A Vaught's conjecture toolbox

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A dichotomy:

- If T is not small, then there is a perfect set of complete types, hence $I(T, \aleph_0) = 2^{\aleph_0}$ [in fact, a perfect set of pairwise non-isomorphic models].
- If T is small, then T has a countable, saturated model and a prime model, which is also the unique countable atomic model.

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However... DLS can be recovered by restricting to reasonable countable fragments.

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- If Δ is a reasonable countable fragment, then for any L-structure M, there is a countable $M' \preceq_\Delta M$.

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Definition (Keisler)

Let Δ be any reasonable countable fragment of $L_{\omega_1,\omega}$.

- A set $T \subseteq \Delta$ of sentences is consistent if there is a model $M \models T$;
- A consistent set $T \subseteq \Delta$ is Δ-complete if T decides ψ for every Δ-sentence ψ.
- A complete Δ-n-type $p(\overline{x})$ with respect to T is a maximal consistent (w.r.t. T) set of Δ-formulas with at most $(x_1 \ldots, x_n)$ free.
- A Δ-complete theory T is small if $S_n(T, \Delta)$ is countable for all $n \geq 1$.

Theorem (Keisler)

Let Δ be any reasonable countable fragment of $L_{\omega_1,\omega}$ and let T be Δ-complete.

- *If T is not small, then there is a perfect set contained in $S_n(T,\Delta)$ for some n [hence a perfect set of pairwise non-isomorphic models];*
- *If T is small, then there is a unique (up to isomorphism) Δ-prime model, which is also the unique countable, Δ-atomic model.*

Definition (Morley)

An $L_{\omega_1,\omega}$-sentence Φ is scattered if $S_n(\Phi, \Delta)$ is countable for every (reasonable) countable fragment Δ.

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Proposition

TFAE for a sentence Φ of $L_{\omega_1,\omega}$:

1. *Φ is scattered;*
2. *$Mod(\Phi)$ does not contain a perfect set of pairwise non-isomorphic models.*

Polish space of L-structures

Fix a (countable) vocabulary L with at least one binary relation or function symbol.

$$X_L = \{\text{all } L\text{-structures } M \text{ with universe } \omega\}$$

Basic open sets $U_{\varphi(\overline{m})} = \{M \in X_L : M \models \varphi(\overline{m})\}$.

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Then:

- X_L is a standard Borel space;
- For any $\Phi \in L_{\omega_1,\omega}$, $Mod(\Phi)$ is a Borel subset of X_L;
- The isomorphism relation $\cong_\Phi$ is a $\mathbf{\Sigma}^1_1$-subset of $X_L \times X_L$ ($M \cong N$ iff $\exists f(\ldots))$.

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Whether $\cong_\Phi$ is Borel or not will be an important distinction!

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Fix a countable M. A potential back-and-forth system $\mathbf{F}$ is a set of finite, partial functions $f : \overline{a} \to \overline{b}$ satisfying:

- $\mathbf{F}$ is closed under restrictions;
- If $f : \overline{a} \to \overline{b}$ is in $\mathbf{F}$, then $qftp(\overline{a}) = qftp(\overline{b})$; and
- If $\sigma \in Aut(M)$, then each restriction $\sigma|_{\overline{a}} \in \mathbf{F}$.

Examples: All $f : \overline{a} \to \overline{b}$ with:

- $qftp(\overline{a}) = qftp(\overline{b})$ (i.e., no additional restrictions); OR
- The first-order types $tp(\overline{a}) = \mathrm{tp}(\overline{b})$; OR
- For any reasonable fragment Δ, $\mathrm{tp}_{\Delta}(\overline{a}) = \mathrm{tp}_{\Delta}(\overline{b})$.

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- $(M, \overline{a}) \sim_0 (M, \overline{b})$ iff $f : \overline{a} \mapsto \overline{b} \in \mathbf{F}$;
- For λ limit, $(M, \overline{a}) \sim_\lambda (M, \overline{b})$ iff $(M, \overline{a}) \sim_\alpha (M, \overline{b})$ for all $\alpha < \lambda$;
- $(M, \overline{a}) \sim_{\alpha+1} (N, \overline{b})$ iff
  1. For all $c \in M$ there is $d \in M$ such that $(M, \overline{a}c) \sim_\alpha (M, \overline{b}d)$; AND
  2. For all $d \in M$ there is $c \in M$ such that $(M, \overline{a}c) \sim_\alpha (M, \overline{b}d)$.

Scott heights

Note: If $(M,\bar{a}) \sim_{\alpha+\gamma} (M,\bar{b})$ then $(M,\bar{a}) \sim_{\alpha} (M,\bar{b})$.

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Proposition

TFAE for any $M, \bar{a}, \bar{b}$ and **F**:

1. $\{\alpha < \omega_1 : (M, \bar{a}) \sim_{\alpha} (M, \bar{b})\}$ *is uncountable;*
2. *For all* $\alpha < \omega_1$, $(M, \bar{a}) \sim_{\alpha} (M, \bar{b})$*;*
3. *There is* $\sigma \in Aut(M)$ *satisfying* $\sigma(\bar{a}) = \bar{b}$.

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3. *There is $\sigma \in Aut(M)$ satisfying $\sigma(\bar{a}) = \bar{b}$.*

Thus: For every M and $\mathbf{F}$, there is a least $\alpha^* = \alpha^*(M, \mathbf{F}) < \omega_1$ such that for all $\bar{a}, \bar{b}$ from M,

$$(M, \bar{a}) \sim_{\alpha^*} (M, \bar{b}) \text{ iff there is } \sigma \in Aut(M) \text{ with } \sigma(\bar{a}) = \bar{b}.$$

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When $\mathbf{F}$ consists of *qftp*-preserving partial maps, $\alpha^*(M,\mathbf{F}) := SH(M)$, the Scott height of M.

Now suppose $\Phi \in L_{\omega_1,\omega}$ and **F** is any of the above.

Put: $\alpha^*(\Phi, \mathbf{F}) := \sup\{\alpha^*(M, \mathbf{F}) : M \models \Phi\}$. We say Φ has bounded Scott heights if $\alpha^*(\Phi, \mathbf{F}) < \omega_1$ for some/every **F**.

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Theorem (Morley)

Let $\Phi \in L_{\omega_1,\omega}$ be scattered. Then:

- *$I(\Phi, \aleph_0) \leq \aleph_1$ always; and*
- *$I(\Phi, \aleph_0)$ is countable if and only if $\cong_\Phi$ is Borel.*

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Empirical fact: There are relatively few (known!) complete, first order T so that $\cong_T$ is not Borel (without being Borel complete).

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$T = Th$(Binary splitting, refining eq. relations) has $\cong_T$ non-Borel.

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Open (1989): If T is small and every countable universal model is saturated, must every countable weakly saturated (realize all n-types over $\emptyset$) model be saturated?

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In December, 1986 Harrington stated that "Vaught's conjecture for superstable theories is the major open problem in stability theory." Newelski and Buechler have made progress on this.