Borel complexity of complete, first order theories (status report)

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Recall:

- $X_L = \{\text{all } L\text{-structures with universe } \omega\}$.
- S_∞ induces the logic action on X_L.
- From Sam's talk: A Borel subset $Y \subseteq X_L$ is invariant under this action iff $Y = Mod(\Phi)$ for some $\Phi \in L_{\omega_1,\omega}$.

Theorem (Friedman-Stanley)

With respect to Borel reducibility, among all pairs $(Mod(\Phi), \cong_\Phi)$, there is a maximum Borel degree.

Definition

We say $\cong_\Phi$ is Borel complete if it is Borel equivalent to this maximum degree.

Examples: (Friedman-Stanley) The following classes of structures $(Mod(\Phi), \cong_\Phi)$ are all Borel complete:

- Directed graphs;
- Symmetric graphs;
- Linear orders;
- Fields;
- Subtrees of $\omega^{<\omega}$.

Throughout the whole of this talk, T will denote a complete, first order theory in a countable language.

- Interested in the Borel complexity of $(Mod(T), \cong_T)$.

Jumps: Suppose T is a complete L-theory. Let $L^+ = L \cup \{E\}$ and T^+ be the theory specifying:

- E is an equivalence relation with infinitely many classes;
- Each E-class is a model of T.

Then $\cong_{(T^+)}$ is Borel equivalent to the jump $(\cong_T)^+$.

Friedman-Stanley tower: Let

- $\cong_0$ be $id(\omega)$ [Think: Countably many non-isomorphic models.]
- $\cong_1$ be $id(2^\omega)$ [Countable sets of integers, i.e., reals]
- $\cong_2$ be $(\cong_1)^+$ [Countable sets of reals]

In general, given $\cong_\alpha$, let

- $\cong_{\alpha+1}$ be the jump $(\cong_\alpha)^+$ (i.e., 'countable sets of $\cong_\alpha$')

Note: $\cong_T <_B \cong_0$ iff T has finitely many models.
Of special note: $\cong_2$ is 'Countable sets of reals.'

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Easy: If $\cong_T$ is Borel complete, then $\cong_T$ is properly $\mathbf{\Sigma}^1_1$.

Note: Until recently, all known examples of $\cong_T$ properly $\mathbf{\Sigma}^1_1$ were Borel complete, hence $\geq_B$ every $\cong_{T'}$.

This led me (and maybe others) to think of every instance of $\cong_T$ properly $\mathbf{\Sigma}^1_1$ as being $>_B \cong_{T'}$ whenever $\cong_{T'}$ is Borel.

This is not always the case!

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- Borel complexity is ill-behaved under reducts.
  - There are complete $T_0 \subseteq T_1 \subseteq T_2$ (in languages $L_0 \subseteq L_1 \subseteq L_2$) such that $Mod(T_0)$ is $\aleph_0$-categorical, $Mod(T_1)$ is Borel complete, and $Mod(T_2)$ has countably many models.

- Naming (or deleting) constants is only partially understood.

Throughout most of model theory (e.g., showing $I(T, \aleph_0) = 2^{\aleph_0}$ or the configurations determining the spectrum $I(T, \kappa)$ for $\kappa > \aleph_0$), naming or deleting finitely many constants is free.

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Open: Can Borel completeness be gained or lost by naming a constant?

Best result so far:

Proposition (Rast)

Let T be complete,and $T(c)$ an expansion formed by naming a constant. Then $\cong_T$ is Borel if and only if $\cong_{T(c)}$ is Borel.

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Ulrich: Let M denote the (unique) countable random graph, and let $(M, c_n)_{n\in\omega}$ be any expansion such that $c_i \neq c_j$ for distinct i, j. Then $Th(M)$ is $\aleph_0$-categorical, while $Th((M, c_n)_{n\in\omega})$ is Borel complete.

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A later example will give a complete theory T such that $\cong_T$ is properly $\mathbf{\Sigma^1_1}$, but for any model M, the isomorphism relation $\cong_{El(M)}$ of the elementary diagram of M is Borel.

Only general result to date.

Marker: If T is not small, then $\cong_2 \leq_B \cong_T$, i.e., 'countable sets of reals' Borel reduce to $(Mod(T), \cong_T)$.

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Paradigm: 'Independent unary predicates' $L = \{U_n : n \in \omega\}$, T says 'Every finite boolean combination of $\pm U_n$ is consistent.'

Complete 1-types correspond to branches through $2^{<\omega}$ (i.e., reals) and for each branch, one can choose how many elements realize it.

o-minimal theories

Theorem (Rast/Sahota)

If T is o-minimal, then $\cong_T$ is one of the following:

- $<_B \cong_0$ *(finitely many models);*
- *Borel equivalent to* $\cong_1$ *(reals);*
- *Borel equivalent to* $\cong_2$ *(countable sets of reals);*
- *Borel complete.*

Note: The proof of this theorem would have been massively simpler if one could name a constant!

Complete theories of linear orders with (countably many) unary predicates

Theorem (Rast)

If T is a complete theory of linear orders with unary predicates, then $\cong_T$ is one of the following:

- $<_B \cong_0$ *(finitely many models);*
- *Borel equivalent to* $\cong_1$ *(reals);*
- *Borel equivalent to* $\cong_2$ *(countable sets of reals);*
- *Borel complete.*

ω-stable theories

Note: T ω-stable implies T small ($S_n(\emptyset)$ countable for each n)

Theorem (L-Shelah)

If T is ω-stable and has eni-DOP or is eni-DEEP, then $\cong_T$ is Borel complete.

Note: The proof of this would have been at least 10 pages shorter if one could name a constant!

Theorem (Rast, streamlining Koerwien)

For each ordinal $\alpha < \omega_1$, there is an ω-stable theory T_α such that $\cong_{(T_\alpha)}$ is Borel equivalent to $\cong_\alpha$ (the α'th jump).

ω-stable theories (cont.)

Theorem (Koerwien+Ulrich)

There is an ω-stable, depth 2 theory K for which

- $\cong_K$ *is properly* $\mathbf{\Sigma}^1_1$ *BUT*
- $\cong_K$ *is NOT Borel complete.*

Refining equivalence relations

Let $L = \{E_n : n \in \omega\}$ and consider L-theories T that say:

- Each E_n is an equivalence relation;
- E_0 consists of a single class;
- Each E_{n+1} refines E_n, i.e., $E_{n+1}(a, b)$ implies $E_n(a, b)$.

In order to make T complete, need only say how many classes E_{n+1} partitions each E_n-class into.

Case 1: REF_ω says: Each E_{n+1}-class partitions each E_n-class into infinitely many classes.

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Case 2: REF_2 says: Each E_{n+1}-class partitions each E_n-class into 2 classes.

Theorem (L-Rast-Ulrich)

The isomorphism relation on REF_2 is properly $\mathbf{\Sigma}_1^1$ but is not Borel complete.

Hybrids: Given $m \leq \omega$, let T_m be:

- For $n < m$, E_{n+1} partitions each E_n-class into infinitely many classes;
- For $n \geq m$, E_{n+1} partitions each E_n-class into 2 classes.

Then:

- T_0 is REF_2, T_ω is REF_ω;
- For all m, $\cong_{T_m}$ is properly $\mathbf{\Sigma}^1_1$
- For all m, T_m is small;
- $\cong_{T_0} \ <_B \ \cong_{T_1} \ <_B \ \cong_{T_2} \ <_B \cdots <_B \ \cong_{T_\omega}$.

Suppose $M \models REF_2$ is countable. Then the elementary diagram $El(M)$ is essentially the same as 'Independent unary predicates.' In particular:

- $\cong_{El(M)}$ is Borel equivalent to $\cong_2$ (countable sets of reals);
- Thus, $\cong_{El(M)}$ is Borel; BUT
- Its restriction to $L = \{E_n : n \in \omega\}$ is REF_2 and $\cong_{REF_2}$ is properly $\mathbf{\Sigma}^1_1$

A final thought: It has become empirically clear that 'Vaught's conjecture for superstable T' is much more involved than 'Vaught's conjecture for ω-stable T.'

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Fact: If T is superstable, but not ω-stable, then T is either not small, or else has a type of infinite multiplicity.

A final thought: It has become empirically clear that 'Vaught's conjecture for superstable T' is much more involved than 'Vaught's conjecture for ω-stable T.'

Fact: If T is superstable, but not ω-stable, then T is either not small, or else has a type of infinite multiplicity.

REF_2 is the paradigm of a superstable theory with infinite multiplicity!