb) = \mathrm{cl}_p(C)$ if and only if $\mathrm{cl}_p(Da) \cap \mathrm{cl}_p(Db) = \mathrm{cl}_p(D)$.*

Proof. First, assume there is a 'bad element' e for the triple (a, b, C), that is $e \in \mathrm{cl}_p(Ca) \cap \mathrm{cl}_p(Cb) \setminus \mathrm{cl}_p(C)$. As the existence of such an e is clearly determined by $\mathrm{tp}(ab/C)$, by replacing D by some independent D^* realizing the same strong type as D over Cab, we may assume that $abe \underset{C}{\perp} D$. It follows immediately that $e \in [\mathrm{cl}_p(Da) \cap \mathrm{cl}_p(Db)] \setminus \mathrm{cl}_p(D)$ so e is bad for (a, b, D) as well.

Conversely, if e is a 'bad element' for (a, b, D), let $h := \mathrm{Cb}(De/Cab)$. We first claim that $h \notin \mathrm{cl}_p(C)$. If it were, then as $\mathrm{stp}(ab/C)$ is p-semiregular, we would have $ab \underset{C}{\perp} h$. But, as $ab \underset{Ch}{\perp} De$, this would imply $ab \underset{D}{\perp} e$, contradicting $e \notin \mathrm{cl}_p(D)$. Thus, $h \notin \mathrm{cl}_p(C)$.

So, arguing by symmetry between a and b, it suffices to prove that $h \in \mathrm{cl}_p(Ca)$. Choose a Morley sequence $\langle D_1 e_1, \ldots, D_n e_n \rangle$ in $\mathrm{stp}(De/Cab)$ with $D_1 e_1 = De$ such that $h \in dcl(e_1 \ldots e_n D_1 \ldots D_n)$. The standard argument yields

$$D_1, \ldots, D_n \underset{C}{\perp} ab$$

As well, $h \in \mathrm{acl}(Cab)$, hence $D_1, \ldots, D_n \underset{Ca}{\perp} h$. Because of this, it suffices to prove that $\mathrm{stp}(h/CaD_1 \ldots D_n)$ is hereditarily orthogonal to p, i.e., has p-weight zero. However, for each i, $w_p(e_i/aD_i) = 0$, so $w_p(e_i/CaD_1 \ldots D_n) = 0$ for each i. But $h \in \mathrm{dcl}(e_1 \ldots e_n D_1 \ldots D_n)$, so $w_p(h/CaD_1 \ldots D_n) = 0$. $\square$

Lemma 7.21 *Suppose that $\mathrm{stp}(ab/C)$ is p-semiregular and $\mathrm{cl}_p(Ca) \cap \mathrm{cl}_p(Cb) = \mathrm{cl}_p(C)$. Then $\mathrm{acl}(Ca) \cap \mathrm{acl}(Cb) = \mathrm{acl}(C)$.*

Proof. Choose any $e \in \mathrm{acl}(Ca) \cap \mathrm{acl}(Cb)$. As $\mathrm{acl}(Ca) \subseteq \mathrm{cl}_p(Ca)$ and $\mathrm{acl}(Cb) \subseteq \mathrm{cl}_p(Cb)$, our hypothesis implies that $e \in \mathrm{cl}_p(C)$. However, as $\mathrm{stp}(ab/C)$ is p-semi-regular, this implies $ab \underset{C}{\perp} e$. As $e \in \mathrm{acl}(abC)$, this implies $e \in \mathrm{acl}(C)$ as desired. $\square$

Thanks to Lemma 7.21 we will be able to apply the following general result below about forking to p-disjoint p-semiregular types.

Lemma 7.22 *For all a, b, C, $\mathrm{acl}(Ca) \cap \mathrm{acl}(Cb) = \mathrm{acl}(C)$ if and only if for every set X, if $X \mathop{\smile\!\!\!\!\!\mid}_{Ca} b$ and $X \mathop{\smile\!\!\!\!\!\mid}_{Cb} a$ both hold, then $X \mathop{\smile\!\!\!\!\!\mid}_{C} ab$ holds as well.*

Proof. To ease notation, assume $C = \emptyset$. It suffices to prove this for finite sets X. For left to right, fix an X, and let D denote the canonical base of $\mathrm{tp}(X/ab)$. On one hand, $D \subseteq \mathrm{acl}(a)$, and on the other hand, $D \subseteq \mathrm{acl}(b)$. Thus, by our assumption, $D \subseteq \mathrm{acl}(\emptyset)$, implying that $X \mathop{\smile\!\!\!\!\!\mid} ab$.

For the converse, choose any $h \in \mathrm{acl}(a) \cap \mathrm{acl}(b)$. Then, for trivial reasons we have $h \mathop{\smile\!\!\!\!\!\mid}_{a} b$ and $h \mathop{\smile\!\!\!\!\!\mid}_{b} a$, so by our hypothesis we have $h \mathop{\smile\!\!\!\!\!\mid} ab$. But, as $h \in \mathrm{acl}(ab)$, this implies $h \in \mathrm{acl}(\emptyset)$. $\square$

References

[1] B. Hart, An exposition of OTOP. Classification theory (Chicago, IL, 1985), 107–126, Lecture Notes in Math., 1292, Springer, Berlin-New York, 1987.

[2] B. Hart, E. Hrushovski and M.C. Laskowski, The uncountable spectra of countable theories, Annals of Mathematics, 152, 2000, 207 – 257.

[3] E. Hrushovski, Contributions to stable model theory, doctoral thesis, UC Berkeley, 1986.

[4] E. Hrushovski and S. Shelah, A dichotomy theorem for regular types, APAL, 45 (2), 1989, 157–169.

[5] E. Hrushovski and S. Shelah, Stability and omitting types, Israel J. Math, 74 (2-3), 1991, 289–321

[6] R. Moosa and O.L. Sanchez, Isolated types of finite rank; an abstract Dixmier-Moeglin equivalence, Selecta Mathematica, volume 25 (2019), number 1, Art.10.

[7] A. Pillay, *Geometric Stability Theory*, Oxford University Press, 1996.

[8] B. Poizat, *Stable groups*, Mathematical surveys and monographs, Vol. 87, AMS, 2001.

[9] S. Shelah and S. Buechler, On the existence of regular types, APAL, 45 (3), 1989, 277–308

[10] S. Shelah, *Classification Theory*, (revised edition) North Holland, Amsterdam, 1990.