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THE SCHRÖDER-BERNSTEIN PROPERTY FOR a-SATURATED MODELS

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ABSTRACT. A first-order theory T has the *Schröder-Bernstein (SB) property* if any pair of elementarily bi-embeddable models are isomorphic. We prove that T has an expansion by constants with the SB property if and only if T is superstable and non-multidimensional. We also prove that among superstable theories T, the class of a-saturated models of T has the SB property if and only if T has no nomadic types (see Definition 3.1 below).

1. INTRODUCTION

The classical Schröder-Bernstein theorem asserts that if A and B are bi-embeddable sets, i.e., there exist injections $f : A \to B$ and $g : B \to A$, then there is a bijection between them. It is natural to extend this concept to classes $(\mathbf{K}, \mathbf{Mor})$, where $\mathbf{K}$ is a class of algebraic structures and $\mathbf{Mor}$ is a distinguished class of injections between elements of $\mathbf{K}$. We say that $(\mathbf{K}, \mathbf{Mor})$ has the *Schröder-Bernstein (SB) property* if any pair of bi-embeddable structures in $\mathbf{K}$ (with respect to $\mathbf{Mor}$) are isomorphic. In this paper, we discuss the SB property for classes $\mathbf{K}$ that are subclasses of $\mathbf{Mod}(T)$, the class of models of a complete, first-order theory T. Throughout this paper, $\mathbf{Mor}$ will always be taken to be the class of elementary embeddings (which are necessarily injective). Thus, we say that a theory T has the SB property if any two elementarily bi-embeddable models of T are isomorphic. As examples, the theory of algebraically closed fields of any characteristic has the SB property, but the theory of dense linear order does not.

One motivation for considering the SB property is that it should be a nice litmus test for our understanding of the models of T: Once we have a sufficiently good understanding of these models (knowing that they are classified by some reasonable collection of invariants, or, conversely, knowing that they are "wild" in a suitably precise sense), then we ought to be able to say whether or not T has SB. For instance, by the results of Morley in [4], if T is countable and $\aleph_1$-categorical (e.g., algbraically closed fields of any characteristic), then the models of T are classified

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by a single cardinal number invariant (a dimension) which is preserved by elementary embeddings. This implies that such a T has the SB property. In general, it seems that SB is a fairly strong tameness property, but it is strictly weaker than uncountable categoricity.

Given that we are interested in relating SB to the classification of models of T, it is not surprising that we use tools from the so-called *classification theory* of Shelah (now more commonly called *stability theory*), developed in the 1970's and expounded in [9]. One of the main ideas there was the use of dividing lines amongst theories T (such as superstability, NDOP, NOTOP) to separate those T whose classes of models do admit some kind of classification from those for which this is hopeless. Another idea from [9] that is very useful for the present paper is the development of a local dimension theory for certain classes of elements within a model using the independence notion known as nonforking.

The Schröder-Bernstein property for first-order theories seems to have first been considered by Nurmagambetov in [5] and [6], where he showed that for totally transcendental theories it is equivalent to nonmultidimensionality. Various other results around SB were proved in the first author's thesis [1], such as:

Theorem 1.1. *If T is not superstable, then T does not have SB. Furthermore, if T is unstable, then for any cardinal κ, there is an infinite collection of κ-saturated models of T which are pairwise bi-embeddable but pairwise nonisomorphic.*

Our previous paper [2] gives a characterization of which countable weakly minimal theories have the SB property, and this characterization is precise enough to show that for any fixed T the SB property is absolute under forcing extensions of the set-theoretic universe (which does not seem obvious *a priori*). We still do not have a satisfactory characterization of which theories have SB in general.

In the current paper, we address two questions: When is it the case that T has SB after naming a set of constants (which we call "eventual SB"), and when do the sufficiently saturated models of T have the SB property?

We give a simple complete characterization of eventual SB in Theorem 4.2: It is equivalent to superstability plus nonmultidimensionality. As for SB for κ-saturated models, with Theorem 3.11 we succeed in characterizing SB for a-saturated models when T is superstable. This result, coupled with Theorem 1.1, imply that the only remaining case to consider is when T is strictly stable. To extend our methods to such theories would require additional knowledge about the prevalence of regular types.

A motivating example to keep in mind is the complete theory T of the additive group $(\mathbb{Z};+)$. It turns out that T is the theory of all torsion-free abelian groups G such that $[G : pG] = p$ for every prime p (this follows from a more general result by Szmielew in [10]). Hence any model G of T can be decomposed as $G = H \oplus \mathbb{Q}^\kappa$ where $H \leq \widehat{\mathbb{Z}}$. The theory T is "classifiable" according to Shelah's dichotomies. In fact, T is superstable, non-multidimensional, and weakly minimal; see [8]). The a-saturated models of T are simply the models of the form $G = \widehat{\mathbb{Z}} \oplus \mathbb{Q}^\kappa$ where the cardinal κ is infinite, and it is not hard to see that the class of these models has the SB property. Furthermore, T has the eventual SB property since we can add constants for every element in a copy of $\widehat{\mathbb{Z}}$ in some model. However, the class of *all* models of T is more complex, and this does not have the SB property (this follows from the main theorem of [2]).

Section 2 introduces some concepts and tools necessary for the main results (countable local pre-weight and low substructures). Along the way, we give a new characterization of the a-prime models in any superstable theory (Theorem 2.6). Section 3 gives the characterization of SB for a-saturated models in a superstable theory, and Section 4 gives the characterization of eventual SB (whose proof depends heavily on our analysis of a-saturated models in previous sections).

Throughout the paper, we will assume that T is a complete superstable theory unless otherwise noted, though in a few results we note that T is superstable for emphasis. Given a complete theory T, we always work within a sufficiently saturated model $\mathfrak{C}$ of T (for our purposes, $(2^{|T|})^+$-saturated is enough). Our notation is mostly standard and follows [7] and [9], where the interested reader can find definitions for the terms we use from stability theory ("superstable," "regular type," etc.). As in [7], we call a structure *a-saturated* in place of Shelah's '$\mathbf{F}^a_{\aleph_0}$-saturated,' and we call a model *a-prime* instead of '$\mathbf{F}^a_{\aleph_0}$-prime over $\emptyset$.' When describing dimensions of regular strong types inside models, the notation $\dim(p, A, M) = \kappa$ means that p is based on A and that any maximal A-independent set of realizations of $p|A$ inside M has size κ.

It is noteworthy that none of the results in this paper have any dependence on the cardinality of the language.

2. a-prime models of superstable theories

In this section we focus on the a-prime and a-saturated models of a superstable theory. Recall (from [7]) that a model is a-saturated if it realizes every strong type over a finite subset, and that a model is a-prime just in case it is a-saturated and embeds into any other a-saturated model of its complete theory. We will freely use well-known facts about a-prime and a-saturated models from [7] and [9].

We first focus on proving a characterization of the a-prime models in any superstable theory (Proposition 2.6) which is important for our subsequent results. Note that the main use of superstability in the proof of this proposition is the "ubiquity of regular types." In the remainder of the section, we introduce low substructures and some lemmas on dimensions in a-saturated models that will be useful later.

Our first definition is a variation on the classical notion of *pre-weight* (see [7] or [9]) which measures the size of a set by how many distinct independent elements can fork with it.

Definition 2.1. A set B has *countable local pre-weight* if for every finite set A, every stationary, regular type $p \in S(A)$, and every A-independent set $I \subseteq p(\mathfrak{C})$, there is a countable $I_0 \subseteq I$ such that $(I \setminus I_0) \underset{A}{\downarrow} B$.

Remark 2.2. In the definition above, "local" refers to the fact that we require that the elements of I come from a single regular type. Classically, weight differs from pre-weight in that the weight is the supremum of the pre-weights of all nonforking extensions. Whereas Theorem 2.6 shows that the a-prime model of a superstable theory has countable local pre-weight, Example 2.13 shows that it need not have countable local weight.

Lemma 2.3. *Suppose that M is any a-prime model and $p \in S(\emptyset)$ is stationary and regular. For any countable set A and for any A-independent set I of realizations of*

$p|A$, there is no uncountable, pairwise disjoint family $\{E_i : i \in \omega_1\}$ of subsets of I such that $E_i \not\downarrow_A M$ for each i.

Proof. By way of contradiction, suppose an uncountable family $\{E_i : i \in \omega_1\}$ existed. We can clearly assume that each E_i is finite. For each i choose a finite tuple a_i from A and a finite b_i from M such that $E_i \not\downarrow_{a_i} b_i$ for each i. As A is countable, there is a specific a^* such that $a_i = a^*$ for uncountably many i. Thus, by reindexing, we may assume that every $a_i = a^*$. Let $N = M[a^*]$ be the a-prime model over $M \cup \{a^*\}$. As a^* is finite, N is also a-prime over $\emptyset$. Choose a maximal, independent set $J \subseteq p(N)$. As N is a-prime, J is countable.

Next, for each i, choose a finite $J(i) \subseteq J$ so that $a^* b_i \downarrow_{J(i)} J$. Arguing as above, we may assume that there is a single J^* so that $J(i) = J^*$ for every i. Furthermore, since J^* is finite and the sets $\{E_i\}$ are independent, by eliminating at most finitely many i we may additionally assume that $E_i \downarrow J^*$ for each i.

Now we obtain a contradiction by fixing any remaining i. As N is a-saturated, we can choose $E_i' \subseteq N$ such that $\mathrm{stp}(E_i' a^* b_i J^*) = \mathrm{stp}(E_i a^* b_i J^*)$. Since every element of E_i' realizes $p(N)$ and is independent of J^*, the maximality of J implies that J dominates E_i' over J^*. Since E_i is independent from J^* and forks with $a^* b_i$ over $\emptyset$, it follows that E_i, and hence E_i', forks with $a^* b_i$ over J^*. Combining this with the domination described above implies that $a^* b_i \not\downarrow_{J^*} J$, which is a contradiction.

Lemma 2.4. *Suppose that M is any a-prime model and $p \in S(\emptyset)$ is stationary and regular. For any countable set A and for any A-independent set I of realizations of $p|A$, there is a countable $I^* \subseteq I$ such that $(I \setminus I^*) \downarrow_{AI^*} M$.*

Proof. We first argue that for any countable set A there is a countable $I_0 \subseteq I$ such that $(I \setminus I_0) \downarrow_A M$. To see this, given a countable set A, call a subset $E \subseteq I$ a *minimal witness to forking* if $E \not\downarrow_A EM$, but any proper subset of E is free from M over A. It is clear that every minimal witness is finite, and that if we set I_0 to be the union of all the minimal witnesses, then $(I \setminus I_0) \downarrow_A M$. Thus, it remains to prove that I_0 is countable.

However, if I_0 were uncountable, then we would have uncountably many minimal witnesses $\{E_i\}$. By the Δ-system lemma, there would be a finite set $G \subseteq I$ and an uncountable family $\{E_i : i \in \omega_1\}$ such that $E_i \cap E_j = G$ for distinct i, j. But then, apply Lemma 2.3 with $A' = A \cup G$, $I' = I \setminus G$, and the family $\{F_i\}$, where $F_i = E_i \setminus G$ and obtain a contradiction.

Now, to prove the Lemma, suppose we are given a countable set A. Form a sequence $I_0 \subseteq I_1 \subseteq \dots$ of countable subsets of I by applying the result in the first paragraph successively to the countable sets A, $A \cup I_0$, $A \cup I_1$, et cetera. Then the set $I^* = \bigcup I_n$ satisfies our demands.

Proposition 2.5. *Every a-prime model has countable local pre-weight. In fact, given any sets $B \subseteq A$ with B finite and A countable, and for any stationary, regular $p \in S(B)$, then for every B-independent set $I \subseteq p(\mathfrak{C})$, there is a countable set $I^* \subseteq I$ such that $(I \setminus I^*) \downarrow_{AI^*} M$.*

Proof. First, as A is countable, there is a finite $I_0 \subseteq I$ such that $I \setminus I_0$ is AI_0-independent. Thus, by replacing I by $I \setminus I_0$ and A by $A \cup I_0$, we may additionally assume that I is A-independent.

Next, let $N = M[B]$ be a-prime over $M \cup B$. Then N is also a-prime. But furthermore, as B is finite, N is also a-prime over B. Thus, if we work over B and apply Lemma 2.4, we obtain the requisite I^*.

In fact, the countability of local pre-weight characterizes the a-prime models among the class of all a-saturated models.

Proposition 2.6. *The following are equivalent for an a-saturated model M of a superstable theory:*

1. *M is a-prime over $\emptyset$;*
2. *Every infinite, indiscernible set in M is countable;*
3. *For all finite $B \subseteq M$, every stationary, regular $p \in S(B)$ has countable dimension;*
4. *M has countable local pre-weight.*

Proof. The equivalence of (1) and (2) is given in IV 4.18 of [9], noting that any finite tuple is trivially $\mathbf{F}^a_{\aleph_0}$-atomic over $\emptyset$. (2) implies (3) is immediate.

To see that (3) implies (2), suppose that there is an uncountable indiscernible set $I \subseteq M$. Let $q = Av(I, M)$. By superstability there is a regular type p non-orthogonal to q. Since M is a-saturated, possibly by replacing p by a non-orthogonal regular type, we may assume that p is based and stationary on a finite $B \subseteq M$. Again by superstability, there is a finite $I_0 \subseteq I$ on which q is stationary and moreover, by padding B with a finite Morley sequence in p, we may additionally assume that the types $p', q' \in S(BI_0)$, that are parallel to p, q respectively, are not almost orthogonal. From this and the fact that p' has weight one, it is clear that M must contain an uncountable Morley sequence in p'.

Finally, (4) implies (3) is obvious, and the implication (3) implies (4) is the content of Proposition 2.5.

Definition 2.7. Given an a-saturated model N, M is a *low substructure of N* if $M \preceq N$, M is a-prime, and $\dim(p, M, N) \geq \aleph_0$ for every regular type $p \in S(M)$.

Lemma 2.8. *Every a-saturated model has a low substructure.*

Proof. It suffices to prove that every a-prime model has a low substructure. By the uniqueness of a-prime models, it suffices to construct a single a-prime model N that has a low substructure.

Toward this end, fix M any a-prime model. Let $\Gamma \subseteq S(M)$ be any maximal subset of pairwise orthogonal weight one types over $S(M)$. Let

$$I = \bigcup_{p \in \Gamma} I_p$$

be independent over M such that each I_p is a Morley sequence of length ω built from p, and let N be a-prime over $M \cup I$. By construction, $\dim(q, M, N) \geq \aleph_0$ for every $q \in S(M)$, so it suffices to show that N is a-prime over $\emptyset$. By Lemma 2.6, it suffices to show that as a set, N has countable local pre-weight. To see this, choose any finite set A, any stationary, regular $r \in S(A)$, and any Morley sequence J built from r. There is at most one $p \in \Gamma$ non-orthogonal to r. Choose $M' \preceq N$ to be a-prime over $M \cup I_p$ if such a $p \in \Gamma$ exists, or else let $M' = M$ if $r \perp p$ for every $p \in \Gamma$. Note that in either case, $\mathrm{tp}(N/M')$ is orthogonal to r. Since M is a-prime it has countable local pre-weight, so there is a countable $J_0 \subseteq J$ with $M \mathop{\smile\!\!\!\!\!|}_{AJ_0} J$. By the construction of M', it follows that there is a countable J_1, $J_0 \subseteq J_1 \subseteq J$, satisfying $M' \mathop{\smile\!\!\!\!\!\mid}_{AJ_1} J$. As $\mathrm{tp}(N/M')$ is orthogonal to r, $N \mathop{\smile\!\!\!\!\!\mid}_{AJ_1} J$, so N has countable local pre-weight.

Lemma 2.9. *Suppose that N is an a-saturated model, $X \subseteq N$ is any set, and $p, q \in S(X)$ are stationary, regular types that are not almost orthogonal. Then* $\dim(p, X, N) = \dim(q, X, N)$.

Proof. We first prove that for each $c \in p(N)$, there is $d \in q(N)$ that forks with c over X. To see this, by non-almost orthogonality, for any such c there is $d_0 \in q(\mathfrak{C})$ forking with c over X. Choose $B \subseteq X$ finite such that p and q are based on B and $c \not\mathop{\smile\!\!\!\!\!\mid}_B d_0$. As N is a-saturated, we can find $d \in N$ such that $\mathrm{stp}(d/Bc) = \mathrm{stp}(d_0/Bc)$. To see that d realizes q, it suffices to show that $d \mathop{\smile\!\!\!\!\!\mid}_B X$. However, if it forked, then $r = \mathrm{stp}(d/X)$ would be a forking extension of a strong type parallel to q. But then, by regularity r would be orthogonal to p, which would be a contradiction.

Now let $I \subseteq p(N)$ be any maximal X-independent set. From the argument above, for each $c \in I$, choose $d_c \in q(N)$ that forks with c over X. It follows immediately from the fact that regular types have weight one that the mapping $c \mapsto d_c$ is injective, and moreover that $J = \{d_c : c \in I\}$ is X-independent. Thus, $\dim(p, X, N) \leq \dim(q, X, N)$. By symmetry, this suffices to prove the lemma.

Proposition 2.10. *Suppose M_0, M_1 are both low substructures of an arbitrary a-saturated model N. If $p \in S(M_0)$ and $q \in S(M_1)$ are non-orthogonal regular types, then* $\dim(p, M_0, N) = \dim(q, M_1, N)$.

Proof. By the definition of a low substructure, M_0 and M_1 are both a-prime and each dimension is infinite. If the dimensions are both countable, they are equal. Thus, by symmetry, assume that $\dim(p, M_0, N) = \kappa > \aleph_0$. It suffices to show that $\dim(q, M_1, N) = \kappa$. To see this, first choose a finite $B \subseteq M_0$ on which p is based and stationary. As M_0 is a-prime, $\dim(p, B, M_0)$ is countable, hence $\dim(p, B, N) = \kappa$ as well. Let $I \subseteq N$ be a maximal B-independent set of realizations of $p|B$. By Proposition 2.5 there is a countable $I_0 \subseteq I$ such that $(I \setminus I_0) \mathop{\smile\!\!\!\!\!\mid}_{BI_0} M_1$. By adding at most finitely many additional points to BI_0, we may assume that the types parallel to p and q over M_1BI_0 are not almost orthogonal. Thus, by Lemma 2.9, as $|I \setminus I_0| = \kappa$, it follows that $\dim(q, M_1BI_0, N) = \kappa$. But then, as BI_0 is countable, it follows that $\dim(q, M_1, N) = \kappa$ and we finish.

Lemma 2.11. *T superstable. If $f : M \to N$ is any elementary embedding of a-saturated models, and if M_0 is a low substructure of M, then $f(M_0)$ is a low substructure of N.*

Proof. Clearly, $f(M_0) \preceq N$ and is a-prime, since it is isomorphic to M_0. Furthermore, if $q \in S(f(M_0))$ is regular, then as $p := f^{-1}(q)$ is a regular type over M_0, p has infinite dimension in M, hence q has infinite dimension in N. Thus, $f(M_0)$ is a low substructure of N.

Corollary 2.12. *Suppose that T is superstable, M_0 is a low substructure of an a-saturated model M, and that $f : M \to M$ is an elementary endomorphism. If $p \in S(M_0)$ is regular and non-orthogonal to $f(p)$, then* $\dim(p, M_0, M) = \dim(f(p), f(M_0), M)$.

Proof. This follows immediately from Proposition 2.10 and Lemma 2.11.

We close this section with a remark about Definition 2.1. For the moment, say that a set B has *countable local weight* if for any set A independent from B, any stationary, regular $p \in S(A)$, and any $I \subseteq p(\mathfrak{C})$, there is a countable $I_0 \subseteq I$ such that $(I \setminus I_0) \mathop{\smile\!\!\!\!\!\!\mid}_A B$. The following example shows that having countable local weight is too much to expect for the universe of an a-prime model, even if the theory is weakly minimal and unidimensional.

Example 2.13. *The a-prime model of $Th((\mathbb{Z},+))$ does not have countable local weight. In fact, the a-prime model $(G,+,\dots)$ of any weakly minimal group in a countable language that has a family of $2^{\aleph_0}$ strong types over $\emptyset$, each describing cosets of the principal generic type p does not have countable local weight.*

Proof. Let $(G,+,\dots)$ be the a-prime model of such a theory. As the language is countable, we can inductively find a sequence $\{q_\alpha : \alpha \in \omega_1\}$ of strong types that are 'almost independent over $\emptyset$' i.e., for any choices of b_α realizing q_α, the set $B = \{b_\alpha : \alpha \in \omega_1\}$ is independent over $\emptyset$.

As G is a-saturated, we can choose such a set $B \subseteq G$ as described above. Let $A = \{a_\alpha : \alpha \in \omega_1\}$ be free from G over $\emptyset$, with each a_α realizing q_α. Now let $I = \{c_\alpha : \alpha \in \omega_1\}$, where each $c_\alpha := b_\alpha - a_\alpha$ realizes the principal generic type p. It is easy to see that I is A-independent, but $I_1 \not\mathop{\smile\!\!\!\!\!\!\mid}_A G$ for any non-empty $I_1 \subseteq I$.

3. The SB-property for a-saturated models

In this section, we use the results of the previous section to characterize which superstable theories have the SB property for a-saturated models: they are precisely the ones without nomadic types (see Definition 3.1 and Theorem 3.11). It turns out that this condition is slightly stronger than being non-multidimensional (nmd), and so we found it useful to first establish some facts about a-prime models in any superstable nmd theory.

We continue to assume that T is superstable unless otherwise specified.

Definition 3.1. A non-algebraic strong type p is *nomadic* if there is an automorphism $f \in Aut(\mathfrak{C})$ such that the n-fold iterate $f^{(n)}(p) \perp p$ for each integer n.

It is easy to see that if a superstable theory T has a nomadic type, then it has a regular nomadic type. As well, it is easily seen (using e.g., Lemma 1.4.3.3 of [7]) that any type p that is orthogonal to $\emptyset$ is nomadic, so that any multidimensional theory has nomadic types. However, even some nmd theories have nomadic types:

Example 3.2. *Let T be the theory of a collection $\{E_i : i \in \mathbb{N}\}$ of refining equivalence relations ($E_{i+1}(x,y) \Rightarrow E_i(x,y)$) such that E_0 has exactly two classes and each E_i-class splits into two E_{i+1}-classes. Then T has quantifier elimination and is complete, superstable (in fact, weakly minimal), and nmd, and T has a unique* 1*-type $p(x)$ over $\emptyset$. If $f \in \mathrm{Aut}(\mathfrak{C})$ is chosen so that it permutes the E_i-classes in a single cycle of order 2^{i+1}, then $p \perp f(p) \perp f^2(p) \perp \dots$, so p is nomadic.*

Among nmd theories, then the existence of nomadic types can be obviated by eliminating automorphisms of $\mathrm{acl}(\emptyset)$:

Lemma 3.3. *Suppose T is nmd. If, in $\mathfrak{C}^{\mathrm{eq}}$, $\mathrm{acl}(\emptyset) = \mathrm{dcl}(\emptyset)$, then T has no nomadic types.*

Proof. By the observation above, it suffices to show that no stationary, regular $p \in S(A)$ is nomadic via any automorphism $f \in Aut(\mathfrak{C})$. In fact, we show that $p \not\perp f(p)$ for any such p and f. To see this, let $A' = f(A)$, and choose A'' having the same strong type as A over $\emptyset$, but with A'' free from $A \cup A'$ over $\emptyset$. Note that by our assumptions, $\mathrm{tp}(A/\mathrm{acl}(\emptyset)) = \mathrm{tp}(A''/\mathrm{acl}(\emptyset)) = \mathrm{tp}(A'/\mathrm{acl}(\emptyset))$. Let p'' be the (regular) type conjugate to p over A''. Because T has nmd, $p \not\perp \emptyset$, so by e.g., Lemma 1.4.3.3 of [7], $p \not\perp p''$ and $f(p) \not\perp p''$. Thus, $p \not\perp f(p)$ by the transitivity of nonorthogonality among regular types.

Lemma 3.4. *Suppose that T is nmd and $M \preceq M^* \preceq N$, where M and N are a-saturated and M^* is any model that is not equal to N. Then there is $c \in N \setminus M$ such that $\mathrm{tp}(c/M)$ is regular and $c \underset{M}{\downarrow} N$.*

Proof. By superstability and the fact that $M^* \neq N$, there is a regular $q \in S(M^*)$ that is realized in N (see e.g., Proposition 8.3.2 of [7]). Since T has nmd, q is not orthogonal to M. So, since M is a-saturated, the existence of such a c follows immediately by Proposition 8.3.6 of [7].

Proposition 3.5. *Suppose that T is nmd and that $N \models T$ is any model with an a-saturated elementary submodel M. Then N is a-saturated.*

Proof. Let $N^* = M[N]$ be any a-prime model over $M \cup N = N$. If $N \neq N^*$, then by Proposition 3.4 there would be $c \in N^* \setminus M$ with $c \downarrow_M N$, which contradicts the fact that N^* is dominated by N over M.

Thus, for any set B that contains an a-saturated model M, any model N containing B is automatically a-saturated. Hence, the notions of 'prime over B' and 'minimal over B' are equivalent to the notions 'a-prime over B' and 'a-minimal over B', respectively. These observations will be used extensively in the next section.[1]

Proposition 3.6. *Suppose that T is nmd and that $M \preceq N$ are both a-saturated and J is any maximal, M-independent subset of N consisting of realizations of regular types over M. Then N is a-prime and a-minimal over $M \cup J$.*

Proof. Let $M^* \preceq N$ be any a-prime model over $M \cup J$. To see that N is both a-prime and a-minimal over $M \cup J$, it suffices to prove that $M^* = N$. If this were not the case, then by Lemma 3.4 there would be $c \in N \setminus M^*$ such that $\mathrm{tp}(c/M)$ is regular and $c \downarrow_M M^*$, which would contradict the maximality of J.

Definition 3.7. Suppose that $M_0 \preceq M$ and N are a-saturated. An elementary embedding $f : M \to N$ is *dimension preserving over $M_0 \preceq M$* if $\dim(p, M_0, M) = \dim(f(p), f(M_0), N)$ for every regular $p \in S(M_0)$.

[1]These equivalences also illustrate an error in popular parlance. In the setting of superstable, NDOP theories in a countable language, in Chapter 12 of [9] Shelah proved that NOTOP is equivalent to the '$(\infty, 2)$-existence property'. In several places, this unweildly phrase has been replaced by 'PMOP', an acronym for 'Prime Models Over Pairs.' Although this term sounds better, it is misleading. The results above show that for any superstable, nmd theory (even those with OTOP, see e.g., Example 2.2 of [3]), there is a prime model over $M_1 \cup M_2$, where M_1 and M_2 are any models containing an a-saturated model. The 'correct' meaning of PMOP (i.e., equivalent to $(\infty, 2)$-existence) is that there is a t-constructible model over any independent pair of models.

Corollary 3.8. *Suppose T is nmd. If there is an elementary embedding $f : M \to N$ between a-saturated models that is dimension-preserving over some a-saturated $M_0 \preceq M$, then $M \cong N$. In fact, the isomorphism can be chosen to extend $f|_{M_0}$.*

Proof. Fix a-saturated models M, N and an elementary embedding $f : M \to N$ that is dimension preserving over some a-saturated $M_0 \preceq M$. Let $\Gamma \subseteq S(M_0)$ be a maximal, pairwise orthogonal set of regular types over M_0, and for each $p \in \Gamma$, let I_p be a maximal, independent subset of $p(M)$. Note that $I = \bigcup_{p\in\Gamma} I_p$ is a maximal, M_0-independent set of realizations of regular types in M. Next, for each $p \in \Gamma$, let J_p be a maximal, $f(M_0)$-independent set of realizations of $f(p)$ in N. As $f|_{M_0}$ is an isomorphism between M_0 and $f(M_0)$, it follows that $J = \bigcup_{p\in\Gamma} J_p$ is a maximal, $f(M_0)$-independent set of realizations of regular types in N. Since f is dimension preserving over M_0, $|I_p| = |J_p|$ for each $p \in \Gamma$. For each $p \in \Gamma$, fix any bijection $g_p : I_p \to J_p$. By indiscernibility, for each $p \in \Gamma$, the map $h_p := f|_{M_0} \cup g_p$ is elementary, and by independence over a model, $h := \bigcup h_p : M_0 \cup I \to f(M_0) \cup J$ is elementary as well. By Proposition 3.6, M is a-prime over $\mathrm{dom}(h)$, while N is a-prime over $\mathrm{range}(h)$. It follows from the uniqueness of a-prime models that h extends to an isomorphism $h^* : M \to N$.

The proof of the following Lemma is immediate.

Lemma 3.9. *If $f : M \to N$ is any elementary embedding and $M_0 \preceq M$ is arbitrary, then $\dim(p, M_0, M) \leq \dim(f(p), f(M_0), N)$ for any regular type $p \in S(M_0)$.*

Proposition 3.10. *Assume that T has no nomadic types. Suppose that M and N are both a-saturated and $f : M \to N$ and $g : N \to M$ are elementary embeddings. Then f is dimension preserving over any low substructure M_0 of M.*

Proof. Let $h = g \circ f$ denote the composition. Fix any low substructure M_0 of M and any regular type $p \in S(M_0)$. Since there are no nomadic types, there is a positive integer n so that the n-fold composition $k = h^{(n)}$ satisfies $p \not\perp k(p)$. By Corollary 2.12 we have $\dim(p, M_0, M) = \dim(k(p), k(M_0), M)$. But, by iterating Lemma 3.9 we have

$$\dim(p, M_0, M) \leq \dim(f(p), f(M_0), N) \leq \dim(k(p), k(M_0), M)$$

so $\dim(p, M_0, M) = \dim(f(p), f(M_0), N)$ as required.

Theorem 3.11. *For a superstable theory T, the following are equivalent:*

(1) *T has the Schröder-Bernstein property for a-saturated models;*
(2) *there is no* infinite *collection of pairwise elementarily bi-embeddable, pairwise nonisomorphic a-saturated models of T;*
(3) *T has no nomadic types.*

Proof. $1 \Rightarrow 2$ is trivial. The direction $2 \Rightarrow 3$ is proved in Theorem 4.8 of [2] (whose statement does not mention saturation, but as noted in the proof there, the argument can be used to produce bi-embeddable, nonisomorphic a-saturated models). Finally, for $3 \Rightarrow 1$, note that T superstable with no nomadic types implies that T is nmd as well. Choose a-saturated models M and N and fix elementary embeddings $f : M \to N$ and $g : N \to M$. By Lemma 2.8, choose a low substructure M_0 of M. By Proposition 3.10, f is dimension preserving over M_0, so M and N are isomorphic by Corollary 3.8.

It might be true that having no nomadic types implies the SB property for a-saturated models in *any* stable theory; we know of no counterexample. (For a strictly stable T, "a-saturated" means "all strong types over subsets of size less than $\kappa_r(T)$ are realized.")

4. The eventual SB property for models

Definition 4.1. A complete theory T has the *eventual SB property* if there is a *small* set $A \subseteq \mathfrak{C}$ such that the expansion $Th_A(\mathfrak{C})$ formed by adding a new constant symbol for each element of A has the SB property. (Here "small" means that $|A| < \kappa$ for some cardinal κ such that the universal domain $\mathfrak{C}$ is κ-saturated.)

The goal of this short section is to characterize those theories with the eventual SB property.

Theorem 4.2. *The following are equivalent for a complete theory T:*

1. *T has the eventual SB property;*
2. *For every small subset $A \subseteq \mathfrak{C}$ containing an a-saturated model, $Th_A(\mathfrak{C})$ has the SB property;*
3. *T is superstable and nmd.*

Proof. $2 \Rightarrow 1$: Trivial.

$1 \Rightarrow 3$: We prove the contrapositive. If T is *not* superstable, then the same is true of $Th_A(\mathfrak{C})$ for any small set $A \subseteq \mathfrak{C}$, so $Th_A(\mathfrak{C})$ does not have SB by Theorem 1.1. The other case to consider is when T is stable and multidimensional, in which case $Th_A(\mathfrak{C})$ is again stable and multidimensional, and the failure of SB follows from Theorem 4.8 of [2] (noting that any regular type p orthogonal to $\emptyset$ satisfies the hypothesis of that result).

$3 \Rightarrow 2$: Fix a small set A containing an a-saturated model M, and choose any pair N_1^*, N_2^* of bi-embeddable models $Th_A(\mathfrak{C})$. That is, the reducts N_1 and N_2 to the original language are models of T that are bi-embeddable over A. By Proposition 3.5, both reducts N_1 and N_2 are themselves a-saturated. We argue that any $L(A)$-elementary embedding $f : N_1^* \to N_2^*$, when viewed as an L-elementary embedding from N_1 to N_2 that fixes A pointwise, is dimension preserving over any $M_1 \preceq N_1$ that is a-prime over A. To see this, fix any such function f and choose any $M_1 \preceq N_1$ that is a-prime over A. As f is over A, $f(M_1)$, which we denote by M_2, is also a-prime over A.

Note that for each regular type $q \in S(M)$ and each $i = 1, 2$, the fact that M_i is dominated by A over M implies that the non-forking extension $q|A$ is omitted in M_i. Furthermore, as $q|A$ is fixed by any embedding over A, Lemma 3.9 and the fact that N_1^* and N_2^* are bi-embeddable $L(A)$-structures imply that $\dim(q, A, N_1) = \dim(q, A, N_2)$. It follows that $\dim(q, M_1, N_1) = \dim(q, M_2, N_2)$ for every regular $q \in S(M)$. However, it follows from nmd and the fact that M is an a-saturated model that any regular type $p \in S(M_1)$ is non-orthogonal to some regular $q \in S(M)$. Moreover, $f(p)$ is non-orthogonal to the same type q. Thus, applying Lemma 2.9 on both sides yields

$$\dim(p, M_1, N_1) = \dim(q, M_1, N_1) = \dim(q, M_2, N_2) = \dim(f(p), M_2, N_2)$$

Thus, f is dimension-preserving over M_1. By Corollary 3.8, the L-structures N_1 and N_2 are isomorphic via an isomorphism h extending $f|A$. As f fixes A pointwise, h is an $L(A)$-isomorphism between N_1^* and N_2^*.

Question 4.3. *What conditions on a set A are needed to ensure that $Th_A(\mathfrak{C})$ has SB for a classifiable, nmd theory?*

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