WORST CASE EXPANSIONS OF COMPLETE THEORIES

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Abstract. Given a complete theory T and a subset $Y \subseteq X^k$, we precisely determine the *worst case complexity* of an expansion (M, Y) by Y of a model M of T with universe X. Although by definition monadically stable/NIP theories remain tame under arbitrary monadic expansions, we show that monadically NFCP (equivalently, mutually algebraic) theories are the largest class robust under anything beyond monadic expansions. We also exhibit a paradigmatic structure for the failure of each of these monadic properties, and prove each of these paradigms definably embeds into a monadic expansion of a sufficiently saturated model of any theory without the corresponding property.

1. Introduction

The theory ACF_0 of algebraically closed fields of characteristic zero is model theoretically very well behaved. In particular, it is strongly minimal, hence uncountably categorical. Moreover, the model $\mathfrak{C} = (\mathbb{C}, +, -, \cdot, 0, 1)$ has many interesting expansions formed by adding a unary predicate U. If one interprets U as the algebraics $\tilde{\mathbb{Q}}$, then we get a theory of pairs $(\mathfrak{C}, \tilde{\mathbb{Q}})$, whose models are also well behaved. Interpreting U as the reals $\mathbb{R}$ gives an interesting structure $(\mathfrak{C}, \mathbb{R})$, whose theory is unstable, but is otherwise rather tame. Things are worse if one interprets U as the integers $\mathbb{Z}$, as the theory $(\mathfrak{C}, \mathbb{Z})$ admits rather unpleasant Gödel phenomena. But artificial expansions can be even worse. Let I be any countably infinite, linearly independent subset and let U_S be interpreted as $I \cup S$, where S is a carefully chosen subset of the sum set $I + I$. By choosing S appropriately, one can definably encode any given countable graph into an expansion $(\mathfrak{C}, U_S)$ of this form. We codify this behavior in the following definition.

Definition 1.1. A theory T *monadically codes graphs* if for every graph G, there is some $M \models T$ and M^* and expansion of T by unary predicates such that G definably embeds into M^*, in the sense of Definition 3.1.

Thus, whether or not a theory T monadically codes graphs depends on the *worst case* complexity of a monadic expansion of a model of T. In the example above, even though ACF_0 has some tame expansions, ACF_0 monadically codes graphs.

To ease notation, in what follows assume that T is a complete theory in a countable language (with an infinite model) and we consider models of T

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with universe ω. Now, given such a T and a subset $Y \subseteq \omega^k$ for some $k \geq 1$, we ask:

> What is the worst-case complexity of $Th(M, Y)$ among all models M of T with universe ω?

To measure this, for P any property of theories, we say (T, Y) *is always* P if $Th(M, Y)$ has P for all models M of T with universe ω. It turns out that among all complete, countable theories T and all $Y \subseteq \omega^k$, there are very few combinations we need to consider.

Definition 1.2. A complete theory T is *purely monadic* if, for every model $M \models T$ with universe ω, every definable (with parameters) $Y \subseteq \omega^k$ is definable in a monadic structure $(\omega, U_1, \dots, U_n)$.

A complete theory T is *monadically NFCP* if every monadic expansion $(M, U_1, \dots, U_n)$ of any model of T has NFCP, with T being *monadically stable* and *monadically NIP* being defined analogously. (The definitions of NFCP, stability, and NIP are recalled in the next section.)

We remark here that T is monadically NFCP if and only if T is mutually algebraic (Definition 4.1). It is well known that $NFCP \Rightarrow$ stable $\Rightarrow NIP$, hence we have the implications

T purely monadic $\quad\Rightarrow\quad$ mon. NFCP $\quad\Rightarrow\quad$ mon. stable $\quad\Rightarrow\quad$ mon. NIP

Definition 1.3. A subset $Y \subseteq \omega^k$ is *monadically definable* if it is definable in some monadic structure $(N, U_1, \dots, U_n)$.

$Y \subseteq \omega^k$ is *monadically NFCP definable* if it is definable in some monadically NFCP structure N. Analogously, Y is *monadically stable/monadically NIP definable* if it is definable in some monadically stable/monadically NIP structure N.

Equivalently, a subset $Y \subseteq \omega^k$ is monadically definable (respectively, any of monadically NFCP, stable, NIP) if and only if the structure $N = (\omega, Y)$ in a language with a single k-ary predicate symbol, is purely monadic (respectively, monadically NFCP, stable, NIP).

Thus, we have the implications

Y mon. definable $\Rightarrow$ mon. NFCP-def $\Rightarrow$ mon. stable-def $\Rightarrow$ mon. NIP-def

Our main theorem, Theorem 1.6, characterizes the worst-case complexity of (T, Y). To make the statement more readable, note that two special cases are immediate, simply by unpacking the definitions.

Fact 1.4. *Let T be a complete theory and $Y \subseteq \omega^k$.*

1. *If T is purely monadic and Y is mon. def/mon. NFCP def/mon. stable def/mon. NIP def, then (T, Y) is always purely monadic/mon. NFCP/ mon. stable/mon. NIP.*
2. *If T is any of purely monadic/mon. NFCP/mon. stable/mon. NIP and $Y \subseteq \omega^k$ is monadically definable, then (T, Y) is always purely monadic/mon. NFCP/mon. stable/mon. NIP, respectively.*

It is also helpful to record the following equivalents of a theory monadically coding graphs that are proved by Baldwin and Shelah [1].

Fact 1.5. *The following are equivalent for a complete theory T.*

(1) T does not monadically code graphs;
(2) T does not monadically admit coding (see Definition 2.2); and
(3) T is monadically NIP.

With this in hand, we can state our main result. Note that (T, Y) is not always monadically NIP if and only if there is some model M of T with universe ω such that (M, Y) monadically codes graphs.

Theorem 1.6. *Suppose a complete theory T is not purely monadic and $Y \subseteq \omega^k$ is not monadically definable. Then (T, Y) is always monadically NIP if and only if both T is monadically NFCP and Y is monadically NFCP definable. Furthermore, in that case, (T, Y) is always monadically NFCP.*

A consequence of Theorem 1.6 is that monadically stable and monadically NIP theories are aptly named. That is, every monadic expansion of any model of such a theory has the same complexity, but for any non-monadically definable Y, some expansion (M, Y) of a model of T monadically codes graphs.

The new material begins in Section 3, where we discuss complete theories that are not monadically NFCP. As noted above, some are monadically stable, others are monadically NIP but not monadically stable, and some are not monadically NIP. We exhibit paradigms of structures that are in these differences, and also prove that each paradigm definably embeds into some monadic expansion of some model of any theory of that class.

In Section 4, we characterize the sets Y that are monadically NFCP-definable, but not monadically definable, and in Section 5 we put these results together and give the proof of Theorem 1.6.

We remark that in the discussion above, we considered theories in a countable language and sets $Y \subseteq \omega^k$, but this was not necessary. In what follows, we consider complete theories in languages L of arbitrary size, and sets $Y \subseteq \lambda^k$ for any cardinal $\lambda \geq ||L||$.

2. Preliminaries

We recall the following well known conditions on a partitioned formula $\phi(\overline{x}, \overline{y})$, when we are working in a sufficiently saturated model $\mathfrak{C}$ of a complete theory T: $\phi(\overline{x}, \overline{y})$ has the *finite cover property (FCP)* if, for arbitrarily large n, there are $\langle \overline{a}_i : i < n \rangle$ in $\mathfrak{C}$ such that,

$$\mathfrak{C} \models \neg \exists \overline{x} (\bigwedge_{i<n} \phi(\overline{x}, \overline{a}_i)) \wedge \bigwedge_{\ell<n} \exists \overline{x} (\bigwedge_{i<n, i \neq \ell} \phi(\overline{x}, \overline{a}_i))$$

$\phi(\overline{x},\overline{y})$ has the *order property* if, for each n, there are $\langle \overline{a}_i : i < n\rangle$ in $\mathfrak{C}$ such that, for each $k < n$,

$$\mathfrak{C} \models \bigwedge_{k<n} \left[\exists \overline{x} (\bigwedge_{i<k} \phi(\overline{x},\overline{a}_i) \wedge \bigwedge_{k\le i<n} \neg\phi(\overline{x},\overline{a}_i)) \right]$$

$\phi(\overline{x},\overline{y})$ has the *independence property* if, for each n, there are $\langle \overline{a}_i : i < n\rangle$ in $\mathfrak{C}$ such that,

$$\mathfrak{C} \models \bigwedge_{s\subseteq[n]} \left[\exists \overline{x} (\bigwedge_{i\in s} \phi(\overline{x},\overline{a}_i) \wedge \bigwedge_{i\in n\setminus s} \neg\phi(\overline{x},\overline{a}_i)) \right]$$

A complete theory T is *NFCP* if no partitioned formula $\phi(\overline{x},\overline{y})$ has the FCP, T is *stable* if no partitioned formula $\phi(\overline{x},\overline{y})$ has the order property, and T is NIP if no partitioned formula $\phi(\overline{x},\overline{y})$ has the independence property.

There are many equivalents to monadic NFCP (see e.g., [2, 4, 5]) and monadic NIP (see [1, 3, 7]. What we use is encapsulated in the following facts.

Fact 2.1 ([5, Theorem 3.3])**.** *The following are equivalent for a complete theory T.*

(1) T is monadically NFCP;
(2) T is mutually algebraic (see Definition 4.1 below); and
(3) T is weakly minimal and trivial, i.e., for any pair $M \preceq N$ of models, every non-algebraic 1-type $p \in S_1(M)$ has a unique non-algebraic extension $q \in S_1(N)$ and, for every model M, $\mathrm{acl}(A) = \bigcup_{a\in A} \mathrm{acl}(a)$ for every subset $A \subseteq M$.

Finally, we will also make use of the following sufficient condition from [1] for monadically coding graphs, or equivalently by Fact 1.5, for the failure of monadic NIP. The proof that this implies monadically coding graphs will be the content of the first part of Theorem 3.2.

Definition 2.2. A structure M *admits coding* if there are infinite subsets $A, B, C \subseteq M^1$ and a formula $\phi(x, y, z)$ whose restriction to $A \times B \times C$ is the graph of a bijection $f : A \times B \to C$. A theory T *monadically admits coding* if some monadic expansion M^* of some model M of T admits coding.

3. Finding paradigms of non-monadically NFCP theories

In this section, we show the following classical structures will always witness the failure of monadic NIP/stability/NFCP in a suitable monadic expansion.

- The random graph, sometimes called the Rado graph, $\mathcal{R} = (A, E)$ is the standard example of a structure whose theory has the independence property. In particular, its theory is not monadically NIP.

- Dense linear order (DLO), the theory of $(\mathbb{Q}, \leq)$, is one of the simplest non-stable theories as $\leq$ visibly witnesses the order property. Thus, DLO is not monadically stable, but it is monadically NIP, as can be seen by the classification of colored linear orders.
- Let $\mathcal{E} = (X, E)$, where $X = \omega \times \omega$ (so each element of X can be uniquely written as $(a, b) \in \omega^2$) and $E((a_1, b_1), (a_2, b_2))$ holds if and only if $a_1 = a_2$. Thus, $\mathcal{E}$ is the (unique) model of the ω-categorical theory of an equivalence relation with infinitely many classes, with each class infinite. The theory $Th(\mathcal{E})$ is monadically stable, but it is not monadically NFCP. To the the latter, one can add a single unary predicate whose interpretation contains exactly n elements from the n^{th} E-class. This expanded structure is a paradigm of a stable structure with the finite cover property.

We next show that these paradigms all *definably embed* into a monadic expansion of any model of its class. In the definitions below, it is crucial that the embedding be into the universe M, as opposed to a cartesian power.

Definition 3.1. Let $\mathcal{A} = (A, R)$ be any structure in a language with a binary relation, and let M be an L-structure in some arbitrary language. We say $\mathcal{A}$ *definably embeds into* M if there are L-definable $X \subseteq M^1$ and $R' \subseteq X^2$ and a bijection $f : A \to X$ such that for all $a, b \in A$, $\mathcal{A} \models R(a, b)$ iff $M \models R'(f(a), f(b))$. [Informally, (X, R') is an 'isomorphic copy of $\mathcal{A}$'.]

A definable embedding $f : (A, R) \to (X, R')$ is *type-respecting* if, in addition, for any tuples $\overline{a}, \overline{a}' \in A^n$, if $\mathrm{qftp}_{\mathcal{A}}(\overline{a}) = \mathrm{qftp}_{\mathcal{A}}(\overline{a}')$, then $\mathrm{tp}_M(f(\overline{a})) = \mathrm{tp}_M(f(\overline{a}'))$.

Theorem 3.2. *Let T be a complete L-theory that is not monadically NFCP.*

(1) If T is not monadically NIP, then the random graph $\mathcal{R}$ definably embeds into some monadic expansion M^* of a model M of T.

(2) If T is monadically NIP but not monadically stable, then there is a definable, type-respecting embedding of $(\mathbb{Q}, \leq)$ into some monadic expansion M^* of a model M of T.

(3) If T is monadically stable, then there is a definable, type-respecting embedding of $\mathcal{E}$ into some monadic expansion M^* of a model M of T.

(4) For any (non-monadically NFCP) theory T, $\mathcal{E}$ definably embeds into some monadic expansion M^* of a model M of T.

Proof. (1) Assume T is not monadically NIP. By either [1] or [3], there is an expansion M^* of a model of T that admits coding, i.e., there are infinite sets A, B, C and a 3-ary L^*-formula $\phi(x, y, z)$ coding the graph of a bijection from $A \times B$ to C. By adding more unary predicates, we may assume each of A, B, C are definable in M^* and are countably infinite, and by replacing ϕ by $\phi(x, y, z) \wedge A(x) \wedge B(y) \wedge C(z)$, the graph of ϕ is precisely the bijection. Now add a unary predicate $D \subseteq C$ so that for every $b \in B$ there are exactly two $a \in A$ such that $\exists (d \in D) \phi(a, b, d)$. Thus, in this expansion, one can think of B as coding (symmetric) edges of A via this formula. For the whole of D, we get a complete graph on A, but for any predetermined graph $\mathcal{G}$ with universe A, one can add a single unary predicate $E \subseteq D$ so that for any $a_1, a_2 \in A$,

$$\exists y \exists z (E(z) \wedge \phi(a_1, y, z) \wedge \phi(a_2, y, z)) \iff a_1, a_2 \text{ are edge-related in } \mathcal{G}$$

In particular, we get a definable embedding of $\mathcal{R}$ into this expansion of M^*.

(2) By passing to a monadic expansion, we may assume T itself is unstable. [In fact, any monadically NIP, non-monadically stable theory must itself be unstable, but we don't need this.] By [8], after adding parameters, there is a formula $\phi(x, y)$ with the order property, where x and y are both singletons. Thus, by adding an additional unary predicate for each of the parameters c (with interpretation $\{c\}$) there is an expansion M^* of a model of T with a 0-definable L^*-formula $\psi(x, y)$ with the order property.

By Ramsey and compactness and by passing to an L^*-elementary extension, we may assume there are order-indiscernible subsets $A = \{a_i : i \in \mathbb{Q}\}$ and $B = \{b_j : j \in \mathbb{Q}\}$ of M^* such that $M^* \models \phi(a_i, b_j)$ iff $i \leq j$. By replacing M^* by a mondadic expansion of itself, we may additionally assume there are predicates for A and B. But now, the ordering $a_i \leq' a_j$ is definable on A via the 0-definable L^*-formula $(\forall b \in B)[\phi(a_j, b) \to \phi(a_i, b)]$. Then $(A, \leq')$ witness that there is a type-respecting, definable embedding of $(\mathbb{Q}, \leq)$ into M^*.

(3) By [1, Lemma 4.2.6], if T is monadically stable, then T has trivial forking. Thus, as T is not monadically NFCP, it follows from the characterization in [5] that T is not weakly minimal.

Fact 3.3. *If T is stable but not weakly minimal, then, working in a large, saturated model $\mathfrak{C}$ of T, there is a model $M \preceq \mathfrak{C}$ and singletons a, b such that* $\operatorname{tp}(a/Mb)$ *is not algebraic, but forks over M.*

Proof. As T is not weakly minimal, there are $M_0 \preceq N$ and $p \in S_1(M_0)$ that has two non-algebraic extensions to $S_1(N)$. As p is stationary, this implies there is a non-algebraic $q \in S_1(N)$ that forks over M_0. Let a be any realization of q, and choose Y to be maximal such that $M \subseteq Y \subseteq N$ and $a \mathop{\smile\!\!\!\!\!\!\mid}_{M_0} Y$. As $\operatorname{tp}(a/N)$ forks over M_0, $Y \neq N$, so choose any singleton $b \in N \setminus Y$. By the maximality of Y, $a \not\mathop{\smile\!\!\!\!\!\!\mid}_{Y} b$. To complete the proof, choose a model $M \supseteq Y$ with $M \mathop{\smile\!\!\!\!\!\!\mid}_{Y} ab$. It follows by symmetry and transitivity of non-forking that $a \not\mathop{\smile\!\!\!\!\!\!\mid}_{M} b$. Also, since $\operatorname{tp}(a/N)$ is non-algebraic, so is $\operatorname{tp}(a/Yb)$. But, as $\operatorname{tp}(a/M)$ does not fork over Yb, $\operatorname{tp}(a/M)$ is non-algebraic as well. $\square$

Fix a, b, M as in Fact 3.3 and choose an formula $\phi(x, y) \in \operatorname{tp}(ab/M)$ (with parameters from M) that witnesses the forking over M.

Let $r = \operatorname{tp}(b/M)$ and choose a Morley sequence $B = \{b_n : n \in \omega\}$ in r. Let $q = \operatorname{stp}(a/Mb)$ and, for each n, let q_{b_n} be the strong type over Mb_n conjugate to q. Recursively construct sets $\{I_n : n \in \omega\}$ where each $I_n = \{a_{n,m} : m \in \omega\}$ is a Morley sequence of realizations of the non-forking extension $q^*_{b_n}$ of q_{b_n} to $M \cup B \cup \bigcup\{I_k : k < n\}$. It follows by symmetry and transitivity of non-forking that each I_n is independent and fully indiscernible over $MB \cup \bigcup\{I_k : k \neq n\}$.

Let $A = \{a_{n,m} : n, m \in \omega\}$. Now, any permutation $\sigma \in Sym(B)$ is L_M-elementary, and in fact, induces an L_M-elementary permutation $\sigma^* \in Sym(AB)$. Let $L^* = L \cup \{A, B, C_1, \ldots, C_n\}$ and let $\mathfrak{C}^*$ be the natural monadic expansion of $\mathfrak{C}$ formed by interpreting A and B as above, and interpreting each C_i as $\{c_i\}$, where $\{c_1, \ldots, c_n\}$ are the parameters occurring in ϕ. [We silently replace $\phi(x, y)$ by the natural 0-definable L^*-formula formed by replacing each c_i by C_i.] Finally, define an L^*-definable binary relation E on A^2 by:

$$E(a, a') \iff (\exists b \in B)[\phi(a, b) \wedge \phi(a', b)]$$

It is easily checked that E is an equivalence relation, whose classes are precisely $\{I_n : n \in \omega\}$. Thus, (A, E) is the image of a type-respecting, definable embedding of $\mathcal{E}$ into $\mathfrak{C}^*$.

(4) We prove this by cases. If T is not monadically NIP, then as some monadic expansion codes graphs, any structure will be definably embeddable in some further monadic expansion, so assume T is monadically NIP. If T is also monadically stable, we are done by (3), so assume T is not monadically stable. Then, by (2), there is a type-respecting, definable embedding of $(\mathbb{Q}, \leq)$ into some monadic expansion M^* of a model of T. Thus, it suffices to prove that $\mathcal{E}$ definably embeds into some monadic expansion of $(\mathbb{Q}, \leq)$. But this is easy. Let $A = \mathbb{Q} \setminus \mathbb{Z}$. Then A is 0-definable in the monadic expansion $(\mathbb{Q}, \leq, A)$, as is the relation $E \subseteq A^2$ given by

$$E(a, a') \iff \forall x([a < x < a' \vee a' < x < a] \rightarrow A(x))$$

It is easily checked that (A, E) is isomorphic to $\mathcal{E}$. $\square$

We close this section by stating one 'improvement' of Theorem 3.2(4) that will be used in Section 5. Whereas Theorem 3.2 speaks about a definable embedding of $\mathcal{E}$ into some monadic expansion of some model of T, we isolate the following corollary, which describes a weaker configuration that can be found in arbitrary models of T in the original language.

Corollary 3.4. *Suppose T is a complete L-theory that is monadically NIP, but not monadically NFCP. Then there is an L-formula $\phi(x, y, \overline{z})$ such that, for every model N of T and every $n \geq 1$, there is $\overline{d}_n$ and disjoint sets $B_n = \{b^n_i : i \in \omega\}$, $A_n = \{a^n_{i,j} : i, j < n\}$ that are without repetition such that*

1. *The sets $\{A_n, B_m : n, m \in \omega\}$ are pairwise disjoint;*
2. *For all n, all $i, j, k < n$, one of the following holds.*
    - *(a) T is stable and $N \models \phi(b^n_k, a^n_{i,j})$ if and only if $k = i$;*
    - *(b) T is unstable and $N \models \phi(b^n_k, a^n_{i,j})$ if and only if $k \leq i$.*

Moreover, we may additionally assume that the set $X = N \setminus \bigcup_{n \geq 1}(A_n \cup B_n)$ is infinite.

Proof. As in the proof of Theorem 3.2(2),(3), we split into cases depending on whether or not T is stable. If T is unstable, as in the proof of Theorem 3.2(2), choose an L-formula $\phi(x, y, \overline{z})$ witnessing the order property in large, sufficiently saturated models of T. Now, choose any $N \models T$. As there is some sufficiently saturated $N' \succeq N$ in which $\phi(x, y, \overline{d})$ codes the order property, it follows from elementarity that, for any fixed n, there are $\overline{d}_n \in N^{\lg(\overline{z})}$ and disjoint sets $\{b_i : i < n\}$ and $\{a_{i,j} : i, j < n\}$ such that for all $k, i, j < n$, $N \models \phi(b_k, a_{i,j}, \overline{d}_n)$ if and only if $k \leq i$.

To get the pairwise disjointness, note that if $\{b_i : i < n\}$, $\{a_{i,j} : i, j < n\}$ work for n, then for any subset $s \subseteq n$, the subsets $\{b_i : i \in s\}$, $\{a_{i,j} : i, j \in s\}$ work for $n' = |s|$. Thus, given any fixed finite set F to avoid, given any n, by choosing $m \geq n$ large enough and choosing an appropriate $s \subseteq m$, we can find disjoint sets $\{b_i : i < n\}$ and $\{a_{i,j} : i, j < n\}$, each of which are disjoint from F.

Using this, we can recursively define sequences $\overline{d}_n$ and pairwise disjoint families $B_n = \{b_i^n : i < n\}$ and $A_n = \{a_{i,j}^n : i, j < n\}$ such that for all $k, i, j < n$, $N \models \phi(b_k, a_{i,j}, \overline{d}_n)$ if and only if $k \leq i$. By passing to an infinite subsequence, using the remarks above, and reindexing we can shrink any family $\{B_n, A_n : n \in \omega\}$ to one satisfying the Moreover clause.

If T is stable, then as in the proof of Theorem 3.2(3), T monadically NIP but not monadically NFCP implies T is not weakly minimal. Thus, as in the proof of Theorem 3.2(3), there is a sufficiently saturated elementary extension $N' \succeq N$ and a formula $\phi(x, y, \overline{z})$ that witnesses forking. That is, in N' there are $\{b_i : i \in \mathbb{Z}\}$, $\{a_{i,j} : i, j \in \mathbb{Z}\}$, and $\overline{d}$ such that for all $i, j, k \in \mathbb{Z}$, $N' \models \phi(b_k, a_{i,j}, \overline{d})$ if and only if $k = i$.

Now, using this configuration, the methods used in the unstable case apply here as well. $\square$

4. Sets definable in purely monadic and monadically NFCP structures

Fact 2.1 asserts that a theory is monadically NFCP if and only if it is mutually algebraic, so we recall what is known about sets definable in a mutually algebraic structure. Throughout this section, fix an infinite cardinal λ and think of the set $\lambda = \{\alpha : \alpha \in \lambda\}$ as being a universe. The goal of this section is Lemma 4.5, which gives a configuration that is present in any structure M whose theory is monadically NFCP but not purely monadic.

Definition 4.1. Fix any infinite cardinal λ and any integer $k \geq 1$.

- A subset $Y \subseteq \lambda^k$ is *mutually algebraic* if there is some integer m so that for every $a \in \lambda$, $\{\overline{a} \in Y : a \in \overline{a}\}$ has size at most m.
- A subset $Y^* \subseteq \lambda^{k+\ell}$ is *padded mutually algebraic* if, for some permutation $\sigma \in Sym(k + \ell)$ of the coordinates, there is a mutually algebraic $Y \subseteq \lambda^k$ and $Y^* = \sigma(Y \times \lambda^\ell)$.

- A model M with universe λ is *mutually algebraic* if, for every n, every definable (with parameters) $D \subseteq \lambda^n$ is a boolean combination of definable (with parameters) padded mutually algebraic sets.
- A complete theory T is *mutually algebraic* if some (equivalently, all) models of T are mutually algebraic.

Trivially, every unary subset $Y \subseteq \lambda^1$ is mutually algebraic.

Fact 4.2 ([6, Theorem 2.1])**.** *An L-structure M is mutually algebraic if and only if every atomic L-formula $\alpha(x_1, \dots, x_n)$ is equivalent to a boolean combination of quantifier-free definable (with parameters) padded mutually algebraic sets.*

It follows immediately that any purely monadic structure $M = (\lambda, U_1, \dots, U_n)$ is mutually algebraic.

In this section, our goal is to obtain a particular configuration, described in Lemma 4.5, appearing in any mutually algebraic structure whose theory is not purely monadic. This will be used in the proof of Theorem 1.6, when a non-monadically definable Y induces a mutually algebraic structure.

We begin by characterizing which mutually algebraic sets $Y \subseteq \lambda^k$ are monadically definable. Obviously, every $Y \subseteq \lambda^1$ is monadically definable, so we concentrate on $k \geq 2$. As notation, let $\Delta_k = \{(a, a, \dots, a) \in \lambda^k : a \in \lambda\}$ denote set of constant k-tuples.

Lemma 4.3. *Fix any infinite cardinal λ and any integer $k \geq 2$. A mutually algebraic subset $Y \subseteq \lambda^k$ is monadically definable if and only if $Y \setminus \Delta_k$ is finite.*

Proof. First, suppose $Y \setminus \Delta_k$ is finite. Let $F = \bigcup(Y \setminus \Delta_k) = \{a_1, \dots, a_n\} \subseteq \lambda$ and let $Z = \{a \in \lambda : (a, a, \dots, a) \in Y\}$. Let $N = (\lambda, U_1, \dots, U_n, U_{n+1})$ be the structure in which U_i is interpreted as $\{a_i\}$ for each $i \leq n$ and U_{n+1} is interpreted as Z. Then Y is definable in N, so Y is monadically definable.

Conversely, suppose Y is mutually algebraic and definable in some monadic $N = (\lambda, U_1, \dots, U_n)$. It is easily seen that N admits elimination of quantifiers in a very nice way. Collectively, the unary predicates U_i color each element $a \in \lambda$ into one of 2^n colors. Some of these 2^n colors will have infinitely many elements of λ, while other colorings have only finitely many elements. Let $F = \{a \in \lambda :$ there are only finitely many $b \in \lambda$ such that $N \models \bigwedge_{i=1}^n U_i(a) \leftrightarrow U_i(b)\}$. The set F is clearly finite. Now, the elements of $\lambda \setminus F$ are partitioned into finitely many infinite chunks, each of which is fully indiscernible over its complement. Thus, it follows that $F = \mathrm{acl}_N(\emptyset)$ and for any $a \in \lambda$, $\mathrm{acl}_N(a) = F \cup \{a\}$. To show $Y \setminus \Delta_k$ finite, it suffices to prove the following.

Claim. $Y \subseteq F^k \cup \Delta_k$.

Proof of Claim. Choose any $\overline{a} \in \lambda^k \setminus (F^k \cup \Delta_k)$. Since $\overline{a} \notin F^k$, choose a coordinate $a^* \in \overline{a}$ with $a^* \notin F$. Since the k-tuple $\overline{a}$ is not constant, choose

$b \in \overline{a}$ with $b \neq a^*$. Now, by way of contradiction, suppose $\overline{a} \in Y$. As Y is mutually algebraic, $a^* \in \mathrm{acl}_N(b) = F \cup \{b\}$, which it isn't. $\Diamond$

$\square$

Lemma 4.4. *Suppose M is a mutually algebraic structure with universe λ such that $Th(M)$ is not purely monadic. Then, for some $k \geq 2$ there is some L_M-definable, mutually algebraic $Y \subseteq \lambda^k$ with $Y \setminus \Delta_k$ infinite.*

Proof. Fix such an M and assume that no such L_M-definable, mutually algebraic set existed. By Lemma 4.3 we would have that for every k, every L_M-definable, mutually algebraic subset of λ^k is monadically definable. From this, it follows easily that every L_M-definable, padded mutually algebraic set would be monadically definable, as would every boolean combination of these. But then, as M is mutually algebraic, by Fact 4.2 we would have that the solution set of L-atomic formula is monadically definable. From this, it would follow that every L_M-definable set is monadically definable, contradicting $Th(M)$ not being purely monadic. $\square$

We now obtain our desired configuration.

Lemma 4.5. *Suppose M is a mutually algebraic structure with universe λ whose theory is not purely monadic. Then there is some $k \geq 2$, some L_M-definable $Y \subseteq \lambda^k$ and an infinite set $\mathcal{F} = \{\overline{a}_n : n \in \omega\} \subseteq Y \setminus \Delta_k$ such that*

(1) For each $n \in \omega$, $(\overline{a}_n)_1 \neq (\overline{a}_n)_2$ (the first two coordinates differ); and
(2) $\overline{a}_n \cap \overline{a}_m = \emptyset$ for distinct $n, m \in \omega$.

In particular, if $F = \bigcup \mathcal{F}$, then for every $a \in F$ there is exactly one $\overline{a} \in Y$ with $\overline{a} \subseteq F$ (and hence $(\overline{a})_1 \neq (\overline{a}_2)$).

Proof. By Lemma 4.4, choose $k \geq 2$ and an L_M-definable, mutually algebraic $Y \subseteq \lambda^k$ such that $X := Y \setminus \Delta_k$ is infinite. By mutual algebraicity, choose an integer K such that for every $a \in \lambda$, there are at most K k-tuples $\overline{a} \in Y$ with $a \in \overline{a}$. As each element of X is a non-constant k-tuple, by the pigeonhole principle we can find an infinite $X' \subseteq X$ and $i \neq j \in [k]$ such that $(\overline{a})_i \neq (\overline{a})_j$ for each $\overline{a} \in X'$. By applying a permutation $\sigma \in Sym([k])$ to Y, we may assume $i = 1$ and $j = 2$, so after this transformation (1) holds for any $\overline{a} \in X'$. But now, as $X' \subseteq Y$ is infinite, while every element $a \in \lambda$ occurs in only finitely many $\overline{a} \in X'$, it is easy to recursively construct $\mathcal{F} = \{\overline{a}_n : n \in \omega\} \subseteq X'$. $\square$

5. Monadically stable and monadically NIP are aptly named

In this section, we prove Theorem 1.6. We actually prove this without assuming the language L of T is countable, and where the set Y is a subset of λ^k for an arbitrary cardinal $\lambda \geq ||L||$. The positive part, that (T, Y) is always monadically NFCP whenever both T is and $Y \subseteq \lambda^k$ is monadically NFCP-definable, is immediate from the following.

Lemma 5.1. *Suppose N_1 and N_2 are structures, both with universe λ, in disjoint languages L_1 and L_2. If both N_1 and N_2 are monadically NFCP (=mutually algebraic) then the expansion $N^* = (N_1, N_2)$ is monadically NFCP as well.*

Proof. By replacing each function and constant symbol by its graph, we may assume both L_1 and L_2 only have relation symbols. As the languages are disjoint, this implies that every $L_1 \cup L_2$-atomic formula is either L_1-atomic or L_2-atomic. Thus, by Fact 4.2, every atomic formula in N^* is either equivalent to a boolean combination of either L_1-definable or L_2-definable padded, mutually algebraic formulas. As the notion of a set $Y \subseteq \lambda^k$ being padded mutually algebraic is independent of any structure, the result follows by a second application of Fact 4.2. $\square$

The negative directions are more involved. To efficiently handle the various cases, we first prove two propositions, from which all of the negative results follow in Theorem 5.4.

For the following proposition, first note that a structure with two cross-cutting equivalence relations admits coding. We will essentially encode this configuration, but since we don't want to assume that either N_1 or N_2 is saturated for our eventual application, we must work with the finitary approximations to an equivalence relation with infinitely many infinite classes provided by Corollary 3.4.

Proposition 5.2. *Suppose L_1 and L_2 are disjoint languages, $\lambda \geq ||L_1 \cup L_2||$ a cardinal, N_1 is an L_1-structure with universe λ, and N_2 is an L_2-structure with universe λ. If both $Th(N_1)$ and $Th(N_2)$ are monadically NIP but not monadically NFCP, then there is a permutation $\sigma \in Sym(\lambda)$ such that the $L_1 \cup L_2$-structure $(N_1, \sigma(N_2))$ has a theory that is not monadically NIP.*

Proof. Apply Corollary 3.4 to both N_1 and N_2. This gives an L_1-formula $\phi(x, y, \overline{z})$ and, for each n, pairwise disjoint sets $A_n = \{\alpha^n_{i,j} : i, j < n\}$, $B_n = \{\beta^n_i : i < n\}$ and $\overline{r}_n$ as there, with exceptional set $X = \lambda \setminus \bigcup_{n \geq 1}(A_n \cup B_n)$. Note that as each A_n, B_n is finite, $|X| = \lambda$. On the L_2-side, choose an L_2-formula $\psi(x, y, \overline{w})$ such that, for all $n \geq 1$, there is $\overline{s}_n \in \lambda^{\lg(\overline{w})}$ and pairwise disjoint sets $C_n = \{\gamma^n_{i,j} : i, j < n\}$ and $D_n = \{\delta^n_i : i < n\}$ as there.

Now choose $\sigma \in Sym(\lambda)$ to be any permutation satisfying: For all $n \geq 1$,

(1) $\sigma(D_n) \subseteq X$; and
(2) σ maps C_n bijectively onto A_n via $\sigma(\gamma^n_{i,j}) = \alpha^n_{j,i}$.

Note that there are many permutations σ satisfying these constraints. Choose one, and let $\sigma(N_2)$ be the unique L_2 structure with universe λ so that σ is an L_2-isomorphism.

Claim. The $L_1 \cup L_2$-theory $Th(N_1, \sigma(N_2))$ is not monadically NIP.

Proof of Claim. We will produce M^*, a monadic expansion of an $L_1 \cup L_2$-elementary extension $\overline{M} \succeq (N_1, \sigma(N_2))$ that admits coding, which suffices.

To do this, we first argue that by compactness, there is an $L_1 \cup L_2$-elementary extension $\overline{M} \succeq (N_1, \sigma(N_2))$ that contains disjoint sets $A = \{a_{i,j} : i, j \in \mathbb{Z}\}$, $B = \{b_i : i \in \mathbb{Z}\}$, $D = \{d_j : j \in \mathbb{Z}\}$, and tuples $\overline{r}, \overline{s}$ such that, for all $k, i, j \in \mathbb{Z}$, either (if $Th(N_1)$ is unstable) $\overline{M} \models \phi(b_k, a_{i,j}, \overline{r})$ if and only if $k \leq i$, or (if $Th(N_1)$ is stable) $\overline{M} \models \phi(b_k, a_{i,j}, \overline{r})$ if and only if $k = i$; and dually, either (if $Th(N_2)$ is unstable) $\overline{M} \models \psi(d_k, a_{i,j}, \overline{s})$ if and only if $k \leq j$, or (if $Th(N_2)$ is stable) $\overline{M} \models \psi(d_k, a_{i,j}, \overline{s})$ if and only if $k = j$.

To see that such an $\overline{M}$ exists, consider an expansion of $L_1 \cup L_2$, adding constants for all $a_{i,j}, b_i, d_j, \overline{r}, \overline{s}$, and considering a theory that contains the $L_1 \cup L_2$-elementary diagram of $(N_1, \sigma(N_2))$ and the conditions on the constants described above. By compactness, it suffices to show $(N_1, \sigma(N_2)) \models T_0$ for any finite subset T_0 of this theory. But, for any such T_0, there is an n and some $\{\beta_i^n : i < n\}$, $\{\alpha_{i,j}^n : i, j < n\}$, and $\{\sigma(\delta_j^n) : j < n\}$ from λ that realize the requisite sentences in $(N_1, \sigma(N_2))$ because of the identification $\sigma(\gamma_{i,j}^n) = \alpha_{j,i}^n$.

Now, given $\overline{M}$, let $L^* = L_1 \cup L_2 \cup \{A, B, D\}$ and let M^* be the natural monadic expansion of $\overline{M}$ described by A, B, D above. To show that M^* admits coding, we need to rectify the ambiguity between the stable and unstable cases. Specifically, we claim that there is an L^*-formula $\phi^*(x, y, \overline{z})$ such that for all $b_i \in B$, the solution set $\phi^*(b_i, M^*, \overline{r})$ is $\{a_{i,j} : j \in \mathbb{Z}\}$. If $Th(N_1)$ were stable, this is easy, just take $\phi^*(x, y, \overline{z}) := A(y) \wedge \phi(x, y, \overline{z})$. However, when $Th(N_1)$ is unstable, we need some more L^*-definability in M^*. Specifically, note that in this case, the natural ordering on B is L^*-definable via

$$b_i \leq b_j \text{ if and only if } \forall y[(A(y) \wedge \phi(b_j, y, \overline{r})) \rightarrow \phi(b_i, y, \overline{r})]$$

As the ordering on B is discrete, every element $b \in B$ has a unique successor, $S(b)$, and this operation is L^*-definable since $\leq$ is. Using this, the L^*-formula

$$\phi^*(x, y, \overline{z}) := B(x) \wedge A(y) \wedge \phi(x, y, \overline{z}) \wedge \neg\phi(S(x), y, \overline{z})$$

is as desired.

Arguing similarly, there is an L^*-formula $\psi^*(x, y, \overline{w})$ such that for all $d_j \in D$, the solution set $\psi^*(d_j, M^*, \overline{s})$ is $\{a_{i,j} \in A : i \in \mathbb{Z}\}$. Putting these together, let $\theta(u, v, y, \overline{z}, \overline{w})$ be the L^*-formula

$$B(u) \wedge D(v) \wedge A(y) \wedge \phi^*(u, y, \overline{z}) \wedge \psi^*(v, y, \overline{w})$$

Then the solution set of $\theta(u, v, y, \overline{r}, \overline{s})$ is precisely the graph of a bijection from $B \times D$ onto A. Thus, M^* admits coding, which suffices. $\diamondsuit$

$\square$

The proof of the next proposition is in many ways similar. Here our ideal infinitary configuration consists of an equivalence relation with infinitely many infinite classes, with each tuple from the configuration in Lemma 4.5 pairing two classes by intersecting them. But again, instead of our ideal equivalence relation, we must restrict ourselves to the finitary approximations from Corollary 3.4.

Proposition 5.3. *Suppose L_1 and L_2 are disjoint languages, $\lambda \geq ||L_1 \cup L_2||$ a cardinal, N_1 is an L_1-structure with universe λ, and N_2 is an L_2-structure with universe λ. If $Th(N_1)$ is monadically NIP but not monadically NFCP, and if $Th(N_2)$ is monadically NFCP but not purely monadic, then there is a permutation $\sigma \in Sym(\lambda)$ such that the $L_1 \cup L_2$-structure $(N_1, \sigma(N_2))$ has a theory that is not monadically NIP.*

Proof. Apply Corollary 3.4 to N_1, obtaining an L_1-formula $\phi(x, y, \overline{z})$ and, for each n, pairwise disjoint sets $A_n = \{\alpha^n_{i,j} : i, j < n\}$, $B_n = \{\beta^n_i : i < n\}$ and $\overline{r}_n$ as there, with exceptional set $X = \lambda \setminus \bigcup_{n \geq 1}(A_n \cup B_n)$. Note that as each A_n, B_n is finite, $|X| = \lambda$. For the N_2 side, apply Lemma 4.5, getting an N_2-definable $Y \subseteq \lambda^k$ and a distinguished set $\mathcal{F} = \{\overline{e}_\ell : \ell \in \omega\} \subseteq Y$ as there. Say Y is defined using parameters $\{c_1, \ldots, c_n\}$. Let $L_2^V = L_2 \cup \{V, C_1, \ldots, C_n\}$ and let N_2^V be the monadic expansion of N_2, interpreting V as $F = \bigcup \mathcal{F}$ and each C_i as $\{c_i\}$. Note that in N_2^V, the subsets $F_1 = \{(\overline{e})_1 : \overline{e} \in \mathcal{F}\}$, $F_2 = \{(\overline{e})_2 : \overline{e} \in \mathcal{F}\}$ of F are L_2^V-definable (without parameters), along with the bijection $f : F_1 \to F_2$ given by: $f(x) = (\overline{e})_2$, where $\overline{e}$ is the unique element of $\mathcal{F}$ containing x. Fix an enumeration $\{\gamma_\ell : \ell \in \omega\}$ of $F_1 \subseteq \lambda$.

We now choose a permutation $\sigma \in Sym(\lambda)$ that satisfies:

- For all $n \geq 1$ and all distinct $i < j < n$, there is some (in fact, unique) $\ell \in \omega$ such that $\sigma(\gamma_\ell) = \alpha^n_{i,j}$ and $\sigma(f(\gamma_\ell)) = \alpha^n_{j,i}$.

Let $\sigma(N_2^V)$ be the L_2^V-structure with universe λ so that σ is an L_2^V-isomorphism and let $M_0^V = (N_1, \sigma(N_2^V))$ be the expansion of N_1 to an $L_1 \cup L_2^V$-structure. So M_0^V has universe λ and satisfies:

- For all $n \geq 1$ and $i < j < n$, $f(\alpha^n_{i,j}) = \alpha^n_{j,i}$; and
- The relationships given by N_1.

Let M_0 be the $L_1 \cup L_2$-reduct of M_0^V.

Claim. The $L_1 \cup L_2$-theory of M_0 is not monadically NIP.

Proof of Claim. We show that the $L_1 \cup L_2^V$-theory of M_0^V is not monadically NIP, which suffices. For this, the strategy is similar to the proof of Proposition 5.2. We will find an $L_1 \cup L_2^V$-elementary extension $\overline{M}$ of M_0^V and then find a monadic expansion M^* of $\overline{M}$ that admits coding. Specifically, choose an $L_1 \cup L_2 \cup \{V\}$-elementary extension $\overline{M}$ for which there are sets $B = \{b_i : i \in \mathbb{Z}\}$, $A = \{a_{i,j} : i \neq j \in \mathbb{Z}\}$ such that

(1) For all $i < j$ from $\mathbb{Z}$, $f(a_{i,j}) = a_{j,i}$.
(2) One of the following holds.
  (a) $Th(N_1)$ is unstable, and $\overline{M} \models \phi(b_k, a_{i,j}, \overline{r})$ if and only if $k \leq i$.
  (b) $Th(N_1)$ is stable, and $\overline{M} \models \phi(b_k, a_{i,j}, \overline{r})$ if and only if $k = i$.

Given such an $\overline{M}$, let $L^* = L_1 \cup L_2^V \cup \{A, B\}$, and let M^* be the expansion of $\overline{M}$ interpreting A and B as themselves. Exactly as in the proof of

Proposition 5.2, find an L^*-formula $\phi^*(x, y, \overline{z})$ such that for all $b_i \in B$, the solution set $\phi^*(b_i, M^*, \overline{r})$ is $\{a_{i,j} : j \in \mathbb{Z}, j \neq i\}$. Now let $\theta(u, v, y, \overline{z})$ be the L^*-formula

$$B(u) \wedge B(v) \wedge A(y) \wedge \phi^*(u, y, \overline{z}) \wedge \phi^*(v, f(y), \overline{z})$$

Then the formula $\theta(u, v, y, \overline{r}) \vee \theta(v, u, y, \overline{r})$ is the graph of a bijection from $(B \times B)\backslash\{(b, b) : b \in B\}$ onto A, which suffices. $\Diamond$

$\square$

Using Propositions 5.2 and 5.3 we are now able to prove the negative portions of Theorem 1.6. As the positive portion was proved in Lemma 5.1, this suffices.

Theorem 5.4. *Suppose T is a complete L-theory and $Y \subseteq \lambda^k$ with $\lambda \geq ||L||$. Then:*

1. *If T is not monadically NFCP and Y is not monadically definable, then (T, Y) is not always monadically NIP; and*
2. *If T is not purely monadic and Y is not monadically NFCP definable, then (T, Y) is not always monadically NIP.*

Proof. (1) Choose $N_1 \models T$ with universe λ, and let $N_2 = (\lambda, Y)$ be the structure in the language $L_2 = \{Y\}$ with the obvious interpretation. If T is not monadically NIP, then the expansion (N_1, Y) suffices, so assume $Th(N_1)$ is not monadically NIP. Similarly, if $Th(N_2)$ is not monadically NIP, then again (N_1, Y) suffices, so also assume $Th(N_2)$ is also monadically NIP. Now, depending on whether or not $Th(N_2)$ is monadically NFCP or not, apply either Proposition 5.2 or Proposition 5.3 to get a permutation $\sigma \in Sym(\lambda)$ such that $Th(N_1, \sigma(N_2))$ is not monadically NIP. Of course, Y need not be preserved here, so apply σ^{-1}. That is, let $(\sigma^{-1}(N_1), Y)$ be the $L \cup \{Y\}$-structure so that σ^{-1} is an $L \cup \{Y\}$ isomorphism. As $\sigma(N_1) \models T$, this structure witnesses that (T, Y) is not always monadically NIP.

(2) Let $N_1 = (\lambda, Y)$ and let N_2 be any model of T with universe λ. As in (1), if either $Th(N_1)$ or $Th(N_2)$ does not have a monadically NIP theory we are done, so assume both do. Again, by either Proposition 5.2 or Proposition 5.3 (depending on $Th(N_2)$), we get a permutation $\sigma \in Sym(\lambda)$ such that $(N_1, \sigma(N_2))$ has a non-monadically NIP theory. But this structure is precisely $(\sigma(N_2), Y)$ and $\sigma(N_2) \models T$, so again (T, Y) is not always monadically NIP. $\square$

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