ω-stable theories: Do uncountable languages matter?

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ω-stable theories

This work is joint with Saharon Shelah.

Fix a (complete) ω-stable theory.

So:

- If $p \in S(A)$ is stationary, there is a finite $A_0 \subseteq A$ such that p does not fork over A_0 and $p|A_0$ is stationary.
- Thus, M a-saturated $\Leftrightarrow$ M ω-saturated.
- Prime models exist over arbitrary sets. (Unique up to $\cong$, but not unique!)

Definition

A type $p \in S(A)$ is ENI (eventually non-isolated) if p is stationary, regular, and for some finite $A_0 \subseteq A$ over which p is based and stationary, there is a countable $M \supseteq A_0$ with $\dim(p|A_0, M) < \aleph_0$.

We do not require p to be strongly regular!

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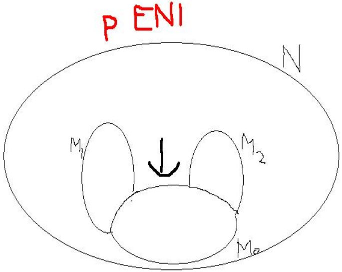
Advantage

The class of ENI types is closed under non-orthogonality and automorphisms of $\mathfrak{C}$.

Definition

A theory T has ENI-NDOP if, for all independent triples $M_0 = M_1 \cap M_2$ of a-saturated models, for all a-prime models $N \supseteq M_1 \cup M_2$, and for all ENI p,

$$p \not\perp N \Rightarrow p \not\perp M_i \quad \text{for some } i < 3.$$



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- *T has ENI-NDOP*
- *The prime model over any independent triple of countable saturated models is saturated*
- *T has NOTOP i.e., there is NO $p(\overline{x}, \overline{y}, \overline{z})$ such that for any graph (G, E) we can find $\{\overline{a}_g : g \in G\}$ and a model M_G such that $p(\overline{x}, \overline{a}_g, \overline{a}_h)$ is omitted in M_G iff $G \models E(g, h)$*

ENI-supportive types

A stationary, regular type $tp(b/A)$ is *ENI-supportive* if $\mathrm{tp}(b/A)$ is ENI OR there is an ENI $q \in S(C)$, $C \supseteq Ab$ and dominated by b over A, with $q \perp A$.

Think: $tp(b/A)$ is ENI or lies below an ENI type in a decomposition tree.

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Fact: T superstable, $\mathbb{P}$ any class of stationary, regular types closed under non-orthogonality and automorphisms of $\mathfrak{C}$. Then:

- T has $\mathbb{P}$-NDOP implies T has $supp(\mathbb{P})$-NDOP
- $q \in supp(\mathbb{P})$ of depth > 0 implies q trivial.

ENI-supportive decompositions

Definition

A *prime decomposition* of M^* sequence $\langle M_\eta, a_\eta : \eta \in I\rangle$ indexed by a tree $(I, \trianglelefteq)$ satisfying:

- $\{M_\eta : \eta \in I\}$ is an independent tree of countable, elementary substructures of M^*
- $M_{\langle\rangle}$ is prime
- For every $\eta \in I$, $\{a_\nu : \nu \in Succ_I(\eta)\}$ is a maximal independent (over M_η) set of ENI-supportive types satisfying $\text{tp}(a_\nu / M_\eta) \perp M_{\eta^-}$ (when $\eta \neq \langle\rangle$)
- For all $\nu \neq \langle\rangle$, M_ν is prime over $M_{\nu^-} \cup \{a_\nu\}$
- M^* is prime over $\bigcup\{M_\eta : \eta \in I\}$.

Existence of decompositions

Theorem (proved independently by Koerwien)

If T is ω-stable with ENI-NDOP, then every model of T has a prime decomposition.

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If T is ω-stable with ENI-NDOP, then every model of T has a prime decomposition.

An ω-stable T with ENI-NDOP is *ENI-deep* if some model of T has a prime decomposition with an infinite branch.

Borel reduciblity

Fix countable vocabularies τ_1, τ_2.

For $i = 1, 2$

- $\mathscr{S}_i = \{$all τ_i-structures with universe $\omega\}$ with the usual topology.
- X_i be a Borel subset of $\mathscr{S}_i$, closed under $\cong$.
- E_i be an equivalence relation on X_i extending $\cong$.

$(X_1, E_1) \leq_B (X_2, E_2)$ iff there is a Borel $f : X_1 \to X_2$ (relative to topologies on $\mathscr{S}_1, \mathscr{S}_2$) such that for all $\mathfrak{A}, \mathfrak{B} \in X_1$

$$\mathfrak{A} E_1 \mathfrak{B} \iff f(\mathfrak{A}) E_2 f(\mathfrak{B})$$

Borel completeness

Fact: $(Graphs, \cong)$ is maximal w.r.t. Borel reducibility.

Definition

(X, E) is *Borel complete* if $(Graphs, \cong) \leq_B (X, E)$.

Routine: If T is ω-stable with ENI-DOP, then $(Mod_{\aleph_0}(T), \cong)$ is Borel complete.

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Routine: If T is ω-stable with ENI-DOP, then $(Mod_{\aleph_0}(T), \cong)$ is Borel complete. Pf: Use OTOP!

A surprise: (Friedman-Stanley) (subtrees of ${}^{<\omega}\omega, \cong$) is Borel complete.

This suggests: T ENI-NDOP, ENI-deep implies $(Mod_{\aleph_0}(T), \cong)$ Borel complete.

Sketch: Given I, form an independent tree $\langle M_\eta, a_\eta : \eta \in I \rangle$ indexed by I in a canonical way, and let M_I be prime over $\bigcup\{M_\eta : \eta \in I\}$.

- If $I \cong J$ as trees, then $M_I \cong M_J$ (easy).
- If $M_I \cong M_J$, then $I \cong J$???

The issue: Given a countable model of T, how unique is its decomposition?

Idea:

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We can use subsets of $R(M^*)$ to 'measure' ENI-supportive types $\text{tp}(c/A)$ with $A \subseteq M^*$ finite and $c \in M^*$.

Definition

A (c, A)-decomposition of M^* is a sequence $\langle M_\eta, a_\eta : \eta \in I \rangle$ such that:

- $(I, \trianglelefteq)$ is a tree in which $\langle 0 \rangle$ is the unique successor of $\langle\rangle$
- $A \subseteq M_{\langle\rangle}$, $c = a_{\langle 0 \rangle}$ and $\mathrm{tp}(c/M_{\langle\rangle})$ does not fork over A
- $\{M_\eta : \eta \in I\}$ is an independent tree of ω-saturated submodels of M^*;
- For all $\nu \neq \langle\rangle$ if $tp(b/M_{\nu^-})$ ENI-supportive and $\mathrm{tp}(b/M_\nu)$ forks over M_{ν^-}, then $\mathrm{tp}(b/M_{\nu^-} \cup \{a_\nu\})$ forks over M_{ν^-}.
- For all $\eta \neq \langle\rangle$ $\{a_\nu : \nu \in Succ_I(\eta)\} \subseteq M^*$ is a maximal independent over M_η set of realizations of ENI-supportive types.
- $(I, \trianglelefteq)$ is maximal with respect to these conditions.

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M^* need not be a-prime over $\bigcup\{M_\eta : \eta \in I\}$!

Fix c, A from M^* with A finite and $\mathrm{tp}(c/A)$ ENI-supportive.

$X(c, A) = \{q \in R(M^*) : q \not\perp M_\eta$ for some (c, A)-decomposition $\langle M_\eta, a_\eta : \eta \in I\rangle$ of M^* and some $\eta \neq \langle\rangle\}$.

Theorem (Shelah)

- *$X(c, A)$ DOES NOT DEPEND on our choice of (c, A)-decompositions (!)*
- *For any ω-saturated decomposition $\langle N_\eta, b_\eta : \eta \in I\rangle$ of M^*, for any $\eta, \nu \in I$,*
  - *If $\eta \trianglelefteq \nu$ then $X(b_\nu, Cb(b_\nu/M_{\nu^-})) \subseteq X(b_\eta, Cb(b_\eta/M_{\eta^-}))$*
  - *If η, ν are incomparable then $X(b_\eta, Cb(b_\eta/M_{\eta^-}))$ and $X(b_\nu, Cb(b_\nu/M_{\nu^-}))$ are disjoint.*

Back to countable models

Definition

A prime decomposition $\langle M_\eta, a_\eta : \eta \in I \rangle$ is *tidy* if for all $\eta \in I$,

- $\{\text{tp}(a_\nu / M_\eta) : \nu \in Succ_I(\eta)\}$ is finite;
- For each $\nu \in Succ_I(\eta)$ there are infinitely many $\nu' \in Succ_I(\eta)$ such that $\text{tp}(a_{\nu'} / M_\eta) = \text{tp}(a_\nu / M_\eta)$;
- If $\nu, \gamma \in Succ_I(\eta)$ and $\text{tp}(a_\nu / M_\eta) \neq \text{tp}(a_\gamma / M_\eta)$, then $\text{tp}(a_\nu / M_\eta) \perp \text{tp}(a_\gamma / M_\eta)$.

Will see: Prime decompositions of a tidy model are 'almost isomorphic'.

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A nonempty subtree $J \subseteq I$ of a tree is *large* if for all $\eta \in J$, $Succ_I(\eta) \setminus J$ is finite.

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Lemma

(subtrees of ${}^{<\omega}\omega, \cong^*$*) are Borel complete.*

Theorem

T ω-stable, ENI-NDOP, $M \models T$. If $\langle M^1_\eta, a_\eta : \eta \in I\rangle$ and $\langle M^2_\eta, b_\eta : \eta \in J\rangle$ are both tidy decompositions of M, then $(I \setminus Leaves(I))$ and $(J \setminus Leaves(J))$ are almost isomorphic.

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- Let M^* be a-prime over $M \cup \bigcup\{N_\nu : \nu \in K_0\}$ and choose an ω-saturated decomposition $\langle N_\nu, c_\nu : \nu \in K\rangle$ of M^* extending $\langle N_\nu, c_\nu : \nu \in K_0\rangle$.

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- For $\eta \in I$, for all but finitely many $\gamma \in Succ_I(\eta) \setminus Leaves(I)$ there is a unique $\nu \in K$ such that $X(a_\gamma/Cb(a_\gamma/M^1_{\gamma^-})) = X(c_\nu/Cb(c_\nu/N_{\nu^-}))$ (and dually for $\langle M^2_\eta : \eta \in J\rangle$).

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- For $\eta \in I$, for all but finitely many $\gamma \in Succ_I(\eta) \setminus Leaves(I)$ there is a unique $\nu \in K$ such that $X(a_\gamma / Cb(a_\gamma / M^1_{\gamma^-})) = X(c_\nu / Cb(c_\nu / N_{\nu^-}))$ (and dually for $\langle M^2_\eta : \eta \in J\rangle$).
- Composing these partial maps gives the almost isomorphism.

Corollary

T ω-stable, ENI-NDOP, ENI-deep $\Rightarrow$ $(Mod_{\aleph_0}(T), \cong)$ is Borel complete.

Question: Are there ω-stable, ENI-shallow theories that are Borel complete?

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Example (Koerwien)

There is an ω-stable, ENI-depth 2 theory for which $\{SH(M) : M \in Mod_{\aleph_0}(T)\}$ is unbounded in ω_1.

Koerwien's example is a 'pun on ω'.

All of the complexity arises from the complicated structure of the automorphisms of $acl(\emptyset)$!

Borel reduciblity

Recall: Fix countable vocabularies τ_1, τ_2.

For $i = 1, 2$

- $\mathscr{S}_i = \{$all τ_i-structures with universe $\omega\}$ with the usual topology.
- X_i be a Borel subset of $\mathscr{S}_i$, closed under $\cong$.
- E_i be an equivalence relation on X_i extending $\cong$.

$(X_1, E_1) \leq_B (X_2, E_2)$ iff there is a Borel $f : X_1 \to X_2$ (relative to topologies on $\mathscr{S}_1, \mathscr{S}_2$) such that for all $\mathfrak{A}, \mathfrak{B} \in X_1$

$$\mathfrak{A} E_1 \mathfrak{B} \iff f(\mathfrak{A}) E_2 f(\mathfrak{B})$$

$\mathscr{S}_2$-Basic Open Sets: $\mathscr{U}_{R,\overline{n}} = \{M_2 \in \mathscr{S}_2 : M_2 \models R(\overline{n})\}$ for every $R \in \tau_2$ and $\overline{n} = (n_1 \ldots, n_k) \in \omega^{\text{arity}(R)}$.

$f : \mathscr{S}_1 \to \mathscr{S}_2$ Borel means f^{-1}(Basic open set in $\mathscr{S}_2$) is a Borel subset of $\mathscr{S}_1$.

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So: For every $R, \overline{n}$ there is a quantifier-free $\Phi_{R,\overline{n}} \in L_{\omega_1.\omega}$ in the vocabulary $\tau_1(\omega)$ such that

$$M_1 \models \Phi_{R,\overline{n}} \iff f(M_1) \models R(\overline{n})$$

λ-Borel reducibility

Generalize to $\lambda \geq \aleph_0$:

$\mathscr{S}_1^\lambda = \{\tau_1$-structures with universe $\lambda\}$
$\mathscr{S}_2^\lambda = \{\tau_2$-structures with universe $\lambda\}$.

$f : \mathscr{S}_1^\lambda \to \mathscr{S}_2^\lambda$ is λ-Borel if for every $R \in \tau_2$ and $\overline{\alpha} = (\alpha_1, \ldots, \alpha_k) \in \lambda^k$ there is a q.f. $L_{\lambda^+,\omega}$-sentence $\Phi_{R,\overline{\alpha}}$ such that

$$M_1 \models \Phi_{R,\overline{\alpha}} \iff f(M_1) \models R(\overline{\alpha})$$

Definition

If, for $i = 1, 2$ X_i is a λ-Borel subset of $\mathscr{S}_i$ and E_i an equivalence relation on X_i extending $\equiv_{\lambda^+,\omega}$, then (X_1, E_1) is *λ-Borel reducible* to (X_2, E_2) if there is a λ-Borel $f : X_1 \to X_2$ such that $\mathfrak{A} E_1 \mathfrak{B} \Leftrightarrow f(\mathfrak{A}) E_2 f(\mathfrak{B})$.

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As before (Graphs on $\lambda, \equiv_{\lambda^+,\omega}$) is maximal w.r.t. λ-Borel reducibility.

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A surprise: (subtrees of ${}^{<\omega}\lambda, \equiv_{\lambda^+,\omega}$) is λ-Borel complete.

Theorem

TFAE for ω-stable theories T:

- *T is ENI-NDOP, ENI-shallow*
- *For some $\lambda \geq \aleph_0$ $(Mod_\lambda(T), \equiv_{\lambda^+,\omega})$ is not λ-Borel complete*
- *For some $\lambda \geq \aleph_0$ $\{SH_{\lambda^+,\omega}(M) : M \in Mod_\lambda(T)\}$ is bounded below λ^+*

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There are ω-stable, ENI-NDOP, ENI-deep theories T with DOP ! Such theories T have 2^κ nonisomorphic models, but fewer than 2^κ $L_{\infty,\omega}$-inequivalent models for most uncountable cardinals κ.

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This partially explains why it is hard to prove "many-models" for theories with DOP.

The real John Baldwin.

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Thank you John for all you have taught me. I am forever grateful.